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Torsion-free sheaves and moduli of generalized spin curves

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Abstract. This article treats compactifications of the space of generalized spin curves. Generalized spin curves, or r-spin curves, are pairs (X, L) with X a smooth curve, and L a line bundle whose rth tensor power is isomorphic to the canonical bundle of X. These are a natural generalization of 2-spin curves (algebraic curves with a theta-characteristic), which have been of interest recently, in part because of their applications to fermionic string theory.

Three different compactifications over $\mathbb{Z}[1/r]$, all using torsion-free sheaves, are constructed. All three yield algebraic stacks, one of which is shown to have Gorenstein singularities that can be described explicitly, and one of which is smooth. All three compactifications generalize constructions of Deligne and Cornalba done for the case when $r = 2$.

Mathematics Subject Classifications: 14H10, 14M30.

Key words: spin curves, compactification, moduli, theta-characteristics.

1. Introduction

In this article we construct and describe three compactifications of the space of generalized spin curves. Generalized spin curves, or r-spin curves, are pairs (X, L) with X a smooth curve, and L a line bundle whose rth tensor power is isomorphic to the canonical cotangent bundle of X. These are a natural generalization of 2-spin curves (algebraic curves with a theta-characteristic), which have been of interest recently, in part because of their applications to fermionic string theory.

More precisely, for positive integers r and g, with $g > 2$, r dividing $2g - 2$, and for a flat family of smooth curves X/T of genus g, an *r-spin structure* on X is a line bundle $\mathcal{L}$, such that $\mathcal{L}^{\otimes r} \cong \omega_{X/T}$. And an *$r$-spin curve* over T is a flat family of smooth curves with an r-spin structure. Now, for a fixed base scheme S over $\mathbb{Z}[1/r]$, let $\mathrm{SPIN}_{r,g}$ be the sheafification of the functor which takes an S-scheme T to the set of isomorphism classes of r-spin curves over T. A compactification of the space of spin curves is a space (scheme or algebraic stack), which is proper over the Deligne–Mumford compactification $\overline{\mathcal{M}_g}$ of the space of smooth curves $\mathcal{M}_g$, and whose fibre over $\mathcal{M}_g$ represents, at least coarsely, the functor $\mathrm{SPIN}_{r,g}$.

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The author has previously [19] constructed a compactification of $\mathrm{S{\small PIN}}_{r,g}$ using geometric invariant theory. Namely, in the style of L. Caporaso [6], for a fixed $d \gg 0$ one can choose a subscheme of the Hilbert scheme $\mathrm{Hilb}^{dz-g}_{\mathbb{P}^N}$ with a geometric quotient that coarsely represents $\mathrm{S{\small PIN}}_{r,g}$. And using results of Gieseker [15, Theorems 1.0.0 and 1.0.1], one can show that the semi-stable closure of the subscheme in $\mathrm{Hilb}^{dz-g}_{\mathbb{P}^N}$ has a categorical quotient that provides a compactification. This compactification is actually a subscheme of Caporaso's compactification of the relative Picard scheme over $\overline{\mathcal{M}_g}$, but it has some weaknesses.

The principle drawback to the GIT compactification is that it is not obviously the solution to a moduli problem, and therefore it is difficult to describe the resulting space and to make the construction work over a general base, rather than only over algebraically closed fields. Moreover, the GIT construction requires some arbitrary choices, and it is not clear that the resulting compactification is completely independent of these choices. To deal with these problems we take a different approach here: we pose moduli problems and then show that the associated stacks are algebraic and that they do indeed compactify $\mathrm{S{\small PIN}}_{r,g}$. This approach permits a fairly detailed look at the local geometry of the compactifications.

Motivation for the moduli problems comes from the compactification of curve-line bundle pairs using geometric invariant theory. The invariant theory approach gives boundary points that correspond to pairs (X, L) with X a semi-stable curve having disjoint exceptional curves (copies of $\mathbb{P}^1$ that intersect the remaining curve in at most two points), and having a line bundle L of degree one on each exceptional curve in X. Contracting all the exceptional curves $\rho: X \to \bar{X}$ makes the underlying curve $\bar{X}$ stable, and the direct image $\rho_* L$ of L is a torsion-free sheaf; namely, it has no associated primes of height one. Furthermore, the torsion-free sheaves on stable curves don't have the problem of having infinite automorphism groups that the line bundles on semi-stable curves have. It is, therefore, natural to expect that torsion-free sheaves will be well-suited to the compactification of the moduli of spin curves, and this is the case, once we add some additional structure.

The additional structure is also motivated by this contraction; namely, there is a canonical $\mathcal{O}_X$-module homomorphism of the rth tensor power of $\rho_* L$ to the canonical dualizing bundle $b: \rho_* L^{\otimes r} \to \omega_{\bar{X}}$ with the property that b is an isomorphism on the open set where $\rho_* L$ is locally free. Thus the most naive approach would be to use a rank-one torsion-free sheaf $\mathcal{E}$, with a suitable $\mathcal{O}_X$-module homomorphism from $\mathcal{E}^{\otimes r}$ to the canonical bundle. But this doesn't quite work, as the resulting space is not separated. A second property of the induced map b is that the length of the cokernel of b is $r-1$ at the points where $\rho_* L$ is not locally free. And the first moduli problem that yields a separated and compact space is given by the functor of *quasi-spin* curves. Namely, a quasi-spin curve is a triple consisting of a flat family of stable curves X/T, a rank-one torsion-free sheaf $\mathcal{E}$ of degree $(2g-2)/r$ on X/T, and an $\mathcal{O}_X$-module homomorphism b that is an isomorphism except where $\mathcal{E}$ is not locally free; and the length of the cokernel of b at those singular points must be $r-1$.

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The main results about quasi-spin curves are summed up in the following three theorems.

THEOREM (Algebraicity). *Quasi-spin curves form a separated algebraic stack of finite type over* $\overline{\mathcal{M}_g}$. (*A more formal statement of this theorem is found in Section* 4.1).

THEOREM (Density). *The stack of smooth spin curves is dense in the stack of quasi-spin curves*.

THEOREM (Properness). *The stack of quasi-spin curves is proper over the stack of stable curves.* (*A formal statement of this theorem is found in Section* 4.2).

The proof of the Density Theorem follows easily from the deformation theory of quasi-spin curves because, as we will see, any quasi-spin curve can be deformed to a smooth spin curve.

The proof of the Properness Theorem uses the valuative criterion of properness, and takes advantage of the fact that it is sufficient to check this criterion when the generic point of the valuation ring in question maps to a point in a dense open subset of the entire space (c.f. [9, pg. 109] or [EGA2, 7.3.10 (ii)]). In this case the dense set is the space of smooth spin curves. It is therefore sufficient to show that a stable curve over the spectrum of a discrete valuation ring, with a smooth spin structure on the generic fibre, must always have an extension to a quasi-spin structure on the whole curve. To make this extension, we will create a semi-stable curve with disjoint exceptional curves that has the same stable model as the first curve, and on which we can define a line bundle that extends the smooth spin structure. This line bundle will also have the special property that on any exceptional curve it has degree one. Such a line bundle pushes down on the contraction to the stable curve to give the desired torsion-free sheaf, and it has a canonical induced homomorphism that extends the smooth spin homomorphism to a quasi-spin homomorphism on the whole stable curve.

The main work in proving the Algebraicity Theorem is in understanding the isomorphisms of quasi-spin curves and in showing that the special condition on the length of the cokernel is a representable condition. In order to do either of these two things, we will have to study the local structure of torsion-free sheaves and the local structure of maps from their rth tensor power to line bundles. This is worked out in Section 3.

The moduli of quasi-spin curves is relatively easy to construct, but is difficult to describe, due to the presence of nilpotent elements. A better moduli problem is that of singular *spin curves*. Namely, even with all of the conditions for a quasi-spin curve, quasi-spin curves may fail to be locally isomorphic to a sheaf-homomorphism pair induced by a contraction of a line bundle on a semi-stable curve as described above. Spin curves are quasi-spin curves whose sheaf-homomorphism pairs are locally isomorphic on X to an induced pair. This additional restriction on the structure map makes it much easier to describe the local structure of the moduli space.

A third moduli problem, that of *pure spin curves*, provides a resolution of singularities for the moduli of spin curves. Pure spin curves are spin curves with additional conditions on the underlying curve to make its local structure well-behaved. Namely, a singularity in the underlying curve X/T can be described locally in the form $xy - \pi$ where x and y are local coordinates on X and π is an element of the base ring $\mathcal{O}_T$. And Faltings [12] has shown that all rank-one torsion-free sheaves near such a singularity are determined by a choice of p and q in $\mathcal{O}_T$ such that $pq = \pi$. Moreover, p and q determine a locally free sheaf precisely in the case that at least one of them is invertible in $\mathcal{O}_T$. Pure spin curves are spin curves which have the property that in Faltings' formulation there exists an element τ in $\mathcal{O}_t$ such that $p = \tau^v$ and $q = \tau^u$. The condition on the cokernel of b for quasi-spin curves ensures that in this case $u + v = r$ and thus $\pi = \tau^r$.

The difficulty with spin curves and pure spin curves is that in order to use the local conditions we need log structures: that is to say, a coherent way to choose local isomorphisms and to impose local conditions so that they make sense globally. Construction of these log structures is done in Section 5.2. But despite the extra work involved, spin curves and pure spin curves have the advantage that the resulting spaces are well-behaved and have nice geometry.

It is worth noting that the only real difference between all of these moduli problems is in the definition of a family. In particular, over a field, all three notions are the same. Indeed, we will show in Section 5.4.1 that the obstructions for quasi-spin curves to be spin all lie in the base ring $\mathcal{O}_T$ and are nilpotent, hence they vanish when T is reduced. Moreover, since in the case that $\mathcal{O}_T$ is a field, p and q vanish, a spin curve is trivially pure over a field.

Similarly, all three notions are the same if the underlying curve is smooth; namely, they are just smooth spin curves $(X, \mathcal{L})$ together with an explicit isomorphism $b: \mathcal{L}^{\otimes r} \xrightarrow{\sim} \omega_X$. Moreover, in the étale topology this extra structure is actually not extra — for any smooth X/T and any two isomorphisms b and b' from $\mathcal{L}^{\otimes r}$ to ω_X, there is an étale cover of T on which the two triples $(X, \mathcal{L}, b)$ and $(X, \mathcal{L}, b')$ are isomorphic.

1.1. Applications and Related Results

Work on this problem in the special case where the base S is $\mathbb{C}$ and $r = 2$ has been done by M. Cornalba in [7] and over a more general base by P. Deligne in [8]. P. Sipe and C. J. Earle have studied rth roots of the canonical bundle on the universal Teichmüller curve (c.f. [11, 26] and [27]). And topological properties of the uncompactified moduli space of 2-spin curves have been studied in many places (e.g. [16, 22]).

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It has recently come to my attention that E. Witten has made a remarkable conjecture related to generalized spin curves; and although it is not stated in exactly these terms, the conjecture amounts to a relationship between intersection theory on the moduli space of pure r-spin curves and Gelfand–Dikii hierachies of order r. This conjecture is a generalization of an earlier conjecture of Witten's, which has been proved by Kontsevich (see [23] and [28]).

Many of the results obtained here are directly applicable to other types of compactifications; most notably, these techniques allow a new approach to the compactification of the universal Picard variety over the moduli of stable curves. Their application in this case results from the fact that although they are stated here only for roots of the canonical bundle, most are true for and are easily generalized to roots of any bundle defined on a family of semi-stable curves. Defining rth-roots of arbitrary line bundles in a manner similar to the way spin and quasi-spin structures are defined here, and allowing r to increase to infinity, produces a stack which, while not of finite type, satisfies the valuative criterion of properness. Moreover, this stack contains the Picard functor over smooth curves as an open substack [2].

1.2. NOTATION AND CONVENTIONS

Unless otherwise stated, both r and g will be fixed integers greater than one. We will use the term *semi-stable curve* of genus g to mean a flat, proper morphism $X \to T$ whose geometric fibres X_t are reduced, connected, one-dimensional schemes, with only ordinary double points, and with $\dim H^1(X_t, \mathcal{O}_{X_t}) = g$. A *stable curve* is a semi-stable curve of genus g (greater than one), with the additional property that any irreducible component which is isomorphic to $\mathbb{P}^1$ meets the rest of the curve in at least three points. Irreducible components of a semi-stable curve which are isomorphic to $\mathbb{P}^1$, but meet the curve in only two points, will be called *exceptional* curves. By *line bundle* we mean an invertible (locally free of rank one) coherent sheaf. An r-spin structure on a smooth curve X/T will be a line bundle $\mathcal{L}$ such that $\mathcal{L}^{\otimes r}$ is isomorphic to the canonical bundle ω_{X_T}. A smooth r-spin curve will be a smooth curve X/S with a spin structure.

2. Quasi-spin curves

2.1. DEFINITIONS

The most basic moduli problem that provides a compactification of $\mathrm{SPIN}_{r,g}$, is the moduli of quasi-spin curves. These are triples of a curve, a rank-one torsion-free sheaf, and a homomorphism. But before we can formally define quasi-spin curves, we need to define torsion-free sheaves carefully.

DEFINITION 2.1.1. A *relatively torsion-free sheaf* (or just torsion-free sheaf) on a semi-stable curve $f: X \to T$, is a coherent sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules, which is of

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finite presentation and flat over T, with the additional property that on each fibre $X_t = X \times_T \operatorname{Spec}(k(t))$ the induced $\mathcal{E}_t$ has no associated primes of height one.

Of course, on the open set where f is smooth, a torsion-free sheaf is locally free.

Now we can define quasi-spin curves and quasi-spin structures.

DEFINITION 2.1.2. A *quasi-spin curve* is a triple $(X/T, \mathcal{E}, b)$ consisting of a flat family of stable curves X/T, a rank-one torsion-free sheaf $\mathcal{E}$, and an $\mathcal{O}_X$-module homomorphism b from the rth tensor power of $\mathcal{E}$ to the canonical dualizing sheaf $b: \mathcal{E}^{\otimes r} \to \omega$, such that the following conditions hold:

(1) $\mathcal{E}$ has degree $(2g-2)/r$.
(2) b is an isomorphism on the open set where $\mathcal{E}$ is locally free.
(3) For each closed point t of the base T, and for each singular point $\mathfrak{p}$ of the fibre X_t where $\mathcal{E}$ is not free, the length of the cokernel of b at $\mathfrak{p}$ is $r-1$.

DEFINITION 2.1.3. A pair $(\mathcal{E}, b)$ consisting of a sheaf and a homomorphism on a curve X/T is called a *quasi-spin structure* if $(X, \mathcal{E}, b)$ is a quasi-spin curve.

DEFINITION 2.1.4. If $\mathfrak{S}/T = (X/T, \mathcal{E}, b)$ and $\mathfrak{S}'/T = (X', \mathcal{E}', b')$ are quasi-spin curves, then an *isomorphism of quasi-spin curves* from $\mathfrak{S}/T$ to $\mathfrak{S}'/T$ is a pair (τ, ε) where $\tau: X \to X'$ is an isomorphism of stable curves over T, and $\varepsilon: \sigma^*\mathcal{E}' \xrightarrow{\sim} \mathcal{E}$ is an isomorphism of $\mathcal{O}_X$-modules which is compatible with b, b' and the canonical isomorphism between $\tau^*\omega_{X'}$ and ω_X.

Note 2.1.5. The requirement that the spin structure map be an isomorphism except on the singular locus of $\mathcal{E}$ is enough to guarantee that the length of the cokernel is at least one at all singular points. Moreover, in the special case of $r = 2$, the condition on the total degree of $\mathcal{E}$ can be seen to guarantee that the length of the cokernel is at most (and hence exactly) one at all singular points. Moreover, in this case, these conditions are equivalent to the condition that the map b induce an isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \omega_X) = \mathcal{E}^\vee \otimes \omega_X$. Thus in the case of $r = 2$ a quasi-spin curve can also be described as a triple $(X/T, \mathcal{E}, \delta)$ where X/T is a stable curve, $\mathcal{E}$ is a rank-one torsion-free sheaf, and δ is an $\mathcal{O}_X$-isomorphism from $\mathcal{E}$ to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \omega_X)$.

This last form is similar to Deligne's definition of a stable super-curve over an ordinary base (c.f. [8]), except that his notation differs somewhat: $\mathcal{E}$ is called $\mathcal{O}_X^-$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \omega_X)$ is called ω_X^-.

3. Local descriptions and tools

We will use two main methods for studying the local nature of rank-one torsion-free sheaves. They are first, a construction of rank-one torsion-free sheaves on stable curves achieved by contracting special line bundles on semi-stable curves (Section 3.1); and second, a local description of the deformations of torsion-free sheaves on (semi-)stable curves due to Faltings (Section 3.2). The important point is that these two methods are essentially the same: Faltings' method assigns to every singularity (point where the sheaf is not locally free) of a rank-one torsion-free sheaf a pair (p,q) of sections of the base ring $\mathcal{O}_T$, and conversely, every pair (p,q) defines a rank-one torsion-free sheaf; similarly, every line bundle on a semi-stable curve having degree one on a fixed exceptional component can also be described by a choice of p and q as above, and will give, when that component is contracted, a rank-one torsion-free sheaf that has the same parameters p and q. This method of contracting semi-stable curves also plays a key role in the proof of properness.

The last of the local tools that we will need is a local description of maps of tensor powers of torsion-free sheaves to locally free sheaves. This is worked out in Section 3.3.

3.1. Contraction and Blowing-up

3.1.1. *The (p,q) semi-stable curve construction*

Let X/B be a stable curve over $B = \text{Spec}(R)$, with R a complete local ring. The completion $A = \hat{\mathcal{O}}_{X,\mathfrak{p}}$ of the local ring of X at a point $\mathfrak{p}$ in the special fibre is of the form $A = R[[x,y]]/(xy-\pi)$ for some choice of π in the ring R. If $\mathfrak{p}$ is singular in the special fibre, then π is in the maximal ideal $\mathfrak{m}$ of R.

Given a semi-stable curve $\tilde{X}$ with stable model X, and such that $\tilde{X}$ has only one exceptional curve over the point $\mathfrak{p}$, where $\mathfrak{p}$ is singular in the special fibre, the local structure of the intersection near the exceptional curve with the rest of the special fibre can be described in a way similar to the way we described the local structure near $\mathfrak{p}$. Namely, we can choose homogeneous coordinates $[\xi;\upsilon]$ on the exceptional curve so that there are elements p and q in the maximal ideal $\mathfrak{m}$ of R and such that the two new nodes have equation $x\upsilon = p\xi$ and $y\xi = q\upsilon$. Moreover, since the contraction of the exceptional curve gives the singularity $\mathfrak{p}$, we must have $pq = \pi$. Conversely, we have the following proposition.

PROPOSITION 3.1.1. *Given an isomorphism of the complete local ring $\hat{\mathcal{O}}_{X,\mathfrak{p}} \cong R[[x,y]]/(xy-\pi)$ and given two elements p and q of $\mathfrak{m}$ such that $pq=\pi$, there is a canonical way to construct a semi-stable curve $\tilde{X} = \tilde{X}(p,q) \xrightarrow{\rho} X$ with stable model X, and with one exceptional curve over $\mathfrak{p}$. Moreover, there is a canonical line bundle $\mathcal{O}_{\tilde{X}}(n)$ for all n, which has degree n on the exceptional curve.*

Ultimately we will see that every rank-one torsion-free sheaf is locally determined by such a choice of p and q, and it is locally isomorphic to the push-forward of the line bundle $\mathcal{O}_{\tilde{X}}(1)$.

Proof. First notice that if p is not a zero divisor, blowing up the ideal $I = (x,p)$ in A to get $\tilde{X}_I := \text{Proj}_X(\oplus_n I^n)$ gives $\tilde{X}_I \cong \text{Proj}_A(A[\xi,\upsilon]/(\upsilon x - p\xi, \upsilon q - \xi y))$.

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Similarly, if q is not a zero divisor, blowing up the ideal $J = (y, q)$ gives $\tilde{X}_J \cong \mathrm{Proj}_A(A[Q, Y]/(Qy - Yq, pQ - xY))$. And these are isomorphic via $\upsilon \mapsto Y, \xi \mapsto Q$. In general, define $\tilde{X}(p, q)$ to be the X-scheme defined locally as

$$\tilde{X} := \tilde{X}(p, q) := \mathrm{Proj}_A(A[\upsilon, \xi]/(\upsilon x - p\xi, \upsilon q - \xi y)) \xrightarrow{\rho} X,$$

regardless of whether or not p or q is a zero divisor. It is straightforward to check that the curve $\tilde{X}(p, q)$ is actually semi-stable of the desired form (only one exceptional curve) and has stable model equal to X.

Let $s = \xi/\upsilon$, and let U in $\tilde{X}$ be the open set $U = \mathrm{Spec} A[s]/(x - ps, ys - q)$. Similarly, setting $t = 1/s = \upsilon/\xi$ let V in $\tilde{X}$ be the open set $V = \mathrm{Spec}(A[t]/(xt - p, y - qt))$. And finally, let $\tilde{A}$ be the ring $\tilde{A} = A[\upsilon, \xi]/(p\xi - x\upsilon, q\upsilon - \xi y)$. The union of U and V is all of $\tilde{X}$, and the canonical line bundles $\mathcal{O}_{\tilde{X}}(n)$ on $\tilde{X}$ are

$$\mathcal{O}_{\tilde{X}}(n) = \begin{cases} \upsilon^n \mathcal{O}_U & \text{on } U \\ \xi^n \mathcal{O}_V & \text{on } V \end{cases}.$$

3.1.2. *Contraction of the (p, q) semi-stable curve*

We will use the results of the following proposition to show that the stack of quasi-spin curves is proper. Namely, in Section 4.2 we will construct an $\tilde{X}(p, q)$ and some line bundles that extend already existing smooth spin curves, and Proposition 3.1.2 shows that these will be torsion-free and will behave as needed to make the valuative criterion of properness hold.

PROPOSITION 3.1.2. *In the notation of the previous section, given any line bundle $\mathcal{L}$ which has degree 1 on the exceptional curve of the special fibre of $\tilde{X}(p, q)$, the following results hold.*

(1) *$\rho_*\mathcal{L}$ is flat over R, commutes with base change, and $R^1\rho_*\mathcal{L} = 0$.*
(2) *$\Gamma(\tilde{X}, \mathcal{L}) \cong \tilde{A}_1$ (the degree-one graded part of A), and $\rho^*\rho_*\mathcal{L} \to \mathcal{L}$ is surjective.*
(3) *$\rho_*\mathcal{L}$ is torsion-free of rank one.*

Proof. The following lemma of M. Cornalba shows that it is enough to prove the proposition in the case that $\mathcal{L}$ is the special line bundle $\mathcal{O}(1)$.

LEMMA 3.1.3. (Cornalba [7]). *If $\mathcal{L}$ is a line bundle on a semi-stable curve $f: X \to T$ such that $\mathcal{L}|_E \cong \mathcal{O}_E$ for some exceptional component E of a special fibre X_0, then there is a neighborhood T' of 0, such that $\mathcal{L}|_{T'}$ on $X_{T'}$ is trivial in a neighborhood of E in $X_{T'}$.*

Now the proposition follows immediately from the following lemma, which shows the special bundles $\mathcal{O}(n)$ are well behaved.

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LEMMA 3.1.4. *In the construction of* $\tilde{X}$ *and* $\mathcal{L} := \mathcal{O}_{\tilde{X}}(n)$ *the following hold.*

(1) $n \geqslant -1$ *implies that* $\rho_*\mathcal{L}$ *is flat over* R, *commutes with base change, and* $R^1\rho_*\mathcal{L} = 0$.
(2) *For* $n \geqslant 0$, $\Gamma(\tilde{X}, \mathcal{L}) \cong \tilde{A}_n$, *the* nth *graded piece of* $\tilde{A}$, *and the natural map* $\rho^*\rho_*\mathcal{L} \to \mathcal{L}$ *is surjective.*
(3) $\rho_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$.
(4) $n = 1$ *implies* $\rho_*\mathcal{L}$ *is torsion-free of rank one.*

Proof of Lemma 3.1.4. It suffices to consider the case $A = R[x,y]/(xy - \pi)$, and in this case $\mathcal{O}_U \cong R[s,y]/(sy - q)$, and $\mathcal{O}_V \cong R[t,x]/(xt - p)$, both of which are flat over R, and $\mathcal{O}_{U\cap V} \cong R[s,t]/(st - 1)$.

Writing out explicitly what the sections of the bundles look like is messy but helpful here: $\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(n)) = \{(g,f) \in \mathcal{O}_U \oplus \mathcal{O}_V | g = s^n f \text{ on } \mathcal{O}_{U\cap V}\}$. So f and g are of the form

$$g = sg_+(s) + g_0 + yg_-(y) \quad \text{and} \quad f = tf_-(t) + f_0 + xf_+(x),$$

with $g_+(s) \in R[s], g_-(y) \in R[y], f_-(t) \in R[t]$, and $f_+(x) \in R[x]$, and

$$sg_+(s) + g_0 + tqg_-(tq) = s^n(tf_-(t) + f_0 + spf_+(sp)).$$

So

$$sg_+(s) + g_0 + tqg_-(tq) = s^{n-1}f_-(t) + s^n f_0 + s^{n+1}pf_+(sp),$$

and rewriting $sg_+(s)$ as $s^{n+1}\gamma(s) + s^n g_n + \cdots + sg_1$, with $g_i \in R$, and $\gamma(s) \in R[s]$, and $tf_+(t)$ as $t^{n+1}\phi(t) + t^n f_n + t^{n-1}f_{n-1} + \cdots + tf_1$, with $\phi(t) \in R[t]$, gives

$$s^{n+1}\gamma(s) + s^n g_n + \cdots + sg_1 + g_0 + tqg_-(tq)$$
$$= t\phi(t) + f_n + \cdots + s^{n-1}f_1 + s^n f_0 + s^{n+1}pf_+(sp).$$

Now $s^{n+1}\gamma(s) = s^{n+1}pf_+(sp)$, and even if p or q is a zero divisor, this implies that $\gamma(s) = pf_+(sp)$, because $\gamma(s)$ is an element of $R[s] \subseteq \mathcal{O}_U$ and $\mathrm{Ann}(s^{n+1}) \cap R[s] = (0)$ (i.e. $R[s] \subseteq \mathcal{O}_U \to \mathcal{O}_{U\cap V}$ is injective). Similarly, $g_{n-i} = f_i$ for $0 \leqslant i \leqslant n$ and $\phi(t) = qg_-(tq)$.

Several things are easy to see from this formulation, namely

(1) $\Gamma(\tilde{X}, \mathcal{L})$ is free over R, hence $\rho_*\mathcal{L}$ is R-flat as long as $n \geqslant -1$.
(2) $\rho_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$.
(3) If (g,f) is an element of $\Gamma(\tilde{X}, \mathcal{L})$, then g can be written as

$$g = s^n xf_+(x) + s^n f_0 + s^{n-1}f_1 + \cdots + sf_{n-1} + f_n + yg_-(y)$$

in $\mathcal{O}_U$ (with $x = sp$), and f can be written as

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$$f = xf_+(x) + f_0 + tf_1 + t^2 f_2 + \cdots + t^n f_n + t^n y g_-(y)$$

in $\mathcal{O}_V$ (with $y = tq$).

The third fact shows that the element $(f_0 + xf_+(x))\xi^n + f_1\xi^{n-1}\upsilon + \cdots + f_{n-1}\xi\upsilon^{n-1} + (f_n + yg_-(y))\upsilon^n$ in $\tilde{A}_n$ maps to (g, f) in $\Gamma(\tilde{X}, \mathcal{L})$, and the natural homomorphism $\tilde{A}_n \to \Gamma(\tilde{X}, \mathcal{L})$ is surjective. Moreover, it is easy to check that if we write $\tilde{A}_n$ as $\tilde{A}_n = \{(F_0\xi^n + F_1\xi^{n-1}\upsilon \cdots + F_n\upsilon^n) | F_i \in A\}$, we can assume that F_0 is in $R[x]$ and F_n is in $R[y]$ and all the remaining terms are in R (i.e. $F_i \in R$ for $0 < i < n$). And thus the homomorphism $\tilde{A}_n \to \Gamma(\tilde{X}, \mathcal{L})$ must actually be injective, hence an isomorphism.

Now since $R^1\rho_*$ is right exact, to see that it vanishes, it suffices to check that it vanishes for each fibre. And $H^1(\tilde{X} \times_X x, \mathcal{O}(n))$ is zero for all x in X, except possibly the singular points of X. But over a singular point we have $H^1(\mathcal{O}(n)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) = 0$. And thus $R^1\rho_*$ is zero, and $\rho_*\mathcal{L}$ commutes with base change if $n \geqslant -1$.

To show that $\rho^*\rho_*\mathcal{L} \to \mathcal{L}$ is surjective, note that this map is locally (on U) just the map taking $\{(g, f) | g = s^n f\} \otimes_A \mathcal{O}_U$ to $\mathcal{O}_U$, given by $(g, f) \otimes z \mapsto gz$. So it suffices to show that there exists $(g, f) \in \rho_*\mathcal{L}$ such that $g = 1$. But $(g, f) = (1, t^n)$ works as long as $t^n \in \mathcal{O}_V$, i.e. if $n \geqslant 0$. A similar computation holds over V, so $\rho^*\rho_*\mathcal{L} \to \mathcal{L}$ is surjective.

To see that $\rho_*\mathcal{L}$ is torsion-free, note that since it is flat and commutes with base change, it suffices to check the case where R is a field and $p = q = \pi = 0$. In this case $\mathcal{O}_U = R[s, y]/sy$ and $\mathcal{O}_V = R[t, x]/tx$ and global sections of $\mathcal{O}(1)$ are $(g, f) \in \mathcal{O}_U \oplus \mathcal{O}_V$ such that $g = sf$. Moreover, $x(g, f) = (0, xf)$ and $y(g, f) = (yg, 0)$, hence if the ideal (x, y) annihilates (g, f), then $x(g, f) = y(g, f) = 0$, and $xf = yg = 0$. Thus $f \in (t), g \in (s)$, and this contradicts the fact that $g = sf$; therefore, $\rho_*\mathcal{O}(1)$ has no associated primes of height one and is torsion-free.

This concludes the proof of Lemma 3.1.4 and of Proposition 3.1.2.

3.1.3. *Induced maps*

Since quasi-spin structures also have a homomorphism that makes them almost an rth root of the canonical bundle, it will be useful to know when the contracted line bundles discussed above have a similar homomorphism. It turns out that a very simple condition guarantees the existence of a canonical homomorphism, and the very best-behaved quasi-spin curves will be those whose structure map is induced by this canonical map.

PROPOSITION 3.1.5. *In the same notation as the previous section, if p and q have the additional property that there exist positive integers u and v with $u + v = r$, and there exists an element w invertible in R, such that $p^u = wq^v$, then there is an induced map from $(\rho_*\mathcal{O}_{\tilde{X}}(1))^{\otimes r}$ to $\mathcal{O}_X$. Moreover, at the singular point $\mathfrak{p}$ the length of the cokernel of the map is $r - 1$.*

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Proof. This follows from the existence of a canonical map on $\tilde{X}$ from $\mathcal{O}_{\tilde{X}}(r)$ to $\mathcal{O}_{\tilde{X}}$; namely, on U the map takes $\Gamma(U, \mathcal{O}(r)) = (A[s]/(ps - x, sy - q)) \cdot \upsilon^{\otimes r}$ to $\Gamma(U, \mathcal{O}) = A[s]/(ps - x, sy - q)$ via $\upsilon^{\otimes r} \mapsto wy^v$. And on V the map is similar: $\Gamma(V, \mathcal{O}(r)) = (A[t]/(p-xt, y-qt)) \cdot (\xi)^{\otimes r}$ maps to $\Gamma(V, \mathcal{O}) = A[t]/(p-xt, y-qt)$ via $\xi^{\otimes r} = s^r \cdot \upsilon^{\otimes r} \mapsto s^r wy^v = x^u$.

The canonical map $\rho^* \rho_* \mathcal{O}(1) \to \mathcal{O}(1)$ induces a map $(\rho^* \rho_* \mathcal{O}(1))^{\otimes r} = \rho^*(\rho_* \mathcal{O}(1)^{\otimes r}) \to \mathcal{O}(1)^{\otimes r} = \mathcal{O}(r)$, and the canonical map $\mathcal{O}(r) \to \mathcal{O}_{\tilde{X}}$ gives a canonical map $\rho^*(\rho_* \mathcal{O}(1))^{\otimes r} \to \mathcal{O}_{\tilde{X}}$. This induces the desired map on the push-forward by adjointness $(\rho_* \mathcal{O}(1))^{\otimes r} \to \rho_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$.

The length of the cokernel of this induced map at the singular point of the special fibre can be computed from the fact that locally the image of the map is generated by x^u and wy^v in $k[[x, y]]/xy$, where w is invertible in k. Hence the cokernel has length $u + v - 1 = r - 1$.

When we define the other moduli problems that compactify the smooth spin curves, they will all be special cases of quasi-spin curves whose structure maps have the property of being locally isomorphic to those induced by the canonical homomorphism.

3.2. FALTINGS' DESCRIPTION OF TORSION-FREE SHEAVES

In this section we recall some results of Faltings' on the local structure of torsion-free sheaves and show how they are related to the description developed in the previous section. As in the previous section, we work with a stable curve X/B, where the base B is the spectrum of a complete local Noetherian ring R; the completion of the ring $\mathcal{O}_{X,\mathfrak{p}}$ at a singular point $\mathfrak{p}$ is isomorphic to $A := R[[x, y]]/(xy - \pi)$, and π is an element of the maximal ideal $\mathfrak{m}$ of R. A torsion-free sheaf $\mathcal{E}$ corresponds to an R-flat A-module, E.

Locally on X, the special torsion-free sheaves obtained by the contraction $\tilde{X}(p, q) \xrightarrow{\rho} X$ of the previous section can be expressed in a nice way.

PROPOSITION 3.2.1. $E := \Gamma(\mathrm{Spec} A, \rho_* \mathcal{O}(1))$ *is naturally isomorphic to the image of the A-homomorphism*

$$\alpha(p, q) = \begin{pmatrix} x & p \\ q & y \end{pmatrix} : A^{\oplus 2} \to A^{\oplus 2}.$$

Proof. We have from Proposition 3.1.2 that $E \cong \tilde{A}_1 = (A[\xi, \upsilon]/(\xi p - \upsilon x, \xi y - \upsilon q))_1 = \{f\xi + g\upsilon | f, g \in A\}$. It is easy to see that we can assume f is in $R[[x]]$, and g is in $R[[y]]$. And so the map $\sigma\colon \rho_* \mathcal{O}(1) \to A^{\oplus 2}$, given by

$$(f\xi + g\upsilon) \mapsto \begin{pmatrix} fx + gp \\ fq + gy \end{pmatrix}$$

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is a well-defined homomorphism of A-modules. Even if p and q are zero divisors, if f is in $R[[x]]$ and g is in $R[[y]]$, then $fx + gp = 0$ implies that $f = 0$. Similarly, $fq + yg = 0$ implies that $g = 0$, so σ is injective.

For any

$$\alpha \begin{pmatrix} f \\ g \end{pmatrix}$$

in the image of α we can assume f is in $R[[x]]$ and g is in $R[[y]]$ because

$$\alpha \begin{pmatrix} y \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ q \end{pmatrix} \quad \text{and} \quad \alpha \begin{pmatrix} 0 \\ x \end{pmatrix} = \alpha \begin{pmatrix} p \\ 0 \end{pmatrix}.$$

Thus the image of σ (and hence E) is A-isomorphic to the image of α and is R-isomorphic to $R[[x]] \oplus R[[y]]$ via the obvious identification.

A result of Faltings' shows that every rank-one torsion-free sheaf is of this form. Namely, let $E(p,q)$ be the image of $\alpha(p,q) : A^{\oplus 2} \to A^{\oplus 2}$, where α is the two-by-two matrix

$$\begin{pmatrix} x & p \\ q & y \end{pmatrix}$$

and p and q are, as before, elements of R such that $pq = \pi$. We saw above that $E(p,q)$ is R-flat, and torsion-free. When p and q are in $\mathfrak{m}$ then $E(p,q)$ is a deformation of the normalization of $A/\mathfrak{m}A$, i.e. of the unique (up to isomorphism) non-free torsion-free sheaf $k[[x]] \oplus k[[y]]$ over the ring $A/\mathfrak{m}A \cong k[[x,y]]/xy$. Faltings' result is the following.

THEOREM 3.2.2 (Faltings [12]). *Any relatively torsion-free E of rank 1 is isomorphic to an $E(p,q)$, for $p,q \in R$ with $pq = \pi$.*

Homomorphisms and isomorphisms of $E(p,q)$'s can be described by their lifts to $A^{\oplus 2}$. Namely, any morphism of A-modules $E \to F$, with E relatively torsion-free, can be lifted to a morphism from $A^{\oplus 2}$ to F. And a homomorphism from E to E' with E and E' both torsion-free lifts to an endomorphism of $A^{\oplus 2}$. More exactly, the following holds.

PROPOSITION 3.2.3 (Faltings [12]). *If $p \equiv q \equiv 0 \pmod{\mathfrak{m}}$, then $E(p,q)$ is isomorphic to $E(p',q')$ if and only if there exist $u,v \in R^{\times}$ such that $p' = upv^{-1}$ and $q' = vqu^{-1}$. In this case the isomorphism is induced by the 'constant' map*

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : A^{\oplus 2} \to A^{\oplus 2}.$$

Moreover, writing a homomorphism Φ *in* $\mathrm{Hom}_A(E(p,q),E(p',q'))$ *as a lift to* $A^{\oplus 2}\to A^{\oplus 2}$ *given by*

$$\begin{pmatrix}\varphi_+ & \psi_+\\ \psi_- & \varphi_-\end{pmatrix},$$

we can take φ_+ *to be in* $R[[x]]$ *and* φ_- *to be in* $R[[y]]$. *Also, the elements* $\varphi_+(x)$ *and* $\varphi_-(y)$ *completely determine* ψ_+ *and* ψ_- *by the relations*

$$\psi_+=\left(\frac{p}{x}\right)(\varphi_+(x)-\varphi_+(0)),\quad \text{and}\quad \psi_-=\left(\frac{q}{y}\right)(\varphi_-(y)-\varphi_-(0))$$

and are subject to the condition that

$$p'\varphi_-(0)=\varphi_+(0)p,\quad \text{and}\quad q'\varphi_+(0)=\varphi_-(0)q.$$

These results also hold for Henselian rings. Suppose now that R is the Henselization of a local ring of finite type over a field or an excellent Dedekind domain, $\mathfrak{m}$ is the maximal ideal of R, $\pi\in\mathfrak{m}$, and A is the Henselisation of $R[x,y]/(xy-\pi)$ at $\mathfrak{m}+(x,y)$. As before, for each pair $p,q\in R$ with $pq=\pi$, define $E(p,q)$.

THEOREM 3.2.4 (Faltings).

(1) *Any torsion-free* E *of rank one over* A *is isomorphic to* $E(p,q)$ *for* $p,q\in R$ *and* $pq=\pi$.
(2) *If* p, q, p' *and* q' *are all in* $\mathfrak{m}$, *then* $E(p,q)$ *and* $E(p',q')$ *are isomorphic if and only if there exist* $u,v\in R^\times$ *with* $p'=upv^{-1}, q'=vqu^{-1}$.

Faltings' results and Section 3.1 show that near any singular point of the special fibre of X/B, any torsion-free sheaf E is uniquely determined by a choice of p and q, and that p and q determine a semi-stable curve over X with a single exceptional curve over the singular point and a line bundle of degree one on the exceptional curve. Contracting the exceptional curve, or pushing down to X, makes the line bundle into a rank-one torsion-free sheaf, which is naturally isomorphic to $E(p,q)$. Moreover, it is easy to see that if $E(p,q)$ is isomorphic to $E(p',q')$, then the induced semi-stable curves $\tilde{X}(p,q)$ and $\tilde{X}(p',q')$ are naturally isomorphic as well. And conversely, if $\tilde{X}(p,q)$ and $\tilde{X}(p',q')$ are isomorphic, then so are the induced torsion-free sheaves. Thus both Faltings' description and the geometric description of the torsion-free sheaves are essentially equivalent, except that semi-stable curves allow many non-trivial automorphisms of the exceptional components that induce the identity automorphism on the stable model and the corresponding torsion-free sheaf.

3.3. Local description of the homomorphisms

The last main tool we need to understand the quasi-spin curves and to prove the main theorems (Density, Properness, and Algebraicity) is a description of the local

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structure of the homomorphisms. In particular, we need to study A-linear maps $b\colon E^{\otimes r} \to A$ of the rth tensor power of a rank-one torsion-free A-module E. Here A is an étale neighborhood of the closed point defined by $(x, y) + \mathfrak{m}A$ in $\mathrm{Spec}(R[[x, y]]/(xy - \pi))$ over the base ring R, where $\mathfrak{m}$ is a maximal ideal of R containing π, p, and q.

The main idea is simply to lift the map $b\colon E^{\otimes r} \to A$ to a map $\mathbf{Sym}^r(A^{\oplus 2}) \to A$ and thus describe b by an $r + 1$-tuple $\tilde{b} = (b_0, b_1, \ldots, b_r)$. We have the following proposition about b.

PROPOSITION 3.3.1. *Locally, any quasi-spin structure* (E, b) *which is not free is of the form* $(E(p, q), b)$ *with* p *and* q *in* $\mathfrak{m}$. *And given such a representation in terms of* p *and* q, *the map* b *from* $E^{\otimes r}$ *to* A *is uniquely determined by a lifting* $\tilde{b}$, *mapping* $\mathbf{Sym}^r(A^{\oplus 2})$ *to* A, *of the form* $\tilde{b} = (b_0, b_1, \ldots, b_r)$, *where* $\tilde{b}$ *acts on an element*

$$\left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \cdot \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \cdots \begin{pmatrix} f_r \\ g_r \end{pmatrix}\right)$$

of $\mathbf{Sym}^r(A^{\oplus 2})$ *by taking it to* $b_0 \prod_{i=1}^r f_i + b_1 \Sigma_{i=1}^r g_i \prod_{j \neq i} f_j + \cdots + b_r \prod_{i=1}^r g_i$ *in* A.

In this case we have for all i *with* $0 \leqslant i \leqslant r - 1$,

$$pb_i = xb_{i+1} \quad \text{and} \quad yb_i = qb_{i+1}. \tag{1}$$

Furthermore, modulo the ideal $\mathfrak{m}A$ *we may write* $(b_0, \ldots, b_r) \equiv (\bar{b}_0, 0, \ldots, 0, \bar{b}_r)$. *And* $\bar{b}_0 = x^u \bar{\beta}_0$ *and* $\bar{b}_r = y^v \bar{\beta}_r$ *for some* $\bar{\beta}_0$ *invertible in* $(A/\mathfrak{m}A)[1/x]$, *but not in the ideal* (x), *and for some* $\bar{\beta}_r$ *invertible in* $(A/\mathfrak{m}A)[1/y]$, *but not in* (y).

And finally, u *and* v *must both be at least one and must sum to* r.

Proof. Any map $b\colon E^{\otimes r} \to A$ that is A-linear, lifts to a map $\tilde{b}$.

$$\begin{array}{ccc} A^{\oplus 2^r} & \xrightarrow{\tilde{b}} & A \\ \Big\downarrow{\scriptstyle \alpha^{\otimes r}} & & \Big\| \\ E^{\otimes r} & \xrightarrow{b} & A. \end{array}$$

Over $A[1/x]$ and over $A[1/y]$ the module E is locally free, thus over these rings any homomorphism $b\colon E^{\otimes r} \to A$ will factor through $\mathbf{Sym}^r(E)$, and its lift to $(A^{\oplus 2})^{\otimes r}$ will factor through $\mathbf{Sym}^r(A^{\oplus 2})$. And since A has no (x, y)-torsion, this holds in general. So if f and $g \in A^{\otimes 2}$ are defined as

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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respectively, we only need to describe $b_i := b(f^{r-i} \otimes g^i)$ for each $0 \leqslant i \leqslant r$ in order to completely describe b and $\tilde{b}$. We will, therefore, denote $\tilde{b}$: $A^{2^r} \to E^{\otimes r} \to A$ by the vector $(b_0, b_1, \ldots, b_r)$.

Now, since

$$\alpha \begin{pmatrix} p \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ x \end{pmatrix} \quad \text{and} \quad \alpha \begin{pmatrix} 0 \\ q \end{pmatrix} = \alpha \begin{pmatrix} y \\ 0 \end{pmatrix},$$

we must have for all i, $0 \leqslant i \leqslant r-1$ the relations $pb_i = xb_{i+1}$ and $yb_i = qb_{i+1}$.

The fact that the map must be an isomorphism off the singular locus of the underlying curve means that b: $E(p,q)^{\otimes r} \to A$ must be an isomorphism on $A[1/x]$ and $A[1/y]$. Over $A[1/x]$ the map $\tilde{b} = (b_0, b_1, \ldots, b_r)$ is completely determined by b_0, namely for any i we have $x^i b_i = p^i b_0$, and thus $b_i = b_0 p^i / x^i$. For $\tilde{b}$ to be surjective over $A[1/x]$ we must have that b_0 is invertible in $A[1/x]$. Similarly, b_r is invertible in $A[1/y]$.

On the special fibre, since p and q are in $\mathfrak{m}$, we have $\tilde{b} \equiv (\bar{b}_0, 0, 0, \ldots, 0, \bar{b}_r)$ (mod $\mathfrak{m}$). Here $\bar{z}$ denotes the image of z modulo $\mathfrak{m}$. And $\bar{b}_0 = x^u \bar{\beta}_0$ and $\bar{b}_r = y^v \bar{\beta}_r$ for some $\bar{\beta}_0$ invertible in $(A/\mathfrak{m}A)[1/x]$, but not in the ideal (x), and for some $\bar{\beta}_r$ invertible in $(A/\mathfrak{m}A)[1/y]$, but not in (y). This makes the length of the cokernel of $\bar{b}$ equal to $u + v - 1$. If u (or similarly v) is zero, then over the ring $A[1/y]$ the fact that $b_0 = b_r q^r / y^r \equiv 0$ (mod $\mathfrak{m}$) implies that $\bar{\beta}_0 = \bar{b}_0 \equiv 0$ (mod $\mathrm{Ann}_{A/\mathfrak{m}A}(y)$), i.e. modulo (x). But this implies that $\bar{\beta}_0 = 0$, and that is a contradiction.

We now have all of the basic tools needed to begin proving the main theorems.

4. The stack $\mathrm{QSPIN}_{r,g}$

The space that we are interested in is the stack of quasi-spin curves over some fixed scheme S. Here S must be a scheme of finite type over a field or over an excellent Dedekind domain so that the standard theorems on algebraic stacks will hold [14] as well as Faltings' results from Section 3.2. We will also require that r be invertible in S. The stack is given by the functor $\mathrm{QSPIN}_{r,g}$, which is the étale sheafification of the functor that takes an S-scheme T to the set of isomorphism classes of quasi-spin curves over T.

The main results of this section are that the functor of quasi-spin curves is an algebraic stack, and that this stack is a compactification of the moduli of smooth spin curves. The first of these results will be proved in Section 4.1, and the second in Section 4.2.

4.1. ALGEBRAICITY

Let $\mathrm{QSPIN}_{r,g}$ be the étale sheafification of the functor taking an S-scheme T to the set of isomorphism classes of quasi-spin curves over T. The following theorem holds.

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THEOREM (Algebraicity). $\mathrm{QS{\small PIN}}_{r,g}$ *is a separated algebraic stack, locally of finite type over* $\overline{\mathcal{M}_g}$, *the moduli space of stable curves.*

To prove that the stack is algebraic, we need to do the following (see, for example, [14, pp. 15–23], or [21]):

(1) Prove that the functor is limit preserving.
(2) Provide a smooth cover U of the stack.
(3) Prove that for a fixed family of curves X over T the functor $\mathrm{I{\small SOM}}_{U_T\times U_T}(pr_1^*, pr_2^*)$ is representable by a scheme (it is clearly a groupoid).
(4) Prove that the stack is separated by showing that $\mathrm{I{\small SOM}}_{U_T\times U_T}(pr_1^*, pr_2^*)$ is actually proper and finite over $U_T \times U_T$.

Step two, that is the existence of a smooth cover, will be worked out in Section 4.1.3. Steps three and four will be done in the section on isomorphisms (Section 4.1.4). The first step is a straightforward consequence of the following theorem of Grothendieck and the fact that the condition on the length of the cokernel is an open condition (c.f. Proposition 4.1.5), hence limit preserving. We will prove the openness of the cokernel condition in Section 4.1.2.

THEOREM 4.1.1 ([EGA4, 8.5.2]). *Given a quasi-compact and quasi-separated scheme* S_0, *and given a projective system* $\{S_\gamma\}$ *of* S_0*-schemes, relatively affine over* S_0, *and given quasi-coherent* $\mathcal{O}_{S_\gamma}$*-modules* $\mathcal{F}_\gamma$ *and* $\mathcal{G}_\gamma$, *with* $\mathcal{F}_\gamma$ *of finite presentation, the canonical homomorphism of groups*

$$\varinjlim(\mathrm{Hom}_{S_\gamma}(\mathcal{F}_\gamma, \mathcal{G}_\gamma)) \to \mathrm{Hom}_S(\mathcal{F}, \mathcal{G})$$

is an isomorphism. Here S, $\mathcal{F}$, *and* $\mathcal{G}$ *are the obvious limit objects.*

Before we can finish step one, we need to make some simple but useful observations about the behavior of torsion-free sheaves under normalization of singularities. These will prove to be useful tools for analyzing everything from the length of the cokernel of the quasi-spin map to isomorphisms of quasi-spin curves.

4.1.1. *Normalization and torsion-free sheaves*

Construction 4.1.2. Given a semi-stable curve Y over a field k and a rank-one torsion-free sheaf $\mathcal{E}$ on Y, we first normalize the curve at the singularities of $\mathcal{E}$ (i.e. just at the points where $\mathcal{E}$ fails to be locally free) to get $\theta: \tilde{Y} \to Y$. We can pull back the sheaf $\mathcal{E}$ and mod out by torsion to get a line bundle, which we call $\theta^\natural\mathcal{E}$.

Note that this construction also works if θ is just the normalization at a subset of the singularities of $\mathcal{E}$, but in that case $\theta^\natural\mathcal{E}$ still fails to be locally free at the singular points that were not normalized. Note also that $\mathcal{E}$ can be recovered from this construction as the push-forward: $\mathcal{E} = \theta_*\theta^\natural\mathcal{E}$.

The degree of the sheaf $\theta^\natural\mathcal{E}$ is also closely related to the degree of $\mathcal{E}$, as the next proposition shows.

PROPOSITION 4.1.3. *If $\mathcal{E}$ is a rank-one torsion-free sheaf and $\pi: X^\nu \to X$ is the normalization of X at a one singular point of $\mathcal{E}$, then* $\deg(\pi^\natural\mathcal{E}) = \deg(\mathcal{E}) - 1$.

Proof. $R^i\pi_*(\mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ on X^ν and for all $i > 0$. So the Leray spectral sequence degenerates, and $H^i(X^\nu, \mathcal{F}) = H^i(X, \pi_*\mathcal{F})$ for all $i \geqslant 0$. Thus in particular, $\chi(\mathcal{E}) = \chi(\pi_*\pi^\natural\mathcal{E}) = \chi(\pi^\natural\mathcal{E})$, and $\chi(\pi_*\mathcal{O}_{X^\nu}) = \chi(\mathcal{O}_{X^\nu})$. Taking Euler–Poincaré characteristics of the exact sequence $0 \to \mathcal{O}_X \to \pi_*\mathcal{O}_{X^\nu} \to k \to 0$ gives $\chi(\pi_*\mathcal{O}_{X^\nu}) = \chi(\mathcal{O}_X)+1$, and thus $\deg(\mathcal{E}) = \deg(\pi^\natural\mathcal{E})+1$.

Now we can proceed to examine the behavior of the cokernel, and in particular, we can finish step one of the proof of the Algebraicity Theorem.

4.1.2. *Behavior of the cokernel under deformation*

DEFINITION 4.1.4. An $\mathcal{O}_X$-linear map $b: \mathcal{E}^{\otimes r} \to \omega$ from the rth tensor power of a rank-one torsion-free sheaf $\mathcal{E}$ to the canonical bundle of a curve X over a field k is said to have *good cokernel* if

(1) the cokernel is supported on the singular points of X, and
(2) for each point $\mathfrak{p}$ of the support of the cokernel C of b we have $\text{length}_{\mathfrak{p}} C = r-1$.

The main result of this section is summarized in the following proposition.

PROPOSITION 4.1.5. *Given $b: \mathcal{E}^{\otimes r} \to \omega$ on $f: X \to T$ (with the cokernel of b supported on the discriminant locus) the set of $t \in T$ such that b_t has good cokernel is open in T.*

The proof of the proposition needs the following lemma.

LEMMA 4.1.6. *If the cokernel of b is supported on the singular locus of $\mathcal{E}$, then the property of having good cokernel is stable under generization.*

Proof of the Lemma. It is enough to consider the case where R is a complete local ring, $E \cong E(p,q)$ is an A-module ($A = R[[x,y]]/(xy-\pi)$), and $\tilde{b} = (b_0, \ldots, b_r): A^{2^r} \to A$ is a lifting of the map $b: E^{\otimes r} \to A$. We can assume that on the special fibre $\overline{b}_0 \in A/\mathfrak{m}A$ is equal to $x^i\overline{\beta}_0$, and $\overline{\beta}_0$ an invertible element of $(A/\mathfrak{m})$. Similarly, $\overline{b}_r \in A/\mathfrak{m}A$ can be written as $\overline{b}_r = y^j\overline{\beta}_r$, and β_r is invertible in A.

For any map $R \to K$ of R into a field, we have the following possible cases.

(1) π does not map to zero in K. In this case, the cokernel is actually zero because Spec$(A \otimes K)$ is regular.
(2) π maps to zero, but at least one of p and q does not. In this case again the cokernel of b is zero.

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(3) π and p and q all map to zero. This is the only interesting case. We have $A \otimes K \cong K[[x,y]]/xy$ and $\tilde{b}_K = (b_0, 0, \ldots, 0, b_r)$. Now $b_0 = x^i\beta_0 + d_0$ and $b_r = y^j\beta_r + d_r$ with $d_0 = x^l\varepsilon_0$ and $d_r = y^m\varepsilon_r$, such that ε_0 is in $\mathfrak{m}A$ but not in (x), and ε_r is in $\mathfrak{m}A$ but not in (y). If, on the one hand, l is larger than $i-1$, then $x^i = b_0/(\beta_0 + x^{l-i}\varepsilon_0)$. The term in the denominator is invertible because β_0 is invertible, and d_0 is in the maximal ideal $\mathfrak{m}A$. If, on the other hand, l is less than i, then $x^i = -x^l\varepsilon_0/\beta_0$, and similarly for y^j. In either case $K[[x,y]]/(xy, x^i, y^j) \cong K < 1, x, x^2, \ldots, x^{i-1}, y, y^2, \ldots, y^{j-1} >$ surjects onto $K[[x,y]]/(xy, b_0, b_r) = A \otimes_R K/\mathrm{im}(b)$.

So the length of the cokernel will be either zero (cases 1 and 2) or bounded above by $i + j - 1 = r - 1$ (case 3).

Thus the length of the cokernel can only decrease under generization, but the degree of $\mathcal{E}_K$ on X_K must be $(2g-2)/r = \deg \theta^\natural\mathcal{E} + \delta$, where δ is the number of singularities of X_K, and $\theta\colon X_K^\nu \to X_K$ is the normalization of X_K at the singularities of $\mathcal{E}_K$. On the other hand, since the cokernel of b is supported on the singular set of $\mathcal{E}$, we have that $\theta^\natural b$ factors

$$\theta^\natural\mathcal{E}_K^{\otimes r} \stackrel{\sim}{\to} \theta^*\omega_{X_K}\left(-\sum u_{\mathfrak{p}}\mathfrak{p}^+ - \sum v_{\mathfrak{p}}\mathfrak{p}^-\right) \hookrightarrow \theta^*\omega_{X_K},$$

where the sum is taken over all $\mathfrak{p}$ in the singular set of $\mathcal{E}_K$, $\theta^{-1}(\mathfrak{p}) = \{\mathfrak{p}^\pm\}$, and for each $\mathfrak{p}$, $u_{\mathfrak{p}} + v_{\mathfrak{p}} - 1 = \mathrm{length}_{\mathfrak{p}}(\mathrm{coker}(b)) \leqslant r-1$. So

$$\deg\mathcal{E}_K = \frac{(2g-2)}{r} = \left(2g - 2 - \sum_{\mathfrak{p}}(u_{\mathfrak{p}} + v_{\mathfrak{p}})\right)/r + \delta,$$

which will be strictly greater than $(2g-2)/r$ unless at each singularity of $\mathcal{E}_K$ the cokernel of b has length $r-1$. Thus the property of having good cokernel is stable under generization.

Now we are ready to prove the openness of the cokernel condition.

Proof of the Proposition. It suffices to show the complement of the set is closed. And since the previous lemma shows this complement is stable under specialization, it suffices to show that the complement is constructible. Let P_m be the property of a geometric point $\bar{t}$ of T that $C := \mathrm{coker}(b)$ has a point of its support over $\bar{t}$ where C has length m. The set we want to show is constructible is the set $\mathfrak{T}_m := \{t \in T | \bar{t} \text{ has } P_m\}$. Actually, the set we are really looking for is

$$\bigcup_{0<m<r-1}\mathfrak{T}_m \cup \bigcup_{r-1<m<N}\mathfrak{T}_m$$

for some very large N. N can be taken to be finite because the degree of the sheaves is fixed, and the sum over all points in a given fibre of the length of the cokernel is bounded, and this bound is determined by the number of singular points and the

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degree. Moreover, the number of singular points is bounded as a function of the genus of the underlying curve, so this number N can be chosen independently of the specific family.

Now to show constructibility we only need to consider one m and one irreducible component of the discriminant locus, say D_0, of X and its image $\rho(D_0) = T_0$, i.e. we only need to show that $\mathfrak{T}_m$ is constructible in T_0. And it is enough to assume T_0 is reduced and irreducible. Since D_0 is proper over T_0, the semi-continuity theorem shows that $\mathfrak{T}_m$ is constructible for any $m \neq r - 1$.

4.1.3. *A smooth cover of* QSPIN$_{r,g}$

Step two of the proof of the Algebraicity Theorem requires that we construct a smooth cover of QSPIN$_{r,g}$. This is provided by the following proposition.

PROPOSITION 4.1.7. *Given a stable curve X/T, there is an integer N and a scheme $U_{X/T}$, which represents all quasi-spin structures $(\mathcal{E}, b)$ on X together with a basis $(e_1, e_2, \ldots, e_n)$ of global sections which generate $\mathcal{E} \otimes \omega^N$. Moreover, U is smooth over the stack of quasi-spin structures on X/T.*

As a part of the proof of the proposition, we will prove the following lemma.

LEMMA 4.1.8. *Given a curve X/B and an integer N, sufficiently large, the functor taking a B-scheme T to the set of all triples $(\mathcal{E}, b, (e_1, \ldots, e_n))$ is representable. Here $(\mathcal{E}, b)$ is a quasi-spin structure on X_T, and $(e_1, \ldots, e_n)$ is a basis for the module $\Gamma(X_T, \mathcal{E}_T \otimes \omega_T^{\otimes N})$ on X_T.*

But before starting the proof of this lemma, we need to prove that quasi-spin curves form bounded families, which means that after tensoring with a suitable very ample bundle $\mathcal{O}(m)$ on X, all higher cohomology vanishes, and global sections generate. This is the key step in proving the lemma. More exactly, since X is projective, we can choose a very ample line bundle $\mathcal{O}(1)$ on X. In general, we will write $\mathcal{F}(m)$ for $\mathcal{F} \otimes \mathcal{O}(1)^{\otimes m}$. It is a straightforward exercise to generalize D'Souza's propositions of [10, Section 3] to the following sublemma.

SUBLEMMA 4.1.9. *If $\mathcal{F}$ is a rank-one torsion-free sheaf on X, there is an integer m_0 depending only on the degree of $\mathcal{F}$ on each irreducible component of X, and on the genus of X, such that for $m \geqslant m_0$ the following holds.*

(1) $H^1(X, \mathcal{F}(m)) = 0$.
(2) $\mathcal{F}(m)$ *is generated by global sections.*

Now to use the previous sublemma for quasi-spin curves we need an additional sublemma that says that the degree of a quasi-spin curve on any given component is uniformly bounded.

SUBLEMMA 4.1.10. *Quasi-spin curves form bounded families. In particular, the degree of any quasi-spin curve on a specific irreducible component of a fibre of a family is bounded below by some integer that depends only on r and the genus of the family g.*

Proof. Any quasi-spin structure $(\mathcal{E}, b)$ on a curve Y over a field k must have total degree $(2g-2)/r$, and on the normalization $\theta: \tilde{Y} \to Y$ we can pull back the sheaf $\mathcal{E}$ and mod out by torsion to get a line bundle, which we call $\theta^\natural \mathcal{E}$. The rank-one torsion-free sheaf $\mathcal{E}$ is just the push-forward $\theta_* \theta^\natural \mathcal{E}$. And the degree of $\mathcal{E}$ on a component Y_i is at least as big as the degree of $\theta^\natural \mathcal{E}$. Now we have

$$\theta^\natural \mathcal{E}^{\otimes r} \cong \theta^* \omega_{Y/k} \left(-\sum (u_i \mathfrak{p}_i^+ + v_i \mathfrak{p}_i^-) \right),$$

where the sum is taken over all singularities $\{\mathfrak{p}_i\}$ of $\mathcal{E}$, and $\{\mathfrak{p}_i^+, \mathfrak{p}_i^-\}$ are the inverse images of $\mathfrak{p}_i$ via θ. In particular, for any given irreducible component Y_j of Y, we have

$$\deg_{Y_j}(\theta^\natural \mathcal{E}) \geqslant \frac{1}{r}(\deg_{Y_j}(\omega_{Y/k}) - r\delta_j),$$

where δ_j is the number of singularities of Y in Y_j. Since $\deg_{Y_j}(\omega_{Y/k})$ is always positive, and since the total number of singularities in a stable curve of genus g is bounded by $3g-3$, we have

$$\deg_{Y_i} \theta^\natural \mathcal{E} \geqslant 1 - \delta_i \geqslant 2 - 3g.$$

Proof of Lemma 4.1.8. Given a very ample line bundle $\mathcal{O}(1)$ on X, we can choose, as in Sublemma 4.1.9, a fixed m_0 depending only on the genus of X such that $\mathcal{E} \otimes \mathcal{O}(m)$ is generated by global sections and has vanishing higher cohomology for all $m \geqslant m_0$. Fix, once and for all, an integer N large enough so that ω^N is very ample and $\mathcal{E} \otimes \omega^N$ has all the desired properties. Now we can represent torsion-free rank-one sheaves with bounded degree on each component by a subscheme of $\mathrm{Quot}_{\mathcal{O}_X^n/X/S}$ for some n, i.e. there exists $U_1 \hookrightarrow \mathrm{Quot}_{\mathcal{O}_X^n/X/S}$ which represents the functor $T \mapsto \{\mathcal{F}, (e_1 \ldots e_n)\}$, where $\mathcal{F}$ is a rank-one, torsion-free sheaf on X_T with bounded degree on each component, and $(e_1 \ldots e_n)$ is a basis of $\Gamma(X_T, \mathcal{F} \otimes \omega^N)$. So over X_1/U_1 there is a universal pair $(\mathcal{E}, (e_1 \ldots e_n))$. And to represent maps $\mathcal{E}^{\otimes r} \to \omega_{X_1/U_1}$, take

$$V := \mathbb{V}(\mathcal{E}^{\otimes r} \otimes \omega^{-1}) := \mathbf{Spec}_{X_1}(\mathbf{Sym}_{\mathcal{O}_{X_1}}(\mathcal{E}^{\otimes r} \otimes \omega^{-1})),$$

so that $\mathrm{Hom}_{X_1}(Y, V) = \mathrm{Hom}_Y(\mathcal{E}_Y^{\otimes r}, \omega_Y)$. So letting $V_T := V \times_{X_1} X_T = V \times_{U_1} T$, we get that

$$\mathrm{Hom}_{X_1}(X_T, V) = \mathrm{Hom}_{X_T}(X_T, V_T) = \left(\prod_{X/S} V/X \right)(T)$$

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is the functor we want, and it is representable because X is flat and projective over S (see [FGA, Section 4.c.]).

Now we have a universal triple $\mathcal{E}, (e_i \dots e_n), b: \mathcal{E}^{\otimes r} \to \omega$ on X_2/U_2 representing all maps $\mathcal{E}^{\otimes r} \to \omega$, and the additional condition that the cokernel of b is supported on the singular locus of X_2 is also representable; namely, it is just the condition that b is an isomorphism on the complement of the discriminant locus, and this is an open condition. Finally, we need to represent the condition on the length of the cokernel, but this condition is open on the base, as proved in Proposition 4.1.5.

Now that the lemma is proved, it is fairly easy to provide the smooth cover of $\text{QS}{\small\text{PIN}}_{r,g}$.

Proof of Proposition 4.1.7. The lemma shows that for an arbitrary stable curve $\mathcal{X}/T$ there is a scheme U representing all quasi-spin structures $(\mathcal{E}, b)$ on $\mathcal{X}_U$ such that $\mathcal{E} \otimes \omega^N$ can be expressed as a quotient of $\mathcal{O}^n_{\mathcal{X}}$, together with a basis for the module $\Gamma(\mathcal{X}, \mathcal{E} \otimes \omega^N)$. Moreover, at any closed point of T, all quasi-spin structures on $\mathcal{X} \times_T \text{Spec}(\mathcal{O}_{T,t})$ can be expressed as such a quotient. Therefore, at each point u of U the complete local ring $\hat{\mathcal{O}}_{U,u}$ is a versal deformation of the quasi-spin structure induced by u. In particular, if the curve $\mathcal{X}$ we begin with is the universal curve over an étale cover T of $\overline{\mathcal{M}_g}$, the moduli stack of stable curves, then U is a cover of the stack $\text{QS}{\small\text{PIN}}_{r,g}$.

All that remains in the proof of Proposition 4.1.7 is to show that U is actually smooth over the stack of quasi-spin curves. And for this we have to show that if an affine T-scheme $Y = \text{Spec}(B)$ has a square-zero ideal $I \subseteq B$ and a quasi-spin structure $(\mathcal{E}, b)$ on $X \times_T Y$ such that $(\mathcal{E}, b)$ restricted to $Y_0 = \text{Spec}(B/I)$ has a basis $(\bar{e}_1, \dots \bar{e}_n)$ for $\Gamma(X \times_T Y_0, \bar{\mathcal{E}} \otimes \bar{\omega}^N)$, then $(\bar{e}_1, \dots, \bar{e}_n)$ lifts to a basis of $\Gamma(X \times_T Y, \mathcal{E} \otimes \omega^N)$ on Y. Namely, it suffices to show that if the projection $pr: X \times_T Y \to Y$ makes $pr_*(\bar{\mathcal{E}} \otimes \bar{\omega}^N)$ free on Y_0, then the locally free sheaf $pr_*(\mathcal{E} \otimes \omega^N)$ is also free of the same rank as $pr_*(\bar{\mathcal{E}} \otimes \bar{\omega}^N)$. But this is straightforward to check.

This completes the proof of Proposition 4.1.7 and step two of the proof of the Algebraicity Theorem. All that remains in the construction of the algebraic stack of quasi-spin curves is an analysis of the isomorphisms of quasi-spin curves.

4.1.4. *Isomorphisms of quasi-spin curves*

4.1.4.1. *Isomorphisms over a field*

The study of isomorphisms over a field k is fairly simple. Any two quasi-spin structures on X/k, say $(\mathcal{E}, b)$ and $(\mathcal{E}', b')$, which are singular at the same points and are the same on X^ν via $\theta^\natural$, must be isomorphic on X. Here $X^\nu \xrightarrow{\theta} X$ is the normalization of X at the singularities of $\mathcal{E}$. And it is easy to see that for a given $\mathcal{E}$ on X, any two quasi-spin structure maps b and b' are the same if and only if $\theta^\natural b = \theta^\natural b'$,

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and this is true if and only if $\text{length}_{\mathfrak{p}}(\text{coker}(\theta^\natural b)) = \text{length}_{\mathfrak{p}}(\text{coker}(\theta^\natural b'))$ for all $\mathfrak{p}$ in the inverse image under θ of each singular point.

PROPOSITION 4.1.11. *Automorphisms of a quasi-spin curve $(X, \mathcal{E}, b)$ that are trivial on X are of the form $\gamma = (\zeta_1, \zeta_2, \dots, \zeta_l)$, where for each i, $\zeta_i^r = 1$, and each ζ_i corresponds to a connected component X_i^ν of the curve X^ν.*

In other words, if μ_r is the group of rth roots of unity in k, and $\Gamma(X^\nu)$ is the dual graph of X^ν, then

$$\text{AUT}_X(\mathcal{E}, b) = H^0(\Gamma(X^\nu), \mu_r).$$

Proof. The normalization $\theta\colon X^\nu \to X$ at all singularities of $\mathcal{E}$ makes $\mathcal{E} \cong \theta_* \theta^\natural \mathcal{E}$, and the two functors $\theta^\natural$ and θ_* induce an equivalence of the categories of torsion-free rank-one $\mathcal{O}_X$-modules which are singular at the double points normalized by θ and invertible sheaves on X^ν. Therefore, it is enough to study $\theta^\natural(\mathcal{E})$. But automorphisms of line bundles on X^ν are just given by ν-tuples of $\zeta \in k^*$; moreover, $\theta^\natural(\mathcal{E})^r = 1$ implies that $\zeta_i^r = 1$ for all i.

Note that in general, isomorphisms of $(\mathcal{E}, b)$ over X/k must be induced by isomorphisms of $(\theta^\natural \mathcal{E}, \theta^\natural b)$ and therefore are always constant on each connected component of X^ν; namely, at a singularity they are of the form

$$\Phi = \begin{pmatrix} \varphi_+ & 0 \\ 0 & \varphi_- \end{pmatrix},$$

with φ_+ and φ_- in the base field k.

4.1.4.2. *Automorphisms of families of quasi-spin structures*

Isomorphisms of families quasi-spin curves are harder to classify than isomorphisms over a field, but if we limit ourselves to automorphisms, we can completely classify these.

Given a quasi-spin structure $(\mathcal{E}, b)$ on X/B, with $B = \text{Spec}(R)$ and R local and complete, we want to study $\text{AUT}_X(\mathcal{E}, b)$. First, we will need to study the local structure, namely, automorphisms of $(E(p,q), b)$ over $A = R[[x,y]]/pq - xy$, with $b = (b_0, \dots, b_r)$.

PROPOSITION 4.1.12. *At a singularity of $E(p,q)$ with at least one of p and q not zero, the automorphism group $\text{AUT}(E(p,q), b)$ is equal to $\mu_r = \{\zeta \in R^\times | \zeta^r = 1\}$. Thus all automorphisms of $(\mathcal{E}, b)$ are in μ_r if no singularities are of type $E(0,0)$. On the other hand, a singularity of type $(0,0)$ has automorphisms given by $(\xi, \zeta) \in \mu_r \times \mu_r$. Normalizing X at each singularity of type $(0,0)$ to get X^ν shows that any automorphism is given by $(\xi_1, \dots, \xi_m)$, with ξ_i in μ_r and with m equal to the number of connected components of X^ν.*

To prove this we first write out the automorphisms explicitly.

Given $\Phi \in \mathrm{AUT}_A(E, b)$, we know

$$\Phi = \begin{pmatrix} \varphi_+ & \psi_+ \\ \psi_- & \varphi_- \end{pmatrix},$$

with $b \circ \Phi^r = b$, and

$$\varphi_+(0)p = \varphi_-(0)p \quad \text{and} \quad \varphi_+(0)q = \varphi_-(0)q,$$

$$\varphi_+ = \varphi_+(0) + x\gamma_+, \psi_+ = p\gamma_+, \quad \text{and} \quad \gamma_+ \in R[[x]],$$

$$\varphi_- = \varphi_-(0) + y\gamma_-, \psi_- = q\gamma_-, \quad \text{and} \quad \gamma_- \in R[[y]].$$

Now if $\mathfrak{m}$ is the maximal ideal of R, we have $b = (b_0, \ldots, b_r) \equiv (\beta_0 x^i, 0, 0, \ldots, 0, \beta_r y^j) \bmod \mathfrak{m}$, $(i + j = r)$ and $\beta_r, \beta_0 \in k^*$, and

$$\Phi \equiv \begin{pmatrix} \overline{\varphi}_+ & 0 \\ 0 & \overline{\varphi}_- \end{pmatrix}.$$

By the results of the previous section, $\varphi_+^r \equiv 1 \equiv \varphi_-^r$. In fact, this will hold for the whole family, i.e. we can replace congruence with equality.

LEMMA 4.1.13.

$$\Phi = \begin{pmatrix} \varphi_+ & 0 \\ 0 & \varphi_- \end{pmatrix},$$

with $\varphi_+^r = 1, \varphi_-^r = 1$.

Proof of the Lemma. Using the step-by-step method we can assume the claim is true $\bmod I$ for some ideal I with $\mathfrak{m}I = 0$. So $\varphi_+^r = 1 + i$, which implies that $(\varphi_+ - (i/r\varphi_+^{r-1}))^r = 1$. So $\varphi_+ = \zeta + i_+$ for some i_+ in $I \cdot R[[x]]$, with $\zeta^r = 1$. Similarly, $\varphi_- = \xi + i_-$ for some i_- in $I \cdot R[[y]]$, with $\zeta^r = 1$. Thus

$$\Phi = \begin{pmatrix} \varphi_+ & 0 \\ 0 & \varphi_- \end{pmatrix}$$

and $b \circ \Phi^r = b$ implies that

$$b_0 = b_0(\zeta + i)^r = b_0(1 + r\zeta^{r-1}i_+), \quad \text{and} \quad b_r = b_r(1 + r\xi^{r-1}i_-).$$

This implies that $b_0 r\zeta^{r-1}i_+ = 0 = b_r r\xi^{r-1}i_-$. But $b_0 \equiv x^i\overline{\beta}_0 \bmod \mathfrak{m}$, with $\overline{\beta}_0 \in k^*$, and since i_+ and i_- annihilate $\mathfrak{m}$, this implies that $b_0 r\zeta^{r-1}i_+ = x^i\beta_0(r\zeta^{r-1})i_+$

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for some $\beta_0 \in R^\times$ lifting $\bar{\beta}_0$, and similarly for b_r. Since $1/r \in R$, $i_\pm = 0$, and $\varphi_+^r = \varphi_-^r = 1$. Moreover, φ_+ and φ_- are in R.

Proof of the Proposition. Note in the previous lemma that $\xi p = \zeta p$, $\xi q = \zeta q$. So $p(\xi - \zeta) = q(\xi - \zeta) = 0$. But if $\gamma := \zeta - \xi$, then $(\xi + \gamma)^r = 1$, which implies that

$$r\gamma\xi^{r-1} + \binom{r}{2}\gamma^2\xi^{r-2} + \cdots = 0.$$

And if (γ) is a proper ideal, then mod(γ^2) we get $r\gamma\xi^{r-1} \equiv 0$, i.e. if $1/r \in R$, $\gamma \in (\gamma^2)$, which implies that $\gamma \in \bigcap_n(\gamma^n) \subseteq \bigcap_n \mathfrak{m}^n = 0$. This implies that $\gamma = 0$.

So either

(1) γ is invertible, hence p and q are zero, or
(2) γ is zero and $\varphi_+ = \varphi_-$.

And at each singularity with at least one of p and q not zero, $\mathrm{AUT}(E(p,q), b) = \mu_r = \{\zeta \in R^\times | \zeta^r = 1\}$. And thus all automorphisms of $(\mathcal{E}, b)$ are also in μ_r if no singularities are of type $E(0,0)$. A singularity of type $(0,0)$ is determined by a choice of $(\xi, \zeta) \in \mu_r \times \mu_r$. And the global automorphisms of $(\mathcal{E}, b)$ are given by a choice of an element from μ_r for each connected component of the curve x^ν, the normalization of X at each singularity of type $(0,0)$.

4.1.4.3. *Properties of the* ISOM *functor*

To complete the proof that the stack of quasi-spin curves is algebraic, we need to finish steps three (representability of the ISOM functor) and four (ISOM is finite and unramified).

For step three we have the following proposition.

PROPOSITION 4.1.14. *For any two quasi-spin structures* $(\mathcal{E}, b)$ *and* $(\mathcal{E}', b')$ *over a stable curve* X/B, *the functor* $T \mapsto \mathrm{ISOM}_{X_T}((\mathcal{E}_T, b_T), ({\mathcal{E}'}_T, {b'}_T))$ *is represented by a quasi-projective* B*-scheme of finite type.*

Proof. For any two quasi-coherent sheaves $\mathcal{E}$ and $\mathcal{E}'$ on a curve X/B the functors $\mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ and $\mathrm{ISOM}(\mathcal{E}, \mathcal{E}')$ are representable (c.f. [EGA3, 7.7.8 and 7.7.9] and [21]). For the B-scheme V and map $\Phi: \mathcal{E}_{X_V} \to {\mathcal{E}'}_{X_V}$ on X_V which represent the functor $\mathrm{ISOM}(\mathcal{E}, \mathcal{E}')$, the condition that Φ^r commutes with b and b' is clearly an open condition, and thus is representable over V. Moreover, the scheme representing the functor $T \mapsto \mathrm{ISOM}_{X_T}((\mathcal{E}_T, b_T), ({\mathcal{E}'}_T, {b'}_T))$ is an open subscheme of $\mathrm{Hom}_{X_T}(\mathcal{E}_T, {\mathcal{E}'}_T)$, which is quasi-projective of finite type.

Moreover, because the ISOM functor for stable curves over S is representable by a quasi-projective S-scheme of finite type ([9, Theorem 1.11]), we actually have that for any two quasi-spin curves $\mathfrak{S}/T$ and $\mathfrak{S}'/T'$ the functor $T \mapsto \mathrm{ISOM}_{T\times T'}(\mathfrak{S}, \mathfrak{S}')$ is also representable by a quasi-projective S-scheme of finite type.

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This completes step three of the proof of the Algebraicity Theorem. And step four is to show that ISOM is also unramified and finite. This is the content of the next two propositions.

PROPOSITION 4.1.15. *For any two quasi-spin curves* $\mathfrak{S} = (X, \mathcal{E}, b)/T$ *and* $\mathfrak{S}' = (X', \mathcal{E}', b')/T'$, *the scheme* $\mathrm{ISOM}_{T\times T'}(pr_1^*\mathfrak{S}, pr_2^*\mathfrak{S}')$ *is unramified over* $T \times T'$.

Proof. It suffices to show that for a ring R with square-zero ideal I and for any two quasi-spin structures $(\mathcal{E}, b)$ and $(\mathcal{E}', b')$ on a stable curve X over R with two isomorphisms from $(\mathcal{E}, b)$ to $(\mathcal{E}', b')$ which agree over $\bar{R} = R/I$, the two isomorphisms must then agree over R. (We do not need to consider isomorphisms of the underlying curve because the ISOM functor for stable curves is unramified.) Since ISOM is a principal homogeneous AUT-space, we are reduced to showing that any automorphism of $(\mathcal{E}, b)$ which is the identity over $\bar{R}$ is the identity over R. But this follows easily from the fact that all automorphisms of quasi-spin curves are constant and have rth power equal to the identity. Therefore, ISOM is unramified.

Since ISOM is of finite type and unramified, it is quasi-finite, so we only need to check that it is proper to see that it is finite.

PROPOSITION 4.1.16. *For any two quasi-spin curves* $\mathfrak{S} = (X, \mathcal{E}, b)/T$ *and* $\mathfrak{S}' = (X', \mathcal{E}', b')/T'$, *the scheme* $\mathrm{ISOM}_{T\times T}(pr_1^*\mathfrak{S}, pr_2^*\mathfrak{S}')$ *is proper* (*hence finite*) *over* $T \times T$.

Proof. We use the valuative criterion. We must show that if we are given two quasi-spin curves $\mathfrak{S} = (X, \mathcal{E}, b)$ and $\mathfrak{S}' = (X', \mathcal{E}', b')$, both over $\mathrm{Spec}(R)$, where R is a discrete valuation ring, and given an isomorphism $\Phi_\eta\colon \mathfrak{S}_\eta \to \mathfrak{S}'_\eta$ defined on the generic fibres, then we can always extend Φ_η to an isomorphism Φ over all of $\mathrm{Spec}(R)$.

We can also assume that R is complete, and since for stable curves the ISOM functor is proper, we can assume that $X = X'$. Now let Y be the $fpqc$ cover of X given by $Y = U \coprod (\coprod_{\mathfrak{p}} \mathrm{Spec}(\hat{\mathcal{O}}_{X,\mathfrak{p}}))$, with the union being taken over all closed points $\mathfrak{p}$ of the singular locus of the special fibre of X, and U the smooth locus of X. If Φ_η extends to all of Y, then it will in fact be constant on all intersections $\mathrm{Spec}(\hat{\mathcal{O}}_{X,\mathfrak{p}}) \times_X U$, and these constant isomorphisms are uniquely determined by Φ_η, hence Φ_Y will descend to an extension of Φ_η on X.

Thus we only need to consider the local situation; namely, about a singular point of the special fibre. This is the case where

$$A = R[[x, y]]/(xy - \pi), \qquad \mathcal{E} = E(p, q) \quad \text{and} \quad \mathcal{E}' = E(p', q'),$$

with $pq = p'q' = \pi$. And we need to show that an isomorphism Φ_η on the fibre over the field of fractions K of R extends to an isomorphism on all of A. Φ_η lifts to a map $\tilde{\Phi}_\eta\colon (A \otimes_R K)^{\oplus 2} \to (A \otimes_R K)^{\oplus 2}$, which induces the isomorphism $\Phi_\eta\colon E(p, q) \otimes_R K \to E(p', q') \otimes_R K$. Since Φ_η is constant, $\tilde{\Phi}_\eta$ is given as a matrix

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$$\tilde{\Phi}_\eta = \begin{pmatrix} \varphi_+ & 0 \\ 0 & \varphi_- \end{pmatrix},$$

with $\varphi_\pm \in K$. It suffices to show that φ_+ and φ_- are actually in R. But to be an isomorphism, $\tilde{\Phi}_\eta$ must be such that $\tilde{b}' \circ \tilde{\Phi}_\eta^{\otimes r} = \tilde{b}$.

And since $\tilde{b} = (b_0, b_1, \dots, b_r)$ and $\tilde{b}' = (b'_0, b'_1, \dots, b'_r)$, we have $b'_0 \varphi_+^r = b_0$ and $b'_r \varphi_-^r = b_r$. But as we have seen (c.f. Prop. 3.3.1), b_0 and b'_0 are both invertible in $A[1/x]$, hence in $A[1/x]$ the constant $\varphi_+^r = b_0/b'_0 \in (R[[x,y]]/(xy - \pi))[1/x]$, and thus $\varphi_+^r \in R$, similarly for φ_-^r. But R is normal, hence the $\varphi_\pm$ are in R. And so Φ_η extends to all of Spec(A), and thus to all of X.

This completes the proof that the stack of quasi-spin curves is algebraic. In the next section we will see that quasi-spin curves actually provide a compactification of the space of smooth spin curves, namely smooth spin curves are dense in the stack of quasi-spin curves, and the stack of quasi-spin curves is proper over the stack of stable curves $\overline{\mathcal{M}}_g$.

4.2. COMPACTNESS OF QSPIN$_{r,g}$

The main result of this section is that the stack of quasi-spin curves provides a compactification of the moduli of smooth spin curves. This result has two parts, which are summarized in the Denseness and Properness Theorems.

THEOREM (Denseness). SPIN$_{r,g}$, *the moduli of smooth spin curves, is dense in* QSPIN$_{r,g}$.

THEOREM (Properness). *The stack* QSPIN$_{r,g}$ *is proper over the moduli of stable curves* $\overline{\mathcal{M}}_g$.

The fact that SPIN$_{r,g}$ is dense in QSPIN$_{r,g}$ follows from the deformation theory of quasi-spin curves. In particular, we will see in Proposition 4.2.2 that any quasi-spin curve can be deformed to a smooth spin curve.

The proof of the properness theorem is accomplished by studying the boundary of the QSPIN$_{r,g}$, i.e. the degeneration of smooth spin curves into quasi-spin structures on stable curves. This is done in Section 4.2.2.

4.2.1. *Deformation theory of quasi-spin curves*

The main result of this section is that any quasi-spin curve can be deformed into a smooth spin curve. A complete description of the deformations of quasi-spin curves is difficult, due to the potential existence of nilpotent elements. This section is limited, therefore, to constructing a specific deformation to a smooth spin curve and thus giving the Denseness Theorem.

PROPOSITION 4.2.1. *Given any quasi-spin curve* $(X/k, \bar{\mathcal{E}}, \bar{b})$, *over a field* k, *there*

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is a canonical deformation to a quasi-spin curve $(\mathcal{X}/\mathcal{M}, \mathcal{E}, b)$ *where* $\mathcal{X}/\mathcal{M}$ *is the pullback of the universal deformation*

$$\mathfrak{X} \to \mathrm{Spec}(\mathfrak{o}_k[[t_1, \ldots, t_n]]) = \mathcal{M},$$

of the curve X/k *along the homomorphism*

$$\mathfrak{o}_k[[t_1, \ldots, t_n]] \to \mathfrak{o}_k[[t_1, \ldots, t_n]],$$

via $t_i \mapsto t_i^r$ *for each* i.

For the purposes of proving that $\mathrm{QSPIN}_{r,g}$ is a compactification of the smooth spin curves, we are particularly interested in the immediate corollary.

COROLLARY 4.2.2. *All quasi-spin structures over a field have a deformation to a smooth spin structure.*

Proof of Proposition 4.2.1. First, we do this locally. Near a point $\mathfrak{p}$ of X for which $\bar{\mathcal{E}}$ is not free, let $R := \mathfrak{o}_k[[t_1, \ldots t_n]]$ and $A := \hat{\mathcal{O}}_{\mathcal{X},\mathfrak{p}} \cong R[[x, y]]/xy - t_i^r$. By Proposition 3.3.1, any quasi-spin structure $(\bar{\mathcal{E}}, \bar{b})$ near $\mathfrak{p}$ on X is completely determined by a pair of integers u and v which sum to r. Locally, the canonical lift $\mathcal{E}_{\mathfrak{p}}$ is just the A-module $E(t_i^v, t_i^u)$. And the map $b_{\mathfrak{p}}$ is induced, as in Proposition 3.3.1, by $\tilde{b} := (x^u, t_i^v x^{u-1}, \ldots, t_i^u y^{v-1}, y^v)$.

Locally, the isomorphisms of these pieces are easy to understand. By Faltings' Theorem, isomorphisms over k are of the form

$$\bar{\Phi} = \begin{pmatrix} \bar{\zeta} & 0 \\ 0 & \bar{\xi} \end{pmatrix},$$

with $\bar{\zeta}, \bar{\xi} \in k^\times$. These lift to isomorphisms

$$\Phi = \begin{pmatrix} \zeta & 0 \\ 0 & \xi \end{pmatrix},$$

for lifts $\zeta, \xi \in R^\times$ of $\bar{\zeta}$ and $\bar{\xi}$, and such lifts always exist in R.

The extension of the quasi-spin structure away from singularities is trivial. Let Z be the set of points where $\bar{\mathcal{E}}$ is not locally free. It is straightforward to see that on the open set $\mathcal{X} - Z$ the line bundle $\bar{\mathcal{E}}$ extends uniquely to a line bundle that is an rth root of $\omega|_{\mathcal{X}-Z}$.

And now any combination of local lifts will patch together into a global one. The main point here is the fact that if $\bar{\sigma}$ is a section of $\mathcal{O}_X^*$ and γ is a section of $\mathcal{O}_{\mathcal{X}}^*$ inducing $\bar{\gamma}$ in $\mathcal{O}_X^*$ such that $\bar{\sigma}^r = \bar{\gamma}$, then $\bar{\sigma}$ lifts uniquely to a section σ of $\mathcal{O}_{\mathcal{X}}^*$ such that $\sigma^r = \gamma$. This is easy to check. We can now use descent to lift $(\bar{\mathcal{E}}, \bar{b})$: we have an *fpqc*-cover of $\mathcal{X}$ by $\mathcal{X} - Z$ and the union of the schemes $\mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{X},\mathfrak{p}})$

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for each singularity $\mathfrak{p}$ of $\bar{\mathcal{E}}$. And we have a covering datum induced by the unique lift of the covering datum on X that makes $\bar{\mathcal{E}}$ a quasi-spin curve. This datum is actually a descent datum because of the uniqueness of rth-root lifts. And since all *fpqc* descent data for coherent sheaves are effective, we have the desired $(\mathcal{E}, b)$ on $\mathcal{X}$ extending $(\bar{\mathcal{E}}, \bar{b})$.

4.2.2. *Extending line bundles and spin-structures*

All that remains to prove that the stack $\mathrm{QSPIN}_{r,g}$ is a compactification of the moduli of smooth spin curves is the Properness Theorem. To prove this we will use the valuative criterion and the fact that smooth spin curves are dense to justify checking the valuative criterion only in the case that the generic fibre is smooth (c.f. [9, p. 109] or [EGA2, 7.3.10(ii)]). Thus we need to show that any smooth spin curve over the generic point of a discrete valuation ring extends (possibly after suitable extension of the valuation field) to a quasi-spin curve on the whole valuation ring.

The idea is to show that there is an extension of a smooth spin curve to a line bundle over a semi-stable curve with the necessary properties to allow us to use the contraction techniques of Propositions 3.1.2 and 3.1.3 from Section 3.1. In this way we produce a quasi-spin curve extending the smooth spin curve and thus show the valuative criterion of properness holds.

Proof of the Properness Theorem. Given a complete, discrete valuation ring R with field of quotients K, and a K-valued point η of $\mathrm{QSPIN}_{r,g}$, corresponding to $\mathfrak{S}_\eta = (X_\eta, \mathcal{L}_\eta, b_\eta)$, with X_η smooth over K, we need to construct a line bundle over a semi-stable curve extending $\mathfrak{S}_\eta$ (up to finite extension of K) which will give the desired extension when contracted to its stable model.

To begin, since $\overline{\mathcal{M}_g}$ is proper, there is a stable curve X extending X_η over R. Take a uniformizing parameter t in R and map R to itself via $t \mapsto t^r$. Pulling back X along this map yields another (singular) curve X_r. Resolving the singularities of X_r by blowing up yields a semi-stable curve $\tilde{X}$ with generic fibre X_η (up to a finite extension of K) and special fibre having chains of $n_i r - 1$ exceptional curves over each singularity of X_r. Here n_i is the order of the corresponding singularity of X, namely if X has local equation $R[[x,y]]/xy - t^n$, then $\tilde{X}$ has $nr - 1$ exceptional curves in a chain over that singularity.

Now, since $\tilde{X}$ is regular, any line bundle on the generic fibre will extend (but not uniquely) to the entire curve. In particular, there is some line bundle $\mathcal{L}$ on $\tilde{X}$ which extends $\mathcal{L}_\eta$. It is well-known that in such a case, any two line bundles which agree on the generic fibre differ only by Cartier divisors supported on the special fibre. In other words, if $\mathcal{M}_\eta \cong \mathcal{N}_\eta$ then $\mathcal{M} \cong \mathcal{N} \otimes \mathcal{O}(\Sigma a_i C_i)$, where C_i are the irreducible components of the special fibre of $\tilde{X}$, and a_i are integers. In our case, therefore, $\mathcal{L}^{\otimes r} \cong \omega_{\tilde{X}} \otimes \mathcal{O}(\Sigma a_i C_i)$ for some integers a_i.

Of course, if $\mathcal{L}$ extends $\mathcal{L}_\eta$ then any line bundle of the form $\mathcal{L} \otimes \mathcal{O}(\Sigma b_i C_i)$ also extends $\mathcal{L}_\eta$. The following lemmas show that there is a choice $\mathcal{O}(\Sigma b_i C_i)$ so that

$\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}(\Sigma b_i C_i)$ is a line bundle with degree zero on all but one exceptional curve per chain, has degree one on the one remaining exceptional curve, and there exists an $\mathcal{O}_{\tilde{X}}$-module homomorphism $\beta: \mathcal{L}'^{\otimes r} \to \omega_{\tilde{X}}$ which is an isomorphism everywhere except on the exceptional curves where $\mathcal{L}'$ has degree one. Contracting all the exceptional curves of $\tilde{X}$ induces a quasi-spin curve on X_r, and hence an R-valued point of $\mathrm{QSPIN}_{r,g}$ extending $\mathfrak{S}_\eta$.

LEMMA 4.2.3. *Given $\mathcal{L}$ on $\tilde{X}$ such that $\mathcal{L}^{\otimes r} \cong \omega_{\tilde{X}} \otimes \mathcal{O}(\Sigma a_i C_i)$, the coefficients a_i which correspond to non-exceptional components of the special fibre can all be assumed divisible by r. In particular, the line bundle $\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}(-(1/r)\Sigma_{C_i \text{ not exceptional}} a_i C_i)$ has $\mathcal{L}'^{\otimes r} \cong \omega \otimes \mathcal{O}(\Sigma e_j E_j)$ where all the E_j are exceptional curves.*

Proof of Lemma 4.2.3. Basic intersection theory shows that, for any curve C_j, the degree of $\mathcal{O}(\Sigma a_i C_i)$ on C_j is $-a_j\delta_j + \Sigma a_i \delta_{ij}$, where δ_j is the number of points in the intersection of C_j with the rest of the special fibre, and δ_{ij} is the number of points in the intersection of C_i and C_j. Now, on any given exceptional curve E in a chain, with E intersecting only two curves C_1 and C_2, we have $\deg_E(\omega(\Sigma a_i C_i)) = \deg_E(\mathcal{O}(\Sigma a_i C_i)) = -2e + c_1 + c_2$, where e, c_1, and c_2 are the coefficients in the sum $\Sigma a_i C_i$ of E, C_1, and C_2 respectively. Moreover, $\deg_{C_i}(\omega(\Sigma a_i C_i)) = r \deg_{C_i} \mathcal{L} \equiv 0 (\mathrm{mod}\, r)$ for every C_i. So, in particular, $c_1 + c_2 \equiv 2e (\mathrm{mod}\, r)$. Now, given a chain of exceptional curves $E_1, \ldots, E_{nr-1}$, and the two non-exceptional curves C and D that the chain joins, if their associated coefficients are $e_1, e_2, \ldots, e_{nr-1}, c$, and d, respectively, then we must have $e_2 \equiv 2e_1 - c$, $e_3 \equiv 2e_2 - e_1 \equiv 3e_1 - 2c$ and $e_i \equiv ie_1 - (i-1)c$, so that $e_{nr-1} \equiv (nr-1)e_1 - (nr-2)c$ and $d \equiv nre_1 - (nr-1)c \equiv c$. Therefore, since the special fibre is connected, and since all of the non-exceptional curves are joined by exceptional chains, all of the coefficients of the non-exceptional curves are congruent to c for some choice of c. But since the divisor (ΣC_i) is trivial, we can assume that at least one of the coefficients of a non-exceptional curve is zero, hence all of them are congruent to zero $(\mathrm{mod}\, r)$, and thus $\mathcal{L}' := \mathcal{L} \otimes (-\Sigma_{C_i \text{ not exceptional}} (a_i/r) C_i)$ is a line bundle extending $\mathcal{L}_\eta$ such that $\mathcal{L}'^{\otimes r} \cong \omega(\Sigma e_i E_i)$ and the E_i are all exceptional curves.

LEMMA 4.2.4. *If $\mathcal{L}$ is a line bundle on $\tilde{X}$ such that $\mathcal{L}^{\otimes r} \cong \omega(\Sigma e_i E_i)$, with all of the E's exceptional curves in the special fibre, then there is a choice of coefficients $\{e'_i\}$ such that $e'_i \equiv e_i (\mathrm{mod}\, r)$ for every i, and the degree of $\omega(\Sigma e'_i E_i)$ is zero on every exceptional curve except perhaps one per chain, where it has degree r. In particular the bundle $\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}(\Sigma((e'_i - e_i)/r) E_i)$ has degree zero on every exceptional curve except perhaps one per chain, where it has degree one. And $\mathcal{L}'^{\otimes r} \cong \omega(\Sigma e'_i E_i)$.*

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Proof of Lemma 4.2.4. Because $\mathcal{L}^{\otimes r} \cong \omega(\Sigma e_i E_i)$, the degree of $\mathcal{O}(\Sigma e_i E_i)$ on each E_i must be congruent to zero $(\operatorname{mod} r)$, and so for any particular chain $E_1, \ldots E_{nr-1}$ we have $e_2 \equiv 2e_1$, $e_3 \equiv 3e_1$, and $e_i \equiv ie_1$ for each i. Choose $0 \geqslant e'_1 > -r$ with $e'_1 \equiv e_1(\operatorname{mod} r)$, and let $e'_i = ie'_1$ for $1 \leqslant i \leqslant n(r+e'_1)$. Choose $e'_i = ie'_1 + r$ for $n(r+e'_1)+1 \leqslant i \leqslant nr-1$. This gives $e'_i \equiv e_i(\operatorname{mod} r)$ for all i, $\deg_{E_j} \mathcal{O}(\Sigma e'_i E_i) = 0$ for all $j \neq n(r+e'_1)$ and on $E_{nr+ne'_1}$, the degree is $-2e'_{n(r+e'_1)} + e'_{n(r+e'_1)-1} + e'_{n(r+e'_1)+1} = (-2n(r+e'_1) + n(r+e'_1) - 1 + n(r+e'_1)+1) + r$, which is r.

Note that all of the e'_i in the previous proposition were negative, thus there is a canonical inclusion map $I\colon \omega(\Sigma e'_i E_i) \hookrightarrow \omega$.

Note also that we can contract all of the exceptional curves, except the one on which $\mathcal{L}'$ has degree one, and we still have a line bundle on a semi-stable curve $\bar{X}$. But now the semi-stable curve has only one exceptional curve per chain, so we are in the situation described in Section 3.1. In this special case, the singularity of the stable model has local equation $R[[x,y]]/xy - t^{nr}$. In other words, using the notation of Sections 3.1 and 3.2, we have $\pi = t^{nr}$. Moreover, the parameters p and q can be taken to be t^{nv} and t^{nu} respectively, and $u+v=r$. This means that Proposition 3.1.3 holds, and the induced map on $\mathcal{L}'^{\otimes r}$ is locally isomorphic to the the canonical inclusion map above. In other words, if λ is the isomorphism $\mathcal{L}' \xrightarrow{\lambda} \mathcal{O}_{\bar{X}}(1)$, and if i is the induced map $\mathcal{O}_{\bar{X}}(1)^{\otimes r} \xrightarrow{i} \mathcal{O}_{\bar{X}}$, then it is easy to see that locally there exist isomorphisms $\phi\colon \mathcal{L}'^{\otimes r} \xrightarrow{\sim} \omega(\Sigma e'_i E_i)$ and $\sigma\colon \omega \xrightarrow{\sim} \mathcal{O}_{\bar{X}}$ which make the following diagram commute.

$$\begin{array}{ccc}
\mathcal{L}'^{\otimes r} & \xrightarrow{I\circ\phi} & \omega \\
\Big\downarrow{\scriptstyle \lambda^{\otimes r}} & & \Big\downarrow{\scriptstyle \sigma} \\
\mathcal{O}_{\bar{X}}(1)^{\otimes r} & \xrightarrow{i} & \mathcal{O}_{\bar{X}}
\end{array}$$

Now it is clear that the valuative criterion holds, because by Proposition 3.1.2, the line bundle and canonical map constructed above contract to give a quasi-spin curve which is an extension of the smooth spin curve over the generic fibre.

This concludes the proof of the Properness Theorem, and the construction of the first compactification, $\mathrm{QSPIN}_{r,g}$. There are, however, two other compactifications that are closely related to $\mathrm{QSPIN}_{r,g}$ and which have some better geometric properties.

5. Singular and pure spin curves

The deformation theory of quasi-spin curves is messy, due to some nilpotent elements in the structure map; thus is it difficult to describe the local geometry of the stack of quasi-spin curves. An alternative compactification of the moduli of smooth spin curves can be constructed using quasi-spin curves with additional restrictions on the structure map to prevent these nilpotent elements. The functor of interest will be called the functor of *generalized spin curves*, *singular spin curves*, or just *spin curves*. Although the stack of spin curves is harder to construct than the stack of quasi-spin curves was, it yields a compactification which is somewhat better behaved than $\mathrm{QSPIN}_{r,g}$. We will not only be able to describe the singularities of the stack of singular spin curves, but we will even be able to provide a resolution of those singularities via a third moduli problem, namely, *pure spin curves*. The biggest difficulty in dealing with singular spin curves and pure spin curves is that their definition depends on their description as an $E(p,q)$, and this requires the use of what we will call *local coordinate systems*.

5.1. DEFINITIONS: $\overline{\mathrm{SPIN}}_{r,g}$, $\mathrm{PURE}_{r,g}$, AND LOCAL COORDINATES

A local coordinate system is really just a way of choosing parameters x, y, and π, but to choose them carefully requires some work. From the deformation theory of stable curves, we know that the complete local ring $\hat{\mathcal{O}}_{X,\mathfrak{p}}$ over $\hat{\mathcal{O}}_{T,t}$ is of the form $\hat{\mathcal{O}}_{X,\mathfrak{p}} \cong \hat{\mathcal{O}}_{T,t}[[x,y]]/(xy-\pi)$ for some $\pi \in \hat{\mathcal{O}}_{T,t}$. And on some étale neighborhood T' of t, there is an étale neighborhood U of $\mathfrak{p}$ in $X \times_T T'$ with sections x and y in $\mathcal{O}_U$ such that

(1) $xy = \pi \in \mathcal{O}_{T',t}$.
(2) The ideal generated by x and y has the discriminant locus of U/T' as its associated closed subscheme.
(3) The obvious homomorphism $(\mathcal{O}_{T',t}[x,y]/(xy-\pi)) \to \mathcal{O}_{U,\mathfrak{p}}$ induces an isomorphism on the completions $(\hat{\mathcal{O}}_{T',t}[[x,y]]/(xy-\pi)) \xrightarrow{\sim} \hat{\mathcal{O}}_{U,\mathfrak{p}}$.

DEFINITION 5.1.1. We call such a system a *local coordinate* for X/T near $\mathfrak{p}$.

Note that a local coordinate is not uniquely determined. It is only determined up to the equivalence relation generated by the following operations:

(1) Pullback to étale covers.
(2) Change by units: namely $x' = \tilde{u}x, y' = \tilde{v}y, \pi' = \tilde{w}\pi$ with $\tilde{u},\tilde{v} \in \mathcal{O}^*_{X'}$, and $\tilde{u}\tilde{v} = \tilde{w} \in \mathcal{O}^*_{T'}$.
(3) Switching branches: namely, interchanging x and y.

Now we can define a spin curve. Intuitively, a spin curve is just a quasi-spin curve that is locally isomorphic to one of the bundle-map pairs induced by contraction of an exceptional curve. To say this more formally, we have the following definition.

DEFINITION 5.1.2. A *spin curve* is a stable curve X/T with quasi-spin structure $(\mathcal{E}, b)$, such that for any singularity $\mathfrak{p}$ of any fibre X_t/t of X/T there is a local coordinate (U, T', x, y, π) and a trivialization of the canonical bundle $\sigma\colon \omega|_U \xrightarrow{\sim} \mathcal{O}_U$

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such that the sheaf $\mathcal{E}$ is isomorphic to $E(p, q)$ for some p and q in $\mathcal{O}_T$ with $pq = \pi$, and the composite map $b: \mathcal{E}^{\otimes r} \to \omega \xrightarrow{\sim} \mathcal{O}_U$ agrees with the canonical induced morphism. In other words, if λ is the isomorphism $\mathcal{E}|_U \xrightarrow{\lambda} E(p, q)$, and if i is the induced map $E(p, q)^{\otimes r} \xrightarrow{i} \mathcal{O}_U$, then the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{E}|_U^{\otimes r} & \xrightarrow{\;b\;} & \omega|_U \\
\Big\downarrow{\scriptstyle \lambda^{\otimes r}} & & \Big\downarrow{\scriptstyle \sigma} \\
E(p, q)^{\otimes r} & \xrightarrow{\;i\;} & \mathcal{O}_U
\end{array}$$

A pure spin curve is a spin curve with the restriction that the p and q in the description of the structure bundle as an $E(p, q)$ are both powers of the same parameter.

DEFINITION 5.1.3. A *pure spin curve* is a spin curve $(X/T, \mathcal{E}, b)$ such that for each singularity $\mathfrak{p}$ of any fibre X_t/t of X/T there is a local coordinate (U, T', x, y, π), a choice of τ in $\mathcal{O}_T$, and non-negative integers u and v summing to r such that $\mathcal{E}|_U$ is isomorphic to $E(\tau^v, \tau^u)$.

Note 5.1.4. It is easy to see that all spin curves are pure over a field, because in that case p and q both vanish. And so we can take $\tau = 0$.

Isomorphisms of spin curves and pure spin curves are just isomorphisms of the underlying quasi-spin curves. The functors that we are interested in are $\overline{\text{SPIN}}_{r,g}$ and $\text{PURE}_{r,g}$, which are the étale sheafifications of the functors taking an S-scheme T to the set of isomorphism classes of spin curves and pure spin curves, respectively.

One comment about these definitions is in order; namely, if these conditions hold for one local coordinate, they will hold for any. But to show that the associated functors are algebraic stacks will require that we choose a local coordinate for all of X in such a way that the local coordinates make sense globally, i.e. we need a log structure.

5.2. LOG-STRUCTURES

A log structure is simply a way of choosing local coordinates coherently.

DEFINITION 5.2.1. A *log structure* for X/T is given by étale covers X' and T' of X and T,

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$$\begin{array}{ccc} X & \longleftarrow & X' \\ \big\downarrow & & \big\downarrow \\ T & \longleftarrow & T' \end{array}$$

and for each irreducible component of the singular locus of X' a choice of $\pi \in \mathcal{O}_{T'}$ and a choice of x and y in $\mathcal{O}_{X'}$ with the three properties listed above for a local coordinate, and with descent data related to the equivalence relation. Namely, on $X'' = X' \times_X X'$ over $T'' = T' \times_T T'$ with projection maps pr_1 and pr_2, there are 1-cocycles u, v in $\mathcal{O}^*_{X''}$ and w in $\mathcal{O}^*_{T''}$ such that: $\mathrm{pr}_2^*(x) = u\,\mathrm{pr}_1^*(x), \mathrm{pr}_2^*(y) = v\,\mathrm{pr}_1^*(y), uv = w$, and $\mathrm{pr}_2^*(\pi) = w\,\mathrm{pr}_1^*(\pi)$ with the cocycle condition that on $X''' = X' \times_X X' \times_X X'$, u, v, and w are all compatible with the different projections, i.e. $\mathrm{pr}_{12}^*(u)\mathrm{pr}_{23}^*(u) = \mathrm{pr}_{13}^*(u)$, and so forth.

As in the local case, we also impose the equivalence relation on the log structures generated by pullback to étale covers and by change by units compatible with the descent data. That is to say, two log structures (X', T', x, y, π) and (X', T', x', y', π') are equivalent if there exist $\tilde{u}, \tilde{v}$ in $\mathcal{O}^*_{X'}$ and $\tilde{w}$ in $\mathcal{O}^*_{T'}$ such that $x' = \tilde{u}x, y' = \tilde{v}x, \pi' = \tilde{w}\pi$, with $\tilde{u}\tilde{v} = \tilde{w}$; and if (u, v, w) and (u', v', w') are the cocycles corresponding to the two log structures, the units $\tilde{u}, \tilde{v}$ and $\tilde{w}$ must be compatible with them as well. By this we mean $u' = (\mathrm{pr}_1^*(\tilde{u})/\mathrm{pr}_2^*(\tilde{u}))u$, and $v' = (\mathrm{pr}_1^*(\tilde{v})/\mathrm{pr}_2^*(\tilde{v}))v$, and $w' = (\mathrm{pr}_1^*(\tilde{w})/\mathrm{pr}_2^*(\tilde{w}))w$.

As discussed above, given any two log structures with distinguished branches (x) and (y), we will have the relations $x' = \tilde{u}x$ and $y' = \tilde{v}y$, etc. And they will be almost equivalent, namely $\mathrm{pr}_1^*(x)(u' - (\mathrm{pr}_1^*(\tilde{u})/\mathrm{pr}_2^*(\tilde{u}))u) = 0$, and so forth; thus, if π is not a zero divisor (and hence x and y also) all log structures are equivalent. In particular, since a versal deformation of a stable curve has no zero divisors, the versal deformation has a unique log structure.

Switching of branches $(x \mapsto y, y \mapsto x)$ and switching of double points (i.e. interchange the different π_i) results in an action of the nth symmetric group (n is the number of double points) and the group $(\mathbb{Z}/2\mathbb{Z})^n$ on the log structures. But for our purposes this is not a problem; namely, we are interested in expressing $\mathcal{E}$ as an $E(p, q)$ and this switching just interchanges p and q or the different π_i. So given a log structure on a stable curve, we can use the methods of Faltings to describe rank-one torsion-free sheaves; namely, any such sheaf $\mathcal{E}$ is isomorphic to an $E(p, q)$, and the results on homomorphisms and isomorphisms still hold.

In general the choice of a log structure is unique up to the automorphisms $x \mapsto ux$, $y \mapsto vy$, and $\pi \mapsto w\pi$, for $uv = w$, but locally this might not be all of the automorphisms of the Henselization of the ring $R[x, y]/(xy - \pi)$. In other words, on a curve $C \to B$ we might have different log structures induced by different maps of B to the versal deformation. Nevertheless, we can get around this by considering the problem globally instead. Namely, let $\mathcal{S}/\mathcal{R} = \overline{\mathcal{M}_g}$ be a presentation of the stack of of stable curves, i.e. $\mathcal{S}$ is étale over $\overline{\mathcal{M}_g}$, and $\mathcal{R}$ is the étale equivalence relation (ISOM). $\mathcal{R}$ is smooth and has no zero divisors, so the two pull-backs to $\mathcal{R}$ of the universal curve with its unique log structure over $\mathcal{S}$ are canonically isomorphic. Hence any curve over any base has a canonical log structure induced by the unique log structure on the universal curve over $\mathcal{S}$.

Note that given a choice of p and q in $\mathcal{O}_T$, the descent data for the canonical log structure determine gluing data for the various blowings up of Section 3.1. Thus this choice of p and q yield a globally-defined semi-stable curve $\tilde{X}(p,q)$ over X, a rank-one torsion-free sheaf $\mathcal{E}(p,q) = \rho_* \mathcal{O}(1)$ on X, and a canonical map $b: \mathcal{E}^{\otimes r} \to \mathcal{M}$, for some line bundle $\mathcal{M}$.

5.3. ALGEBRAICITY

In this section we will prove that the two new moduli problems actually provide algebraic stacks. Using the canonical log structure, it is fairly easy to generalize the results of Section 4.1 on quasi-spin curves. The theorem is as follows.

THEOREM 5.3.1. $\overline{\text{SPIN}}_{r,g}$*, and* $\text{PURE}_{r,g}$ *both form separated algebraic stacks of finite type over* $\overline{\mathcal{M}_g}$*, and* $\text{SPIN}_{r,g}$ *is dense in both of these.*

Proof. First we prove the algebraic nature of $\overline{\text{SPIN}}_{r,g}$. To construct a smooth cover of $\overline{\text{SPIN}}_{r,g}$ we can consider, as in the construction of the versal deformation of $\text{QSPIN}_{r,g}$, a curve X/T and a relatively torsion-free sheaf $\mathcal{E}$, so that the pair $(X/T, \mathcal{E})$ is versal for stable curves with rank-one, torsion-free sheaves with bounded degree on each component. Using the canonical log structure and constructing the canonical induced map $\mathcal{E}^{\otimes r} \to \mathcal{M}$, we can take the scheme representing the property that $\mathcal{M}$ is isomorphic to $\omega_{X/T}$ to be our cover. Since the property of being spin is independent of choice of log structure, this is a cover of $\overline{\text{SPIN}}_{r,g}$. Moreover, because it represents all spin structure maps for $(X/T, \mathcal{E})$, it is smooth over $\overline{\text{SPIN}}_{r,g}$.

The representability of ISOM, as well as the other properties (finite and unramified), all follow from the case of $\text{QSPIN}_{r,g}$, hence $\overline{\text{SPIN}}_{r,g}$ is an algebraic stack, locally of finite type over S.

Now to construct the algebraic stack $\text{PURE}_{r,g}$, take, as above, the cover V of $\overline{\text{SPIN}}_{r,g}$ together with its canonical log-structure on the universal curve X_V and isomorphisms $\mathcal{E} \cong E(p,q)$ for p, and q in $\mathcal{O}_V$. The condition that $\mathfrak{S}_V$ is pure is representable by the relatively affine V-scheme $W := \mathbf{Spec}_V(\mathcal{O}_V[\tau]/(p-\tau^v, q-\tau^u))$. Again it is easy to verify that W is a smooth cover of $\text{PURE}_{r,g}$, and that $\text{PURE}_{r,g}$ is algebraic.

Finally, the fact that $\text{SPIN}_{r,g}$ is dense in both of the stacks follows immediately from the construction in Proposition 4.2.1.

Note 5.3.2. The last step of the proof is true because the construction in Proposition 4.2.1 actually provides a deformation from a pure spin curve to a smooth (hence pure) spin curve.

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We have already seen that over a field all spin curves are pure (Note 5.1.4). We will see in Corollary 5.4.9 that over a reduced base all quasi-spin curves are spin curves. Thus the difference between all three of the functors is strictly in the types of families they have. That is, over a field, all three functors are the same. But the deformation from one quasi/singular/pure spin curve to another must be done using families that have the properties required of that type. In the case of Proposition 4.2.1, the deformation is done via a pure family, so the conclusion is valid for all three functors.

5.4. GEOMETRY OF PURE AND SINGULAR SPIN CURVES

The most important geometric fact we will prove about $\overline{\mathrm{SPIN}}_{r,g}$ and $\mathrm{PURE}_{r,g}$ is that they are both compactifications of $\mathrm{SPIN}_{r,g}$. We will also give an explicit characterization of the singularities of $\overline{\mathrm{SPIN}}_{r,g}$, and a proof that $\mathrm{PURE}_{r,g}$ is smooth over $\overline{\mathcal{M}}_g$.

The main results of this section are the following:

PROPOSITION 5.4.1. $\overline{\mathrm{SPIN}}_{r,g}$ *is a closed substack of* $\mathrm{QSPIN}_{r,g}$*, hence proper over* $\overline{\mathcal{M}}_g$*.*

PROPOSITION 5.4.2. $\mathrm{PURE}_{r,g}$ *is proper over* $\overline{\mathcal{M}}_g$*.*

PROPOSITION 5.4.3. *The universal deformation of a spin curve* (X, E, β) *over a field* k *is the obvious formal spin curve* $(\mathcal{X}/\mathcal{M}, \mathcal{E}, b)$*. Here* $\mathcal{M}$ *is* $\mathrm{Spec}(\mathfrak{o}_k[[P_1, Q_1, \ldots, P_l, Q_l, t_{l+1}, \ldots, t_n]]/(P_i^{u_i} - Q_i^{v_i}))$ *and* $\mathcal{X}$ *is the pullback of the universal deformation* $\mathfrak{X} \to \mathfrak{o}_k[[t_1, \ldots, t_n]]$ *of the curve* X/k *along the homomorphism*

$$\mathfrak{o}_k[[t_1, \ldots, t_n]] \to \frac{\mathfrak{o}_k[[P_1, Q_1, \ldots, P_l, Q_l, t_{l+1}, \ldots, t_n]]}{(P_i^{u_i} - Q_i^{v_i})},$$

via $t_i \mapsto P_i Q_i$ *for* $i \leqslant l$*.* u_i *and* v_i *are uniquely determined by the map* β *at each singularity of* E *on the central fibre.*

PROPOSITION 5.4.4. *The universal deformation of a pure spin curve* (X, E, β) *over a field* k *is the obvious formal spin curve* $(\mathcal{X}/\mathcal{N}, \mathcal{E}, b)$*. Here* $\mathcal{N}$ *is* $\mathrm{Spec}(\mathfrak{o}_k[[\tau_1, \tau_2, \ldots, \tau_l, t_{l+1}, \ldots, t_n]])$ *and* $\mathcal{X}$ *is the pullback of the universal deformation* $\mathfrak{X} \to \mathfrak{o}_k[[t_1, \ldots, t_n]]$ *of the curve* X/k *along the homomorphism*

$$\mathfrak{o}_k[[t_1, \ldots, t_n]] \to \mathfrak{o}_k[[\tau_1, \tau_2, \ldots, \tau_l, t_{l+1}, \ldots, t_n]],$$

via $t_i \mapsto \tau_i^r$ *for* $i \leqslant l$*. Here the parameters* p_i *and* q_i *are simply* $\tau_i^{v_i}$ *and* $\tau_i^{u_i}$ *respectively, and again* u_i *and* v_i *are uniquely determined by the map* β *at each singularity of* E *on the central fibre.*

The immediate corollary of Proposition 5.4.4 is the following.

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COROLLARY 5.4.5. *The stack* $\mathrm{PURE}_{r,g}$ *provides a resolution of the singularities of* $\overline{\mathrm{SPIN}}_{r,g}$*; namely, the completion of any of its local rings is of the form* $\hat{\mathcal{O}}_{S,s}[[\tau_1, \tau_2, \ldots, \tau_l, t_{l+1}, \ldots, t_n]]$*, which is smooth over the base* $\hat{\mathcal{O}}_{S,s}$.

All of these results require that we further develop our study of the structure maps of spin curves begun in Section 3.3. The specific tool needed is the expansion of the structure maps in terms of power series. The power series expansions make it easy to prove Proposition 5.4.1, and they also allow us to study the deformation theory of spin curves and pure spin curves, which will yield Propositions 5.4.3 and 5.4.4.

5.4.1. *Power series expansions*

The main result of this section is that there is a certain relation (Equation (2)) that holds among the coefficients of a quasi-spin structure map exactly when the quasi-spin curve is a spin curve. Moreover, the obstructions to this relation's holding are nilpotent elements of the base ring, and thus any quasi-spin curve over a reduced base is a spin curve. These results are stated explicitly in Propositions 5.4.6 and 5.4.8, and Corollary 5.4.9.

In this section we will work with a quasi-spin curve $(X/B, \mathcal{E}, b)$, where the base B is the spectrum of a complete local Noetherian ring R; the completion of the ring $\mathcal{O}_{X,\mathfrak{p}}$ at a singular point $\mathfrak{p}$ is isomorphic to $A := R[[x,y]]/(xy - \pi)$, and π is an element of the maximal ideal $\mathfrak{m}$ of R. The spin-structure sheaf $\mathcal{E}$ corresponds to an $E(p,q)$, and the map b corresponds to a lift $\tilde{b} = (b_0, \ldots, b_r)$, as in Section 3.3.

The map $\tilde{b}$ has a power series expansion $b_i = \Sigma_{n\geqslant 0} b_{in} x^n + \Sigma_{m>0} b_{i,-m} y^m$. And the relations $pb_i = xb_{i+1}$ and $yb_i = qb_{i+1}$ of Proposition 3.3.1 imply the relations $p^i b_0 = x^i b_i$ and $q^{r-i} b_r = y^{r-i} b_i$, which in turn imply that

$$p^i b_{0,n+i} = b_{i,n} \text{ for } n \geqslant 0, \quad \text{and} \quad b_{i,-m} = q^{r-i} b_{r,-m-(r-i)} \quad \text{for } m \geqslant 0.$$

And in particular

$$p^j b_{0,j} = q^{r-j} b_{r,j-r} \quad \text{for all } j, 0 \leqslant j \leqslant r.$$

Moreover, since b induces a quasi-spin structure on the central fibre, there are non-negative integers u and v such that $(\operatorname{mod} \mathfrak{m})$ $\bar{b}_0 = x^u \bar{\beta}_0$ and $\bar{\beta}_0 \in (\bar{A}_x)^{\times}$. This implies that $\bar{b}_0 = \Sigma_{n\geqslant u} \bar{b}_{0,n} x^n$ with $\bar{b}_{0,u} \neq 0$, hence $b_{0,u}$ is not in $\mathfrak{m}$ and is invertible in R. Similarly, $b_{r,-v} \in R^{\times}$. So, in particular, $p^u = q^v b_{r,-v}/b_{0,u}$. Letting $w = b_{r,-v}/b_{0,u} \in A^{\times}$, we have the relation

$$p^u = q^v w.$$

In the special case that π is not a zero divisor, the relations $p^i b_{0,i} = q^{r-i} b_{r,i-r}$ for $0 \leqslant i \leqslant u$ imply that

$$b_{0,i} = w^{-1} p^{u-i} q^{u-i} b_{r,i-r} = \frac{\pi^{u-i}}{w} b_{r,i-r}.$$

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Similarly,

$$b_{r,i-r} = w\pi^{i-u} b_{0,i}, \quad \text{for } \leqslant i \leqslant r.$$

But even when π is a zero divisor

$$b_{0,0} = \pi^u b_{r,-r}, \qquad b_{r,0} = \pi^v b_{0,r}, \qquad wb_{0,u} = b_{r,-v},$$

and

$$\begin{aligned} b_{0,i} &= \frac{\pi^{u-i}}{w} b_{r,i-r} + \sigma_i \quad \text{for } 0 < i < u, \quad \text{and} \\ b_{r,i-r} &= w\pi^{i-u} b_{0,i} + \sigma_i \quad \text{for } u < i < r. \end{aligned}$$

PROPOSITION 5.4.6. *The 'bad' terms σ_i are all nilpotent elements, and can be nonzero only if π is a zero divisor.*

Proof. The fact that π must be a zero divisor for the σ_i to be nonzero has already been shown. So we only need show that the σ_i are nilpotent.

On the one hand, for any prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ such that p (and hence q) is in $\mathfrak{p}$, we have that $b_i \equiv 0 (\operatorname{mod} \mathfrak{p})$ for $0 < i < r$. And $b_0 \equiv x^u \beta$, and $b_r \equiv y^v \gamma$, with β and γ invertible elements of $R[[x]]$ and $R[[y]]$ respectively. Accordingly, $b_{0,i} \in \mathfrak{p}$ for $0 < i < u$, and $b_{r,i-r} \in \mathfrak{p}$ for $u < i < r$, and thus $\sigma_i \in \mathfrak{p}$ for $0 < i < r$ (σ_u is zero). On the other hand, if p (and therefore q) is not in $\mathfrak{p}$, then p and q are not zero divisors in $R_{\mathfrak{p}}$; hence, as demonstrated before, $\sigma_i \in \mathfrak{p}$ for $0 < i < r$. Thus σ_i is contained in the nilradical of R for every i.

In particular, for any quasi-spin structure over a reduced, complete local ring the relations

$$\begin{aligned} b_{0,i} &= \frac{\pi^{u-i}}{w} b_{r,i-r} \quad \text{for } 0 \leqslant i \leqslant u, \\ b_{r,i-r} &= w\pi^{i-u} b_{0,i} \quad \text{for } r \geqslant i \geqslant u \end{aligned} \tag{2}$$

hold.

Note 5.4.7. In the special case when $r = 2$, every singularity has $u = 1$, and thus $\sigma_0 = \sigma_1 = \sigma_2 = 0$. Therefore, the relations (2) hold for all quasi-spin structures with $r = 2$.

PROPOSITION 5.4.8. *If at each singularity of $\mathcal{E}$ the relations* (2) *hold on $\tilde{b}$, then the quasi-spin curve $(X/B, \mathcal{E}, b)$ is actually a spin curve. And conversely the relations* (2) *hold for any spin curve.*

COROLLARY 5.4.9. *When $r = 2$ all quasi-spin curves are spin curves. And if X/T has a reduced base T then any quasi-spin structure on X/T is a spin structure.*

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The proposition depends on the following simple lemma.

LEMMA 5.4.10. *When the relations* (2) *hold, we can write* b_0 *and* b_r *as the following products:*

$$b_0 = ax^u \quad \text{and} \quad b_r = awy^v \quad \text{for some } a \in A^\times.$$

In particular, given u, v, *and* w, *the fact that the specified relations* (2) *hold means that* b *is completely determined by* $a \in A^\times$.

Proof.

$$b_0 = x^u \sum_{n \geqslant 0} b_{0,n+u} x^n + 1/w \sum_{u > m \geqslant 0} \pi^{u-m} b_{r,m-r} x^m + q^r \sum_{l > 0} b_{r,l-r} y^l.$$

And thus

$$a = \sum_{n \geqslant 0} b_{0,n+u} x^n + 1/w \sum_{m > 0} b_{r,-m-v} y^m.$$

The calculation is similar for b_r.

Proof of Proposition 5.4.8. The first part of the proposition simply means that near each singularity, b is isomorphic to the map induced on $E(p,q)^{\otimes r} = (\rho_* \mathcal{O}_{\tilde{X}(p,q)}(1))^{\otimes r}$. That is to say, if λ is the isomorphism $\hat{\mathcal{E}}_{\mathfrak{p}} \xrightarrow{\lambda} E(p,q)$ and if i is the induced map $E(p,q)^{\otimes r} \xrightarrow{i} \hat{\mathcal{O}}_{\mathfrak{p}}$, then there is an isomorphism σ making the following diagram commute.

$$\begin{array}{ccc} \hat{\mathcal{E}}_{\mathfrak{p}}^{\otimes r} & \xrightarrow{b} & \hat{\omega}_{\mathfrak{p}} \\ \downarrow{\scriptstyle \lambda^{\otimes r}} & & \downarrow{\scriptstyle \sigma} \\ E(p,q)^{\otimes r} & \xrightarrow{i} & \hat{\mathcal{O}}_{\mathfrak{p}} \end{array}$$

Both parts of the proposition are easy to see if we write out explicitly the map

$$(A^2)^{\otimes r} \xrightarrow{\sim} A^{2^r} \xrightarrow{\alpha^r} (E(p,q))^r \xrightarrow{\psi^r} (\rho_* \mathcal{O}(1))^r \xrightarrow{\varphi} \rho_* \mathcal{O}(r) \xrightarrow{\gamma} \mathcal{O}_X = A.$$

The composite map is

$$\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} f_r \\ g_r \end{pmatrix} \mapsto x^u \prod_{1 \leqslant i \leqslant r} (f_i + t g_i),$$

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but this is just the map

$$\begin{aligned} b &= (x^u, tx^u \dots, t^r x^u) = (x^u, px^{u-1}, \dots, p^u, p^u t, \dots, p^u t^v) \\ &= (x^u, px^{u-1}, \dots, p^u, wq^{v-1}y, \dots, wy^v). \end{aligned}$$

This composite map depends only on the choice of isomorphism $\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}) \tilde{\to} A$, and any element a in $A^\times$ induces an automorphism of A, so by Proposition 5.4.10 any quasi-spin structure with the additional relations (2) is actually an induced map. And clearly the above induced map has the relations (2).

Note 5.4.11. All maps for which the relations (2) hold are determined, étale locally, by p, q, u, v, and an element of $A^\times$. This is because we can always map $E(p,q) \tilde{\to} E(p',q')$ with $p' = \lambda p, q' = \lambda^{-1} q$. So if $p^u = wq^v$, then $p'^u = \lambda^u p^u = \lambda^u wq^v = \lambda^r wq'^v$. So $w \to \lambda^r w$. And w has an rth root in k if the base ring R has its residue field k algebraically closed, hence by the step-by-step method the rth root of w will lift to all of R, i.e. if $\mathfrak{m}I = 0$ there is an $i \in I$ such that $w + i = \lambda^r$, which implies that $(\lambda - i/(r\lambda^{r-1}))^r = w$. Thus if the central fibre has residue field $k = \bar{k}$, then w can be taken to be one, and, in general, R has an étale cover on which we can take w to be one.

An immediate consequence of the characterization of spin curves in terms of the power series expansions is the fact that the stack of $\overline{\text{SPIN}}_{r,g}$ is a closed substack of $\text{QSPIN}_{r,g}$. In particular, the property of being a spin structure depends only on the map b, and b is a spin structure map if and only if $\sigma_i = 0$ for all i. Thus $\overline{\text{SPIN}}_{r,g}$ is the closed substack defined by the ideal of all the σ_i's. The definition of this ideal requires, of course, the use of a log structure, but again the result is independent of the choice. In particular, since there is a canonical log structure on any curve, this gives an alternate proof that $\overline{\text{SPIN}}_{r,g}$ is an algebraic stack, as any closed substack of an algebraic stack is algebraic [21, Proposition 3.4. (ii)].

5.4.2. *Deformation theory*

Using the power series results, we are now in a position to study the deformation theory of spin and pure spin curves. In this section we will prove Propositions 5.4.3 and 5.4.4.

Recall that Proposition 5.4.3 states that the the universal deformation of a spin curve (X, E, β) over a field k is the formal spin curve $(\mathcal{X}/\mathcal{M}, \mathcal{E}, b)$ constructed in the following way. Let $\mathcal{M}$ be $\text{Spec}(\mathfrak{o}_k[[P_1, Q_1, \dots P_l, Q_l, t_{l+1}, \dots, t_n]]/(P_i^{u_i} - Q_i^{v_i}))$, then $\mathcal{X}$ is the pullback of the universal deformation $\mathfrak{X} \to \mathfrak{o}_k[[t_1, \dots, t_n]]$ of the curve X/k along the homomorphism

$$\mathfrak{o}_k[[t_1, \dots, t_n]] \to \frac{\mathfrak{o}_k[[P_1, Q_1, \dots, P_l, Q_l, t_{l+1}, \dots, t_n]]}{(P_i^{u_i} - Q_i^{v_i})},$$

via $t_i \mapsto P_iQ_i$ for $i \leqslant l$. The spin-structure $(\mathcal{E}, b)$ is induced on X by the canonical semi-stable curve $X(P_i, Q_i)$ and the induced map, as in Section 3.1.

Proof of Proposition 5.4.3. Given a spin structure $(\bar{\mathcal{E}}, \bar{b})$ on a curve $\bar{X}$ over an Artin local ring $\bar{R}$ with residue field k, and given a deformation R of $\bar{R}$, namely $\bar{R} = R/I$ with $I^2 = 0$, we want to study deformations of $(\bar{X}, \bar{\mathcal{E}}, \bar{b})$ to quasi-spin curves over R.

First, we do this locally. For $\bar{A} := \bar{R}[[x, y]]/(xy - \bar{\pi})$ and $(\bar{E}, \bar{b}) = (E(\bar{p}, \bar{q}), \bar{b})$, a spin structure on $\bar{A}$, we want to lift $\bar{A}$ and $(\bar{E}, \bar{b})$. But any lift corresponds to a choice of β, P and Q such that $P^u = Q^v$ and an isomorphism $(\overline{E(P,Q)}, \bar{\beta}) \tilde{\to} (E(\bar{p}, \bar{q}), \bar{b})$. Here $\overline{E(P,Q)}$ is the module $E(P,Q)/I \cdot E(P,Q) = E(\bar{P}, \bar{Q})$ induced by pulling back $E(P,Q)$ along the canonical map $\mathrm{Spec}(\bar{R}) \to \mathrm{Spec}(R)$, and the map $\bar{\beta}$ is the canonical map induced from β on $\overline{E(P,Q)}$. By Faltings' theorem, isomorphisms over $\overline{R}$ are of the form

$$\bar{\Phi} = \begin{pmatrix} \bar{\zeta} & 0 \\ 0 & \bar{\xi} \end{pmatrix},$$

with $\bar{\zeta}, \bar{\xi} \in \bar{R}^{\times}$. These lift to isomorphisms

$$\Phi = \begin{pmatrix} \zeta & 0 \\ 0 & \xi \end{pmatrix},$$

for lifts $\zeta, \xi \in R^{\times}$ of $\bar{\zeta}$ and $\bar{\xi}$, and such lifts always exist in R. Thus any local spin structure is given simply by a choice of P and Q in R such that $\bar{P} = \bar{p}$ and $\bar{Q} = \bar{q}$ and a choice of β such that the induced map $\bar{\beta}$ on $E(\bar{p}, \bar{q})^{\otimes r} = \bar{E}^{\otimes r}$ differs from $\bar{b}$ only by an automorphism of $\bar{E}$. In particular, $\bar{\beta} = \bar{a}\bar{b}$ with $\bar{a}$ in $\bar{A}^{\times}$, and thus β is uniquely determined by an element $a \in A^{\times}$, i.e. $\beta = a(x^u, px^{u-1}, \ldots, y^v)$. This describes the local deformations.

Any combination of local lifts will patch together into a global one. As in the proof of Proposition 4.2.2, this is a consequence of *fpqc* descent and the uniqueness of rth-root lifts. Given a choice of curve X deforming $\bar{X}$, and X compatible with a given choice of P and Q at each singularity, we need to construct a pair $(\mathcal{E}, b)$ extending $\bar{\mathcal{E}}$ and $\bar{b}$. On U, the complement of the discriminant locus, the line bundle $\bar{\mathcal{E}}$ extends uniquely to a line bundle $\mathcal{E}$ that is an rth root of ω. Given an extension $E(P,Q)$ at each singularity (i.e. at $\mathrm{Spec}(\hat{\mathcal{O}}_{X,x_i})$), we have a covering datum induced by the unique lift of the covering datum on $\bar{X}$ that makes $\bar{\mathcal{E}}$ an rth root of $\bar{\omega}$. This datum is actually a descent datum because of the uniqueness of rth-root lifts. And since all *fpqc* descent data for coherent sheaves are effective, we have the desired $(\mathcal{E}, b)$ on X extending $(\bar{\mathcal{E}}, \bar{b})$. The proposition follows.

Because pure spin curves are defined by a condition on the base, the deformation theory of pure spin curves is easy, given the universal deformation of a spin curve. In particular, it is now easy to see that the universal deformation of

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a pure spin curve (X, E, β) is simply the family of spin curves induced over $\mathrm{Spec}(\mathfrak{o}_k[[\tau_1, \tau_2, \ldots, \tau_l, t_{l+1}, \ldots, t_n]])$ by pulling back along the homomorphism

$$\mathfrak{o}_k[[P_1, Q_1, \ldots, P_l, Q_l, t_{l+1}, \ldots, t_n]]/(P_i^{u_i} - Q_i^{v_i}) \to \mathfrak{o}_k[[\tau_1, \ldots, \tau_l, t_{l+1}, \ldots, t_n]],$$

via $P_i \mapsto \tau_i^{v_i}$ and $Q_i \mapsto \tau_i^{u_i}$ for every $i \leqslant l$. Here again, the integers u_i and v_i are uniquely determined by the central fibre: they are simply the lengths of the cokernel of β on the different branches of the normalization of the singularities of (X, E, β).

5.4.3. *Summary of properties of* $\overline{\mathrm{SPIN}}_{r,g}$ *and* $\mathrm{PURE}_{r,g}$

We have now proven most of the geometric results about $\overline{\mathrm{SPIN}}_{r,g}$ and $\mathrm{PURE}_{r,g}$ that we had set out to prove: Proposition 5.4.1, that the stack $\overline{\mathrm{SPIN}}_{r,g}$ is a closed substack of $\mathrm{QSPIN}_{r,g}$; Proposition 5.4.3, which gives the universal deformation of a spin curve; and Proposition 5.4.4, which gives the universal deformation of a pure spin curve. The immediate consequence of Proposition 5.4.4 is Corollary 5.4.5, which says that $\mathrm{PURE}_{r,g}$ is smooth.

We summarize the main results of all of this work in the following theorem.

THEOREM 5.4.12. *The stacks* $\overline{\mathrm{SPIN}}_{r,g}$ *and* $\mathrm{PURE}_{r,g}$ *are both compactifications of the moduli of smooth spin curves.*

Proof. We have already seen that $\mathrm{SPIN}_{r,g}$ is dense in both of these stacks (Proposition 5.3.1). Since $\overline{\mathrm{SPIN}}_{r,g}$ is a closed subscheme of $\mathrm{QSPIN}_{r,g}$, and since it surjects to $\overline{\mathcal{M}_g}$, it is also a compactification of $\mathrm{SPIN}_{r,g}$ over $\overline{\mathcal{M}_g}$. And since $\mathrm{PURE}_{r,g} \to \overline{\mathcal{M}_g}$ is surjective, and $\mathrm{PURE}_{r,g}$ is proper over $\overline{\mathrm{SPIN}}_{r,g}$, the stack $\mathrm{PURE}_{r,g}$ is also a compactification of $\mathrm{SPIN}_{r,g}$ over $\overline{\mathcal{M}_g}$.

Conclusion

We have constructed three algebraic stacks which compactify the moduli space of spin curves. The stack of quasi-spin curves, which, in some sense, is easiest to construct, is not as easy to describe as the substack of spin curves, which has nice (Gorenstein) singularities. And these singularities are resolved by the stack of pure-spin curves.

The three differ only in the types of families that are permissible. And in the special case of 2-spin curves, all three compactifications coincide, and many of the special considerations of the higher spin curves are no longer necessary. Moreover, this compactification of 2-spin curves agrees with those of Cornalba [7] and Deligne [8].

Finally, it is worth mentioning that determining the connectivity and irreducibility of these stacks is an interesting problem. In the case of $r = 2$ Cornalba has shown that the stack consists of two disjoint irreducible components, but the techniques used in proving that result do not extend easily to the case when r is larger than 2.

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