THE FALSE EQUILIBRIUM OF A FORCE-FREE MAGNETIC FIELD*

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ABSTRACT

It has been a customary assumption that any force-free magnetic field represents an equilibrium field in the solar atmosphere under the extreme condition $8\pi p/B^2 \ll 1$. An example of a force-free magnetic field is presented for which this assumption fails in the sense that no equilibrium is possible for the magnetic field if imposed with an arbitrary ambient pressure, however weak the pressure is. A simple mechanism is proposed for the onset of eruption in the course of otherwise quasi-static evolution of magnetic fields in the solar atmosphere.

1. INTRODUCTION

I show an interesting property of nonlinear force-free magnetic fields. I will first describe the property and then go on to point out its physical implication for the onset of eruptive processes in the solar atmosphere. The mathematical details are reported elsewhere (Low, 1979) and only the results are quoted here.

Consider the force-free equation for a magnetic field $\underline{B}$,

$$(\underline{\nabla} \times \underline{B}) \times \underline{B} = 0. \tag{1}$$

It has been customary to assume that any solution of this equation represents a magnetic field in static equilibrium under the extreme condition of low plasma beta. Let us construct an example below for which this seemingly reasonable assumption fails and find out what it physically means.

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2. SYSTEM INVARIANT IN A GIVEN DIRECTION

For a magnetic field depending on only two Cartesian coordinates y and z, we may express it in the form,

$$\underline{B} = B_o \left(H, \frac{\partial A}{\partial z}, - \frac{\partial A}{\partial y}\right). \tag{2}$$

where B_o is a constant field strength and H and A are scalar functions. If $\underline{B}$ is force-free, equation (1) requires that H be a strict function of A,

$$H(y,z) = H\left[A(y,z)\right], \tag{3}$$

while A satisfies,

$$\nabla^2 A + H(A) \frac{d}{dA} H(A) = 0. \tag{4}$$

In the presence of a gas pressure p, the equation for magnetostatic equilibrium is,

$$\frac{1}{4\pi} (\underline{\nabla} \times \underline{B}) \times \underline{B} - \underline{\nabla} p = 0. \tag{5}$$

With dependence on only y and z, this equation requires that p is also a strict function of A,

$$p(y,z) = p_o\, P\left[A(y,z)\right], \tag{6}$$

where p_o is a constant pressure, while A satisfies,

$$\nabla^2 A + H(A) \frac{d}{dA} H(A) + \beta \frac{d}{dA} P(A) = 0, \tag{7}$$

where the plasma beta is defined to be,

$$\beta = 4\pi p_o / B_o^2 . \tag{8}$$

The question we are addressing here is the following. In the limit of $\beta \to 0$, a solution of equation (7) reduces to a solution of equation (4). In this sense, a force-free field is an approximation of the magnetostatic field in the limit of $\beta \ll 1$. The magnetostatic field differs from the corresponding force-free field by first order amounts needed to accommodate the weak pressure. However, there are force-free fields which have no neighboring magnetostatic equilibria to enable the field to accommodate arbitrarily imposed pressures, no matter how small

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these pressures are. In other words, any one of these force-free fields is not the limit of any solution of equation (7) as β goes to zero. Since realistic pressures are never zero, these types of pathological force-free fields do not represent static fields. There is no equilibrium and force imbalance must develop dynamical motion rapidly in consequence of the large Lorentz force the magnetic field can produce in the low beta environment.

3. THE EXAMPLE

To construct an example of the pathological force-free fields, let us take the plane z = 0 to be the photosphere and confine our interest to the region $z \geq 0$. Consider the force-free field,

$$\underline{B} = B_o \left[y^2 + (z + z_o)^2 + a^2\right]^{-1} (a,\ z+z_o,\ -y) \tag{9}$$

where z_o and a are positive constants. This field extending over the entire y-z plane was first considered by Gold and Hoyle (1960) and I have also used it as a model for evolving bipolar solar magnetic fields (Low, 1977a, b; 1978; 1979). It is a helical field invariant in the x direction such that its field line projections on the y-z plane are concentric circles centered at $y = 0$, $z = -z_o$. In the domain $z \geq 0$, this field appears as bipolar arches with each pair of footpoints rooted in the photosphere z = 0. It is easy to show that $\underline{B}$ is a solution of equation (4) with $A = A_o$ and $H = H_o$ where,

$$A_o = \frac{1}{2} \log \left[y^2 + (z + z_o)^2 + a^2\right], \tag{10}$$

$$H_o(A_o) = a \exp(-A_o). \tag{11}$$

For a fixed value of z_o, suppose $\underline{B}$ given in equation (9) is the lowest order solution of equation (7) in the limit of $\beta << 1$. Then, up to first order in β, we may expand,

$$A = A_o + \beta\, \delta A, \tag{12}$$

$$H = H_o + \beta\, \delta H. \tag{13}$$

Substitute the expansion into equation (7) to get the first order equation,

$$\nabla^2 \delta A + \delta A \frac{d}{dA_o} \left[H_o(A_o) \frac{d}{dA_o} H_o(A_o)\right] + \frac{d}{dA_o} \left[H_o(A_o) \delta H(A_o) + P(A_o)\right] = 0. \tag{14}$$

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If we change variables from y and z to A_o and s where s is the distance along a curve of constant A_o, the connectivity of the field lines requires that δH is related to δA by,

$$\delta H(A_o) = \int_{A_o = \text{constant}} \Lambda(A_o, S)\, \delta A(A_o, s) ds. \qquad (15)$$

The curves of constant A_o are the force-free field lines projected on the y-z plane. In equation (15), the integration is to be performed along a field line projection from one footpoint to the other while Λ is a known function expressible in terms of A_o. Equation (14) then becomes,

$$\nabla^2 \delta A + \delta A \frac{d}{dA_o}\left[H_o(A_o) \frac{d}{dA_o} H_o(A_o)\right] + \int_{A_o = \text{constant}} \frac{\partial}{\partial A_o}\left[H_o(A_o)\, \Lambda(A_o, s)\right] \delta A(A_o, s) ds = -\frac{d}{dA_o} P(A_o). \qquad (16)$$

Field line connectivity also demands that,

$$\delta A = 0 \quad \text{on} \quad z = 0. \qquad (17)$$

Finally, we expect the pressure not to affect regions far above the photosphere so that,

$$\lim_{(y^2 + z^2)^{\frac{1}{2}} \to \infty} \delta A = 0. \qquad (18)$$

Equations (16)-(18) pose a boundary value problem for δA to calculate the first order correction to the force-free field to accommodate the weak pressure P appearing on the right side of equation (16) as a source term. Mathematical analysis of Low (1979) shows that for any choice of P, a unique solution δA always exists if $z_o > 0$. In this case the force-free field is physical in the sense that whatever ambient pressure which is present can be accounted for with a slight local departure of the field from being exactly force-free. This is intuitively what we would expect of most force-free fields. The slight departure from being exactly force-free goes to zero smoothly as the pressure is reduced to zero. On the other hand, the boundary value problem for δA does not have a solution for arbitrarily specified P when $z_o = 0$. The result is independent of the smallness of β. The force-free field in this case

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does not represent true equilibrium since the pressure in a realistic tenuous medium is never zero.

4. CONCLUSION

Consider the evolution of solar magnetic arches given in the lowest order by the force-free field of equation (9) with z_o decreasing with time from a finite value to zero. The evolution may be thought of as the result of a horizontal helical field rising through the photosphere, z=o or the motion of the photospheric footpoints in the model of Low (1977a,b; 1978). At each point of the evolution, so long as $z_o \neq 0$, the ambient pressure of the solar atmosphere has no dynamical effect apart from contributing to first order correction to the force-free field (9). When z_o attains zero, the presence of the ambient pressure becomes crucial and in general, we no longer have equilibrium. The state of non-equilibrium appears in the following form. There is no way of deforming the field by first order amounts from the pathological force-free configuration at $z_o = 0$ to balance pressure gradient everywhere. It was found that force imbalance is not reducible to zero in the upper magnetic arches leading to either upward or downward uniform motion of these arches. The onset of explosive flares or the so-called coronal transient may be due to the appearance of a pathological force-free configuration during the course of quasi-static evolution of nearly force-free solar magnetic fields.

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DISCUSSION

Van Hoven: (i) It is not a general result that one cannot find a nearly force-free equilibrium with pressure gradients; a counter-example is shown in Van Hoven *et al., Ap. J.* (1977). (ii) Ultimately, one must test the stability of any equilibrium to evaluate its reality. I would expect that such configurations as you describe are unstable.

Low: (i) I shall have to read your paper to familiarize myself with it. I would say in general, it is easy to construct nearly force-free

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fields with pressure gradients, contrary to your remark. Take any potential field in the form of arches rooted in an imaginary photospheric plane. It can be shown that putting any small pressure into it merely requires the field to adjust by first order displacement to achieve equilibrium. I believe the non-equilibrium I demonstrate is the first example where such first order adjustment by a force-free field is not possible. (ii) I emphasize that the configuration $z_o = 0$ is not even in equilibrium in the presence of pressure and it is "worse" than an unstable equilibrium. There is just no equilibrium and that is what is so interesting physically.

Tandon: Equation for steady state used by you is

$$-\text{grad } p + \frac{1}{4\pi} (\text{curl } \underline{B}) \times \underline{B} = 0.$$

For $\beta << 1$, the force-free solution is not explicitly a valid one. Virtually, it envisages grad $p \underset{\sim}{\sim} 0$. The only valid solution is pressure balance solution. In the steady state your analysis just approximates the pressure balance solution with a near force-free solution which may not always be possible. Failure to fit a near force-free solution even for very low β with equation (1) merely points out that such a solution is in no way an approximation to pressure balance solution in a steady state. A force-free or near-force-free solution from equation (1) implies that the steady state is not possible until or unless the pressure term is balanced by another potential term (not included) in the left-hand side of equation (1). We should not draw any further conclusions from this analysis.

Low: You have misunderstood my paper. What I said is this. When given a force-free field, it seems reasonable to assume, as we all customarily do, that it approximates a magnetic field in static equilibrium in the limit of $\beta << 1$. I have presented a first example of a force-free field for which I showed rigorously that the above assumption fails. This example has interesting physical implications for the evolution of solar magnetic fields as discussed.

Callebaut: It may be that in your demonstration you have assumed too many requirements. E.g., you have fixed the footpoints, if I understood you correctly. It is *a priori* not excluded that a second-order motion of the footpoints allows a first order solution for the problem to turn up.

Low: Within the physical definition of the problem, the footpoints must be fixed. I am thinking of a force-free field with the footpoints rooted in an imaginary photosphere and asking whether the field can adjust by first order above the photosphere to accommodate an imposed ambient pressure.

Callebaut: (Comment) I have been a strong supporter of the use of force-free fields for solar flares for more than a decade. Since several years I advocate also the importance of nearly force-free fields (i.e., fields for which j is nearly parallel to $\underline{B}$) because it may be that some

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of their properties (e.g., equilibrium, topology, stability, evolution, ...) are drastically different from those of the pure force-free field. E.g., one can think of a linearly stable force-free field and a pressure balanced field very close to it which is unstable. Your calculation, if correct, is very important because it shows that for some equilibria there is either a discontinuity (implying instability) or a need for a nonlinear treatment.

Kuperus: Dr. Low starts with a force-free field and then shows that adding a little pressure ($\beta \ll 1$) the force-free field can easily get out of *equilibrium*. This would indicate that the original ($\beta = 0$) force-free field is most likely already unstable from the very beginning and possibly not a good initial configuration to start with. The analysis could be summarized as an "anti-force free field" theorem.

Low: Let me mention the following result. If we subject each of the above force-free fields to infinitesimal perturbations which depend only on the same 2 Cartesian coordinates and calculate the energy increment ΔE, we find that ΔE has a non-negative minimum. Pressure is neglected here, so each of the force-free fields is stable within this framework. In fact, the critical force-free field with $z_o = 0$ is marginally stable in that min $\Delta E = 0$.

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