Canad. J. Math. Vol. **52** (1), 2000 pp. 172–196

Cubic Base Change for GL(2)

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Abstract. We prove a relative trace formula that establishes the cubic base change for GL(2). One also gets a classification of the image of base change. The case when the field extension is nonnormal gives an example where a trace formula is used to prove lifting which is not endoscopic.

1 Introduction

Let F be a number field. Let E be a cubic extension of F, not necessarily Galois. Let $\mathbf{A}_F$ and $\mathbf{A}_E$ be their Adele rings. Let G' be the group $\mathrm{GL}(2)$, G be the group $\mathrm{Res}_F^E G'$. Then $G(F) = \mathrm{GL}(2,E)$ and $G'(F) = \mathrm{GL}(2,F)$. In this paper, we study the base change from the set of cuspidal representations of $\mathrm{GL}(2,F)$ to the set of automorphic representations of $\mathrm{GL}(2,E)$. The automorphic representations are assumed implicitly to be irreducible.

An automorphic representation $\pi = \bigotimes \pi_v$ of $\mathrm{GL}(2,E)$ is a base change of a cuspidal representation $\pi' = \bigotimes \pi'_v$ of $\mathrm{GL}(2,F)$, if the central character of π' is λ and the central character of π is $\lambda \circ N$ where N is the norm map from E to F, and if for almost all finite places v, π'_v is the principal series of $\mathrm{GL}(2,F_v)$ associated to an unramified character χ, and π_v is the principal series of $\mathrm{GL}(2,E_v)$ associated to the character $\chi \circ N$. With this definition, the base change of π' is unique by the strong multiplicity one theorem for GL(2).

The following Theorem is proved in the work of [J-PS-S]:

Theorem 1 *Any cuspidal representation of* $\mathrm{GL}(2,F)$ *has a base change to an automorphic representation of* $\mathrm{GL}(2,E)$. *A cuspidal representation has a cuspidal base change, unless when* E/F *is nonnormal, and* π' *is of the form* $\pi\bigl(I(\xi)\bigr) \otimes \nu$.

Here ξ is an idele class character on $\mathbf{A}_K^\times$ associated to KE, where KE is the splitting field of E; $\pi\bigl(I(\xi)\bigr)$ is the cuspidal representation associated to ξ, and ν is a character on $\mathbf{A}_F^\times$.

The corresponding result for the Galois extension case is proven by Langlands [L]. His result and that of Jacquet-Piatetski-Shapiro-Shalika [J-PS-S] for the nonnormal extension are used in the proof of the modularity of some Artin L-functions, which implies some cases of Artin's conjecture [L], [T]; the modularity result is also used in Wiles' proof of FLT [Wi].

Langlands uses the twisted trace formula method to prove his result, while the method in [J-PS-S] is the Converse Theorem. From the trace formula, one get a characterization of the image of base change. Namely, when E/F is Galois, let σ be the generator of the Galois group. Then a cuspidal representation π of $\mathrm{GL}(2,E)$ is a base change from $\mathrm{GL}(2,F)$ if and

Received by the editors September 18, 1998; revised September 9, 1999.
The first author was partially supported by NSF DMS 9304580. The second author was partially supported by NSF DMS 7209098.
AMS subject classification: 11F70, 11F72.

only if $\pi \cong \pi^\sigma$. In the nonnormal case, the method of Converse Theorem does not give a characterization of the image.

We will give another proof of the Theorem use a version of relative trace formula. Our method applies to both the Galois and nonnormal case, and one gets a characterization of the image of base change in both cases. Fix a nontrivial additive character ψ on $\mathbf{A}_F/F$. In Section 2, we define a F-group L and a 'Theta function' on $L(\mathbf{A}_F)$ denoted Θ_ψ^ϕ, associated to a Schwartz function ϕ on $\mathbf{A}_F^\times \oplus \mathbf{A}_F \oplus \mathbf{A}_E$. There is a homomorphism $G \to L$ while $L(F)$ is generated by $F^\times$ and the image of $G(F)$. Denote the image of $g \in G$ by g again. Define

$$\Theta_{\psi,\lambda}^\phi(g) = \int_{\mathbf{A}_F^\times/F^\times} \Theta_\psi^\phi(zg)\lambda(z)\, d^\times z, \quad g \in \mathrm{GL}(2, \mathbf{A}_E). \tag{1}$$

We prove:

Theorem 2 *Let π be a cuspidal representation of* GL(2, *E*)*. It is a base change from a cuspidal representation of* GL(2, *F*) *if and only if it satisfies the condition (*): its central character is of the form $\lambda \circ N$, and there exists $\varphi \in \pi$ and ϕ as above, such that*

$$P_\psi(\varphi, \phi) = \int_{\mathbf{A}_E^\times \mathrm{GL}(2,E)\backslash \mathrm{GL}(2,\mathbf{A}_E)} \varphi(g)\Theta_{\psi,\lambda}^\phi(g)\, dg \neq 0. \tag{2}$$

Compare with Langlands' result in the Galois extension case, one gets

Theorem 3 *When E is a cubic cyclic extension of F, the condition (*) for a cuspidal representation π of* GL(2, *E*) *is equivalent to $\pi \cong \pi^\sigma$.*

The relative trace formula we use is of the type introduced in [M-R1]. Let $f \in C_c^\infty\big(\mathrm{GL}(2, \mathbf{A}_E), \lambda \circ N\big)$, *i.e.*, f is smooth of compact support modulo center, and $f(zg) = \lambda^{-1}\big(N(z)\big) f(g)$. Define the kernel function

$$K_f(x, y) = \sum_{\xi \in \mathrm{GL}(2,E)} f(x^{-1}\xi y). \tag{3}$$

Define the distribution $I(f, \phi)$ on $\mathrm{GL}(2, \mathbf{A}_E)$ to be

$$\iint K_f\big(g, n(x)\big)\Theta_\psi^\phi(g)\psi\big(T(-x)\big)\, dx\, dg \tag{4}$$

where the integrations are over $\mathbf{A}_E^\times\, \mathrm{GL}(2, E) \setminus \mathrm{GL}(2, \mathbf{A}_E)$ and $\mathbf{A}_E/E$, $n(x) = \left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right]$; $T(x)$ is the trace of x.

For a function $f' \in C_c^\infty\big(\mathrm{GL}(2, \mathbf{A}_F), \lambda\big)$, we define similarly the kernel function $K_{f'}(x, y)$. Define the distribution $J(f')$ on $\mathrm{GL}(2, \mathbf{A}_F)$ to be

$$\iint K_{f'}\big(n(y_1), n(y_2)\big)\psi(-y_1 + y_2)\, dy_1\, dy_2. \tag{5}$$

Here the integrations are taken over $(\mathbf{A}_F/F)^2$.

For matching functions (f, ϕ) and f', (see Section 4 for a definition of *matching*), we have $I(f, \phi) = J(f')$. From this identity, and its spectral decomposition, we derive the Theorems. Moreover, when a cuspidal representation π is a base change from π' with central character λ, when (f, ϕ) and f' match, we have

$$\sum_{\varphi_i} P_\psi\big(\pi(f)\varphi_i, \phi\big)\bar{W}^{\psi}_{\varphi_i}(e) = \sum_{\varphi'_i} W^{\psi^{-1}}_{\varphi'_i}(e)\bar{W}^{\psi^{-1}}_{\varphi_i}(e) \tag{6}$$

where φ_i and φ'_i are the orthonormal bases of π and π', and

$$W^{\psi}_{\varphi}(e) = \int \varphi\big(n(x)\big)\psi(x)\,dx \tag{7}$$

where the integral is taken over $\mathbf{A}_E/E$ or $\mathbf{A}_F/F$.

We note that the case at hand is not an endoscopic lifting. This is one example that a relative trace formula can be used to treat lifting where the Arthur-Selberg trace formula can not.

The construction of the Theta function $\Theta^{\phi}_{\psi}(g)$ uses the idea of Kazhdan [K] which we summarize in Section 2. In Section 3, we study the distributions $I(f, \phi)$ and $J(f')$ and their spectral decompositions. Here we follow the argument in [M-R]. The spectral decomposition for $I(f, \phi)$ has different forms in the Galois extension and nonnormal extension cases. It is for this reason that in nonnormal extension case, some cuspidal representations have base change which is not cuspidal. In Section 4, we define the matching between (f, ϕ) and f', and prove the existence of matching functions. Here we can use the known results for quadratic base change [J-Y], [J-Y2]. In Sections 5–7, we prove the fundamental lemma, which shows the matching of Hecke functions. In the proof one uses an identity of Kloosterman sum over finite field (identity (53)), and an argument used in [M-R2], where one applies the property of Theta representation to prove the fundamental lemma. We establish the base change in Section 8.

We thank H. Jacquet for many helpful discussions. We thank the Math Research Institute in Ohio State University for their support. The first author thanks the IAS for their hospitality during his visit.

2 The Theta Function

We recall some results in [K]. In this section, unless specified, F is a local field of characteristic 0; E is a 3-dimensional commutative semisimple algebra over F. Let $[a, b]$ be the Hilbert symbol on F. Let N, T be the norm and trace maps $E \to F$. We denote by B_E the bilinear form $(e_1, e_2) \to T(e_1 e_2)$ on E, and by $\delta_{E/F} \in F^\times/F^{\times 2}$ the discriminant of B_E.

For $e \in E$, define $\theta(e)$ as in [K, Section 2]. When e is invertible, we have $\theta(e) = \frac{N(e)}{e}$. Let $\Lambda = F \oplus E$, define a quadratic form on Λ by

$$Q_t(x_0, x) = N(t)x_0^2 + x_0 T\big(\theta(t)x\big) + T\big(t\theta(x)\big).$$

From Lemma 2.1 in [K], we see

$$x_0 Q_t(x_0, x) = N(x + x_0 t) - N(x). \tag{8}$$

Let $\Lambda' = \mathrm{Hom}_F(\Lambda, F)$ be the dual space of Λ. Let $V = \Lambda \oplus \Lambda'$. In [K], Kazhdan constructed a homomorphism ι from $G(F)$ to $\mathrm{GSp}(V)$. (Recall that $G(F) = \mathrm{GL}(2, E)$.) Let $L'(F)$ be the image of the homomorphism. Clearly $F^\times$ is the center of $\mathrm{GSp}(V)$. Let $L(F)$ be the group $F^\times L'(F)$. When $g \in \mathrm{GL}(2, E)$, we will denote the element $\iota(g)$ in $L(F)$ by g.

The main result from [K] we use is the following:

Lemma 1 *There exists a representation σ_ψ of $L(F)$, acting on the space of Schwartz functions $\mathcal{S}(F^\times \oplus \Lambda)$, with:*

$$\sigma_\psi\big(n(t)\big)\phi(y, x_0, x) = \psi\big(y^{-1}Q_t(x_0, x)\big)\phi(y, x_0, x + x_0 t) \tag{9}$$

$$\sigma_\psi(z)\phi(y, x_0, x) = [z, \delta_{E/F}]|z|_F^3\phi(z^2 y, zx_0, zx) \tag{10}$$

$$\sigma_\psi(d_a)\phi(y, x_0, x) = |N(a)|_F^2\phi\big(N(a)y, N(a)x_0, a^{-1}N(a)x\big) \tag{11}$$

$$\sigma_\psi(w)\phi(y, x_0, x) = \zeta|y|_F^{-2}\int_{F\times E}\phi(y, \tilde{x}_0, \tilde{x})\psi\Big(y^{-1}\big(x_0\tilde{x}_0 + T(x\tilde{x})\big)\Big)\, d\tilde{x}_0\, d\tilde{x} \tag{12}$$

where $d_a = \left[\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right]$, $a \in E$ is invertible, $z \in F^\times$, $w = \left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]$.

For the exact value of ζ, see [K, Lemma 3.4]. The above representation is the Weil representation σ_ψ defined for the metaplectic group $\widetilde{\mathrm{GSp}}_4(F)$ restricting to L.

We need some knowledge of the discriminant $\delta_{E/F}$.

Lemma 2 *If E is a cubic Galois extension of F or if $E = F^3$, then $\delta_{E/F}$ is the identity. If $E = F \oplus K$, where $K = F(\sqrt{\tau})$ a quadratic extension of F, then $\delta_{E/F} = \tau$.*

Assume F is a number field, E is a cubic nonnormal extension of F. Let $L = EK$ be the splitting field of E, where $K = F(\sqrt{\tau})$ is a quadratic extension of F, then $\delta_{E/F} = \tau$.

Proof The cases $E = F \oplus K$ and $E = F^3$ are clear. Now assume E is a cubic field extension of F. Let $E = F(\gamma)$ with γ satisfying the irreducible equation $4x^3 - ax - b = 0$. From the definition, $\delta_{E/F} = (a^3 - 27b^2)F^{\times 2}$. Let γ_1, γ_2 be the two other solutions of the cubic equation, then $\delta_{E/F} = [(\gamma - \gamma_1)(\gamma_1 - \gamma_2)(\gamma_2 - \gamma)]^2F^{\times 2}$. Denote the number inside the bracket α.

If E/F is Galois, let σ be the generator of $\mathrm{Gal}(E/F)$. Then $\sigma(\alpha) = \alpha$, thus $\alpha \in F$, and $\delta_{E/F}$ equals identity.

If E/F is nonnormal, then $L = E(\sqrt{\tau})$. Let σ be the generator of $\mathrm{Gal}(L/E)$. Then $\sigma(\alpha) = -\alpha$, thus $\delta_{E/F} = \tau E^{\times 2}$. Since $\delta_{E/F}$ and τ lie in F, we have $\delta_{E/F} = \tau F^{\times 2}$. ∎

Let F be a number field and E a cubic extension of F. Then one can define the group $L(F)$ as before. The representation σ_ψ for the group $L(\mathbf{A}_F)$ is given by Lemma 1. The Theta function used in the introduction is defined as follows:

$$\Theta_\psi^\phi(g) = \sum_{(y,x_0,x)\in F^\times\oplus F\oplus E}\sigma_\psi(g)\phi(y, x_0, x), \quad g \in L(\mathbf{A}_F). \tag{13}$$

We remark that the representation σ_ψ is the usual global theta representation of $\widetilde{\mathrm{GSp}}_4$ restricted to L. In particular, we do not need to use the minimal representation of D_4 constructed in [K]. From the definition of Θ_ψ^ϕ, it is easy to check that it satisfies the moderate growth condition.

3 The Distributions $I(f,\phi)$ and $J(f')$

In this section, F is a number field, E a cubic extension of F. We use v to denote a place of F, $E_v = E \otimes_F F_v$. We fix the measures over E_v or F_v as in [W1, p. 10]. Let Δ_E and Δ_F be the discriminants of the fields E and F respectively. We will identify $F^\times$ with the center of $L(F)$.

3.1 Local Orbital Integrals

We express $I(f,\phi)$ in terms of local orbital integrals. We will use the notation $f_v * \phi_v(y, x_0, x)$ to denote the expression

$$\int_{\mathrm{GL}(2,E_v)/E_v^\times} f_v(g^{-1})\sigma_\psi(g)\phi_v(y, x_0, x)\, dg. \tag{14}$$

Then

Proposition 1 *Let f and ϕ be given as in the introduction,*

$$I(f,\phi) = \sum_{y\in F^\times} \prod_v I_v(y_v, f_v * \phi_v) + |\Delta_E|^{1/2} \prod_v I_v^s(f_v * \phi_v) \tag{15}$$

where

$$I_v(y, f_v * \phi_v) = \int_{E_v}\int_{F_v^\times} \sigma_\psi(z)\lambda_v(z) f_v * \phi_v(y,1,t)\psi\big(y^{-1}N(t) - T(t)\big)\, dt\, d^\times z \tag{16}$$

$$I_v^s(f_v * \phi_v) = \int_{F_v^\times} [z, \delta_{E_v/F_v}]|z|^3_{F_v} f_v * \phi_v(z^2, 0, z)\lambda_v(z)\, d^\times z. \tag{17}$$

Proof We unwind the integral (4) formally. The computation is similar to that in [M-R] and the necessary convergence follows from the argument of Proposition 1 in [M-R]. The integral (4) is

$$\int_{\mathbf{A}_F^\times/F^\times}\int_{\mathbf{A}_E/E}\int_{\mathbf{A}_E^\times \mathrm{GL}(2,E)\backslash \mathrm{GL}(2,\mathbf{A}_E)} \sum_{\gamma\in \mathrm{GL}(2,E)} f\big(g^{-1}\gamma n(x)\big)\Theta_\psi^\phi(zg)\lambda(z)\psi\big(-T(t)\big)\, dg\, dt\, d^\times z.$$

Unwind the integral and make a change of variable $g \to n(x)g$:

$$\int_{\mathbf{A}_F^\times/F^\times}\int_{\mathbf{A}_E/E}\int_{\mathbf{A}_E^\times\backslash \mathrm{GL}(2,\mathbf{A}_E)} f(g^{-1})\Theta_\psi^\phi\big(zn(x)g\big)\lambda(z)\psi\big(-T(t)\big)\, dg\, dt\, d^\times z.$$

Using the formula in Lemma 1 and the notation (14), we get:

$$\int_{\mathbf{A}_F^\times/F^\times}\int_{\mathbf{A}_E/E} \sum_{(y,x_0,x)} [z,\delta_{E/F}]|z|_F^3 f * \phi\big(z^2y, zx_0, z(x + x_0t)\big)$$
$$\cdot\, \psi\big(y^{-1}Q_t(x_0,x)\big)\lambda(z)\psi\big(-T(t)\big)\, dt\, d^\times z.$$

The sum is over $F^\times \oplus F \oplus E$. Consider the contribution I^s from the part $x_0 = 0$; the integration over t is nonzero only when $\theta(x) = y$ in which case it equals $|\Delta_E|^{1/2}$. The condition $\theta(x) = y$ implies $x \in F^\times$ and $x^2 = y$. Thus

$$I^s = |\Delta_E|^{1/2} \int_{\mathbf{A}_F^\times/F^\times} \sum_{x \in F^\times} [z, \delta_{E/F}]|z|_F^3 f * \phi(z^2x^2, 0, zx)\lambda(z)\, d^\times z.$$

Unwind the integral we get the expression $|\Delta_E|^{1/2} \prod_v I_v^s(f_v * \phi_v)$. For the contribution from the part $x_0 \neq 0$, make changes of variables $t \to t - \frac{x}{x_0}$, $z \to zx_0^{-1}$, $y \to yx_0^2$ and using the formulas in Lemma 1, we get the sum over $y \in F^\times$ as in the Proposition. ■

3.2 Spectral Decomposition of $I(f, \phi)$

We now consider the spectral decomposition of $I(f, \phi)$. We follow closely the discussion in [M-R].

At each finite place v, let R_{E_v} be the ring of integers in E_v. Set $K_v = \mathrm{GL}(2, R_{E_v})$. At an infinite place v, let K_v be the unitary group in GL(2). Let $K = \prod K_v$. Let I_2 be the identity matrix. For each idele class character χ, let $V(\chi)$ be the space of functions φ on K such that:

$$\varphi\big(zd_an(x)k\big) = \lambda\big(N(z)\big)\chi(a)\varphi(k), \quad zI_2 \in K, k \in K, d_an(x) \in K.$$

For each $s \in \mathbf{C}$, one may identify $V(\chi)$ with a space of functions on $\mathrm{GL}(2, \mathbf{A}_E)$ by extending a $\varphi \in V(\chi)$ to a function $\varphi(g, s)$ on $\mathrm{GL}(2, \mathbf{A}_E)$, with:

$$\varphi\big(zd_an(x)k, s\big) = \lambda\big(N(z)\big)\chi(a)|a|_E^{s+1/2}\varphi(k), \quad z \in \mathbf{A}_E^\times, k \in K.$$

The group $\mathrm{GL}(2, \mathbf{A}_E)$ acts on $V(\chi)$ by right shift. We get a representation denoted as $\rho_s(\chi)$.

It is well known that

$$K_f(x, y) = \sum_\pi K_{\pi,f}(x, y) + \sum_\chi K_{\chi,f}(x, y) \tag{18}$$

where π is either a cuspidal representation with central character λ or a one dimensional representation, χ is an idele class character; and

$$K_{\pi,f}(x, y) = \sum_{\varphi_i} \pi(f)\varphi_i(x)\bar{\varphi}_i(y) \tag{19}$$

$$K_{\chi,f}(x, y) = \frac{1}{4\pi i} \int_{-i\infty}^{+i\infty} \sum_{\varphi_i} E\big(x, \rho_s(\chi)(f)\varphi_i, s\big)\bar{E}(y, \varphi_i, s)\, ds. \tag{20}$$

The sum is taken over the orthonormal basis of π or $V(\chi)$. The function $E(x, \varphi, s)$ is the Eisenstein series:

$$E(g, \varphi, s) = \sum_{\gamma \in P(E)\backslash \mathrm{GL}(2,E)} \varphi(\gamma g, s).$$

Here $P = TN$ is the parabolic subgroup. The Eisenstein series is defined for Re s large and extended meromorphically to the whole complex plane.

Assume f is a $K-$finite function for the moment. As in [M-R], the integration in (4) and the sum in (18) are interchangeable:

$$I(f,\phi) = \sum_{\pi} I_\pi(f,\phi) + \sum_{\chi} I_\chi(f,\phi) \tag{21}$$

with the distributions:

$$I_\pi(f,\phi) = \int_{\mathbf{A}_{E/E}} \int_{\mathbf{A}_E^\times \text{ GL}(2,E)\backslash \text{GL}(2,\mathbf{A}_E)} K_{\pi,f}\big(g, n(x)\big)\psi\big(T(-x)\big)\Theta^\phi_{\psi,\lambda}(g)\,dg\,dx \tag{22}$$

$$I_\chi(f,\phi) = \int_{\mathbf{A}_{E/E}} \int_{\mathbf{A}_E^\times \text{ GL}(2,E)\backslash \text{GL}(2,\mathbf{A}_E)} K_{\chi,f}\big(g, n(x)\big)\psi\big(T(-x)\big)\Theta^\phi_{\psi,\lambda}(g)\,dg\,dx. \tag{23}$$

Moreover, the sum (21) is absolutely convergent.

One easily verifies $I_\pi(f,\phi) \equiv 0$ if π is a one dimensional representation. For π being any cuspidal representation, with the notations in the introduction,

$$I_\pi(f,\phi) = \sum_{\varphi_i} P_\psi\big(\pi(f)\varphi_i,\phi\big)\bar{W}^\psi_{\varphi_i}(e) \tag{24}$$

where

$$W^\psi_\varphi(g) = \int_{\mathbf{A}_E/E} \varphi\big(n(x)g\big)\psi\big(T(x)\big)\,dx.$$

We now consider $I_\chi(f,\phi)$. One uses the truncation operator. As in [M-R], denote by φ_i' the function $\rho_s(\chi)(f)\varphi_i$, we get $I_\chi(f,\phi)$ equals:

$$\lim_{T\to\infty} \frac{1}{4\pi i}\int_{-i\infty}^{+i\infty} \sum_{\varphi_i} \bar{W}(e,\varphi_i,s) \int_{\mathbf{A}_E^\times \text{ GL}(2,E)\backslash \text{GL}(2,\mathbf{A}_E)} \wedge^T E(g,\varphi_i',s)\Theta^\phi_{\psi,\lambda}(g)\,dg\,ds \tag{25}$$

where

$$W(g,\varphi,s) = \int_{\mathbf{A}_E/E} E\big(n(x)g,\varphi,s\big)\psi\big(T(x)\big)\,dx.$$

Note that $\bar{W}(e,\varphi_i,s)$ extends to a meromorphic function, with no poles on the half plane Re $s < 0$, and for any $c > 0$, there is $\sigma(c) > 0$ sufficiently small, such that the function has no poles in the region $\{s \mid \text{Re}(s) < \sigma(c), |\text{Im}(s)| < c\}$.

We claim that (25) equals

$$\frac{1}{4\pi i}\int_{-i\infty}^{+i\infty} I_{\chi,s}(f,\phi)\,ds + \delta(\chi)I'_\chi(f,\phi) \tag{26}$$

where $I_{\chi,s}$ and I'_χ are some distributions, $\delta(\chi) = 1$ when E/F is nonnormal, and $\chi^2 = \lambda\zeta \circ N$, where ζ is the quadratic character on $\mathbf{A}_F^\times$ associated to the unique quadratic extension K of F such that EK is the splitting field of E; $\delta(\chi)$ is 0 otherwise; the integral in (26) is absolutely convergent.

Proof of the claim The integral (25) has the form:

$$\lim_{T\to\infty} \int_{-i\infty}^{+i\infty} \frac{1}{4\pi i} H(s,T)\, ds$$

where

$$H(s,T) = \sum_{\varphi_i} \bar{W}(e,\varphi_i,s) \int_{\mathbf{A}_E^\times \mathrm{GL}(2,E)\backslash \mathrm{GL}(2,\mathbf{A}_E)} \wedge^T E(g,\varphi_i',s)\Theta^{\phi}_{\psi,\lambda}(g)\, dg.$$

In the notation of $H(s,T)$, we implicitly have the dependence on f, ϕ and χ. As in [M-R], $H(s,T)$ is meromorphic in s, holomorphic on the imaginary line and the above integral is absolutely convergent. Use the following Lemma, we can separate $H(s,T)$ into a sum of four functions as in [M-R].

Lemma 3

$$\int_{\mathbf{A}_E/E} \Theta^{\phi}_{\psi,\lambda}\big(n(x)g\big)\, dx = \sigma_\psi(g)\Phi(0,0) + \sum_{\xi\in F^\times} \int_{\mathbf{A}_E} \sigma_\psi(g)\Phi(\xi,t)\psi\left(\frac{N(t)}{\xi}\right) dt$$

where

$$\Phi(x_0,x) = \int_{\mathbf{A}_E^\times/E^\times} \sum_{y\in F^\times} \sigma_\psi(z)\phi(y,x_0y^{-1},x)\lambda(z)\, d^\times z.$$

The Lemma follows from the computation in [G-R-S]. Note that $\sigma_\psi(z)\Phi(x_0,x) = \lambda(z)^{-1}\Phi(x_0,x)$.

Let $M(s,\chi)$ be the intertwining operator from $V(\chi)$ to $V(\lambda\circ N\cdot\chi^{-1})$. From the Lemma and the explicit expression for $\wedge^T E(g,\varphi_i',s)$, as in [M-R], we get $H(s,T) = H_1(s,T) + H_2(s,T) + H_3(s,T) + H_4(s,T)$, where:

$$H_1(s,T) = \sum_{\varphi_i} \bar{W}(e,\varphi_i,s) \int_0^T \int_{K(\mathbf{A}_E)} a^{s-1/2}\varphi_i'(k)\sigma_\psi(d_ak)\Phi(0,0)\, dk\, d^\times a$$

$$H_2(s,T) = \sum_{\varphi_i} \bar{W}(e,\varphi_i,s) \int_0^T \int_{K(\mathbf{A}_E)} a^{s-1/2}\varphi_i'(k) \times \sum_{\xi\in F^\times} \int_{\mathbf{A}_E} \sigma_\psi(d_ak)\Phi(\xi,t)\psi\left(\frac{N(t)}{\xi}\right) dt\, dk\, d^\times a$$

$$H_3(s,T) = -\sum_{\varphi_i} \bar{W}(e,\varphi_i,s)\int_T^\infty \int_{K(\mathbf{A}_E)} a^{-s-1/2}[M(s,\chi)\varphi_i'](k)\sigma_\psi(d_a k)\Phi(0,0)\,dk\,d^\times a$$

$$H_4(s,T) = -\sum_{\varphi_i} \bar{W}(e,\varphi_i,s)\int_T^\infty \int_{K(\mathbf{A}_E)} a^{-s-1}[M(s,\chi)\varphi_i'](k)$$

$$\times \sum_{\xi\in F^\times}\int_{\mathbf{A}} \sigma_\psi(d_a k)\Phi(\xi,t)\psi\left(\frac{N(t)}{\xi}\right)\,dt\,dk\,d^\times a.$$

We will use the following estimate for the L^2-norm of φ_i' as a function on K. There exists a positive constant L, dependent only on f, such that

$$\|\rho_s(\chi)(f)\varphi_i\| \le e^{L|\operatorname{Re}(s)|} \tag{27}$$

and $\|\rho_s(\chi)(f)\varphi_i\|$ is rapidly decreasing on any vertical line $\operatorname{Re}(s)=\sigma$.

As in [M-R], we do not need to evaluate $H_2(s,T)$ and $H_4(s,T)$. The function $H_4(s,T)$ is homomorphic and rapidly decreasing on the imaginary line; the integral $\int_{-i\infty}^{+i\infty}|H_4(s,T)|\,ds$ converges, and is bounded by a constant independent of T. Meanwhile we can separate $H_2(s,T)$ into a sum of $H_{2,2}(s,T_0,T)$ and $H_{2,1}(s,T_0)$, where $H_{2,2}(s,T_0,T)$ is the part of $H_2(s,T)$ coming from the integration of a over the interval $[T_0,T]$, while $H_{2,1}(s,T_0)$ is the contribution of the integration over $(0,T_0)$. Then $H_{2,2}(s,T_0,T)$ satisfies above mentioned analytic properties for $H_4(s,T)$. The analytic property of $H_{2,1}(s,T_0)$ will follow from these of $H(s,T)$, $H_1(s,T)$ and $H_3(s,T)$. In particular $H_{2,1}(s,T_0)$ may be not holomorphic on the imaginary line, as will be shown below.

We now consider $H_1(s,T)$. Note by our notation, $\varphi_i'(k)=\rho_s(\chi)(f)\varphi_i(k)$. For $a\in\mathbf{R}^+$, write $a=b^2$ with $b\in\mathbf{R}^+$. Thus $H_1(s,T)$ equals

$$\sum_{\varphi_i}\bar{W}(e,\varphi_i,s)\rho_s(\chi)(f)\varphi_i *_K \Phi(0,0)\int_0^{\sqrt{t}} 2b^{2s-1}b\,d^\times b$$

where

$$\varphi *_K \Phi(x_0,x) = \int_K \varphi(k)\sigma_\psi(k)\Phi(x_0,x)\,dk. \tag{28}$$

As in [M-R], we consider the integral

$$\int_{\mathbf{A}_E^1}\varphi_i'(d_a k)\sigma_\psi(d_a k)\Phi(0,0)\,d^\times a \tag{29}$$

where $\mathbf{A}_E^1$ is the set of ideles with norm 1.

Lemma 4 *The expression (29) is nonzero only when (1) E/F is nonnormal, and $\chi^2=(\zeta\lambda)\circ N$ on $\mathbf{A}_E^1$; or (2) E/F is Galois, $\chi^2=\lambda\circ N$ on $\mathbf{A}_E^1$.*

Proof Make a change of variable $a \to ab^2$, $b \in \mathbf{A}_E^1$ in the integral. Observe that $\varphi_i'(d_{ab^2}k) = \chi^2(b)\varphi_i'(d_z k)$. From Lemma 1, one finds

$$\sigma_\psi(d_{ab^2}k)\Phi(0,0) = \lambda^{-1}\big(N(b)\big)[N(b),\delta_{E/F}]\sigma_\psi(d_a k)\Phi(0,0).$$

Thus the integral is nonzero only when $\chi^2(b) = \lambda\big(N(b)\big)[N(b),\delta_{E/F}]$. This translates into the condition in the Lemma. ■

If χ is not as described in the Lemma, then $\rho_s(\chi)(f)\varphi_i *_K \Phi(0,0) = 0$, thus $H_1(s,T) = 0$. If χ is given as in the above Lemma, then $H_1(s,T)$ equals:

$$\sum_{\varphi_i} \bar{W}(e,\varphi_i,s)\frac{T^s}{s}\rho_s(\chi)(f)\varphi_i *_K \Phi(0,0).$$

Similarly, $H_3(s,T)$ is nonzero only when χ is as above, when it equals:

$$\sum_{\varphi_i} \bar{W}(e,\varphi_i,s)\frac{T^{-s}}{-s}[M(s,\chi)\rho_s(\chi)(f)\varphi_i] *_K \Phi(0,0).$$

Thus $H_1(s,T) + H_3(s,T)$ is meromorphic in s, it equals 0 unless χ is as in Lemma 4. If $H_1(s,T) + H_3(s,T)$ have a pole at $s = 0$, then so will $H_{2,1}(s,T_0)$. To compensate for the possible pole, we define:

$$H'(s) = \sum_{\varphi_i} \bar{W}(e,\varphi_i,s)\frac{1}{s}\rho_s(\chi)(f)\varphi_i *_K \Phi(0,0)e^{sM}$$

where M is any positive number that is larger than L in (27). Let $H_0(s,T) = H_1(s,T) + H_3(s,T) - 2H'(s)$, then $H_0(s,T)$ is a holomorphic function on the imaginary line, (here we need the fact that when $\chi^2 = \lambda \circ N$ on $\mathbf{A}_E^1$, $M(0,\chi)$ acts as -1 on $V(\chi)$ [J-Lai]). Since $H(s,T)$ is holomorphic on the imaginary line, from the above discussion, we see $H_{2,1}(s,T_0) + 2H'(s)$ is holomorphic on the imaginary line, and clearly is a function independent of T. As in [M-R], we get

$$\int_{-i\infty}^{+i\infty} |H(s,T) - H_0(s,T)|\,ds = \int_{-i\infty}^{+i\infty} |H_4(s,T) + H_{2,2}(s,T) + [H_{2,1}(s,T_0) + 2H'(s)]|\,ds$$

is bounded by a constant independent of T. By Fatou's Lemma, we see

$$\lim_{T\to\infty} \int_{-i\infty}^{+i\infty} [H(s,T) - H_0(s,T)]\,ds = \int_{-i\infty}^{+i\infty} I_{\chi,s}(f,\phi)\,ds$$

where

$$I_{\chi,s}(f,\phi) = \lim_{T\to\infty} H_4(s,T) + H_{2,2}(s,T) + [H_{2,1}(s,T_0) + 2H'(s)].$$

The above integral is absolutely convergent. We are left to consider the integral, (for a χ in Lemma 4):

$$I'_\chi(f,\phi) = \frac{1}{4\pi i} \lim_{T\to\infty} \int_{-i\infty}^{+i\infty} H_0(s,T)\, ds$$

Recall $H_0(s,T) = [H_1(s,T) - H'(s)] + [H_3(s,T) - H'(s)]$. For the integration of $H_1(s,T) - H'(s)$, we shift the contour to the left, for the integration of $H_3(s,T) - H'(s)$, shift the contour to the right; then

$$I'_\chi(f,\phi) = \frac{1}{4\pi i} \lim_{T\to\infty} \left[\int_C \big(H_1(s,T) - H'(s)\big)\, ds + \int_{C'} \big(H_3(s,T) - H'(s)\big)\, ds \right]$$

where C is the line $\mathrm{Re}(s) = a < 0$, and C' lies in the right half plane such that the left of C' has no poles of the functions $\bar{W}(e,\varphi_i,s)$. Take the limit we get:

$$I'_\chi(f,\phi) = \frac{1}{4\pi i} \left[\int_C \big(-H'(s)\big)\, ds + \int_{C'} \big(-H'(s)\big)\, ds \right].$$

Using the residue theorem, we get

$$I'_\chi(f,\phi) = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} H'(s)\, ds + \frac{1}{2} \sum_{\varphi_i} \bar{W}(e,\varphi_i,0) \rho_0(\chi)(f)\varphi_i *_K \Phi(0,0).$$

As $H'(s)$ is holomorphic on the half plane $\mathrm{Re}(s) < 0$, we can let a tend to $-\infty$; using the estimate (27), we see the integral equals 0. Thus

$$I'_\chi(f,\phi) = \frac{1}{2} \sum_{\varphi_i} \bar{W}(e,\varphi_i,0) \rho_0(\chi)(f)\varphi_i *_K \Phi(0,0).$$

When E/F is Galois, $\lambda \circ N = \chi^2$ on $\mathbf{A}_E^1$, it is well known that $\bar{W}(e,\varphi_i,0) = 0$ for all φ_i. Thus $I'_\chi(f,\phi)$ is always 0. When E/F is nonnormal, $\chi^{-2}\lambda \circ N$ is a quadratic character, and $I'_\chi(f,\phi)$ is a nontrivial distribution. ■

We remark that when χ satisfies Lemma 4,

$$I_{\chi,s}(f,\phi) = \lim_{T\to\infty} H(s,T).$$

We have now obtained the spectral decomposition for $I(f,\phi)$.

Proposition 2 *With above notations, for $f \in C_c^\infty\big(\mathrm{GL}(2,\mathbf{A}_E), \lambda\circ N\big)$, when E/F is Galois,*

$$I(f,\phi) = \sum_\pi I_\pi(f,\phi) + \sum_\chi \int_{-i\infty}^{+i\infty} \frac{1}{4\pi i} I_{s,\chi}(f,\phi)\, ds. \tag{30}$$

When E/F is nonnormal,

(31) $$I(f,\phi) = \sum_{\pi} I_{\pi}(f,\phi) + \sum_{\chi} \int_{-i\infty}^{+i\infty} \frac{1}{4\pi i} I_{s,\chi}(f,\phi)\, ds + \sum_{\mu^2=\zeta\lambda\circ N|_{\mathbf{A}_E^1}} I'_{\mu}(f,\phi).$$

The sum of π is over all cuspidal representations, the sum of χ is over all idele class characters. The sum and integral in (30) and (31) are absolutely convergent.

The proof of the Proposition is as in [M-R]. We note that the only discrete terms in the sum (30) are from the cuspidal representations. That will imply in the cubic Galois extension case, all cuspidal representations of $\mathrm{GL}(2,F)$ base changes to a cuspidal representation of $\mathrm{GL}(2,E)$.

3.3 The Distribution $J(f')$

We now consider the distribution $J(f')$. It is a Kuznietsov trace. The unwinding and the spectral decomposition of $J(f')$ is well known. We will only state the results.

Recall

$$K_{f'}(x,y) = \sum_{\pi} K_{\pi,f'}(x,y) + \sum_{\chi} K_{\chi,f'}(x,y)$$

where π is either a cuspidal representation or one dimensional, and χ is an idele class character on $\mathbf{A}_F^{\times}$. Also

$$K_{\chi,f'}(x,y) = \int_{-i\infty}^{+i\infty} K_{\chi,s,f'}(x,y)\, ds.$$

Define $J_{\pi'}(f')$ and $J_{\chi',s}(f')$ by replacing $K_{f'}$ with the corresponding $K_{\pi',f'}(x,y)$ or $K_{\chi',s,f'}(x,y)$ in the expression (5). Using the estimates in [A], we see

$$J(f') = \sum_{\pi'} J_{\pi'}(f') + \sum_{\chi} \int_{-i\infty}^{+i\infty} J_{\chi,s}(f')\, ds.$$

The sum and integral converge absolutely. We remark that when π' is cuspidal with central character λ, $J_{\pi'}(f')$ is given by the right hand side of (6). It is a nontrivial distribution as π' has a nontrivial Whittaker model. When π' is one dimensional, $J_{\pi'}(f') = 0$.

Similar to Proposition 1, we can write $J(f')$ in terms of orbital integrals. We have

(32) $$J(f') = \sum_{a\in F^{\times}} \prod J_v(a_v, f'_v) + |\Delta_F|^{1/2} \prod J_v^s(f'_v)$$

where

(33) $$J_v(a, f'_v) = \int_{(F_v)^2} f'_v\big(n(x)w^{-1}d_a n(y)\big)\psi(x+y)\, dx\, dy$$

and

(34) $$J_v^s(f'_v) = \int_{F_v} f'_v\big(n(x)\big)\psi(x)\, dx.$$

4 Local Integrals

Fix a local field F_v, we study the space of functions on $F_v^\times$: $\hat{I}_v = \{I_v(a, f_v * \phi_v)\}$ and $\hat{J}_v = \{J_v(a, f_v')\}$. We will drop the reference to the local place v when no confusion occurs. We will show a matching between these two spaces of functions.

First fix some notations. Recall the Hilbert symbol $[a, b]$ takes value 1 if $z^2 = ax^2 + by^2$ has a nonzero solution, and -1 otherwise [Se]. Recall also the Weil's formula [J1]

$$\hat{\phi}(x)\psi\left(\frac{1}{2}ax^2\right) dx = |a|^{-1/2}\gamma(a, \psi) \int \phi(x)\psi\left(-\frac{1}{2a}x^2\right) dx$$

here $\hat{\phi}$ is the Fourier transform of ϕ and $\gamma(a, \psi)$ is the Weil constant. Define

$$\mu(a, \psi) = \frac{\gamma(a, \psi)}{\gamma(1, \psi)}[-1, a].$$

Let $\Delta_{E/F}$ denote the discriminant of R_E as a R_F-module if F is a nonarchimedean field, and 1 if F is archimedean. Then $|\Delta_F| = |\Delta_E| \prod_v |\Delta_{E_v/F_v}|_{F_v}$. From now on, we use $\|$ to denote $\|_{F_v}$.

We say the pair of functions (f, ϕ) match the function f' if

$$I(a^{-1}, f * \phi) = [2a, \delta_{E/F}]|a|\lambda(a)\mu(\delta_{E/F}, \psi) J(a, f') \tag{35}$$

and $I^s(f * \phi) = |\Delta_{E/F}|^{1/2} J^s(f')$. The main result of this section is:

Theorem 4 *Given f', there is a matching pair (f, ϕ). Given (f, ϕ), there is a matching function f'.*

Proof Since $E_v = E \otimes_F F_v$, there are three possibilities to consider: E_v is a field, $E_v = F_v \oplus K_v$ where K_v is a quadratic extension of F_v, and $E_v = F_v^3$. When E/F is Galois, the second possibility does not appear. Note that instead of working with a pair (f, ϕ), we can just consider $I(a^{-1}, \phi)$ and $I^s(\phi)$.

First consider the last two cases. We will denote by K_v either a quadratic extension of F_v or F_v^2. Then $\Lambda_v = F_v \oplus F_v \oplus K_v$. We write an element in Λ_v as (x_0, t, t') with $t' \in K_v$. From the definition, $I(a^{-1}, \phi)$ equals:

$$\iint \phi(a^{-1}z^2, z, zt, zt')\psi(att'\bar{t'} - t - t' - \bar{t'})|z|^3[z, \delta_{E_v/F_v}]\lambda(z)\, d^\times z\, dt\, dt'$$

where $x \to \bar{x}$ is the nontrivial F_v-automorphism on K_v. This is

$$\iint \hat{\phi}\left(a^{-1}z^2, z, \frac{at'\bar{t'} - 1}{z}, zt'\right) \psi(-t' - \bar{t'})|z|^2[z, \delta_{E_v/F_v}]\lambda(z)\, d^\times z\, dt'$$

where $\hat{\phi}$ is the Fourier transform of ϕ with respect to the third variable. Such an integral is an orbital integral considered in [J-Y] and [J-Y2], where the quadratic base change is studied. (The paper [J-Y2] considers the case of GL(3), however it is easy to extract from there the corresponding results in the GL(2) case. The results we use can also be found in an unpublished note [J2].) Similarly $I^s(\phi)$ are the singular orbital integrals considered there. The comparison with $J(a, f')$ and $J^s(f')$ is given in [J-Y] and [J-Y2]. The existence of matching follows from the results there, and the fact that $\delta_{E/F}$ is a square when $K_v = F_v \oplus F_v$ and is a nonsquare over F_v if K_v is a field (by Lemma 2). We will skip the details for these two cases.

In the case E_v is a field, it must be a non-archimedean field and $\delta_{E/F}$ is a square. Here $I(a^{-1}, \phi)$ equals:

$$\iint_{E_v} \phi(a^{-1}z^2, z, zt)\psi\big(aN(t) - T(t)\big)|z|^3\lambda(z)\, d^\times z\, dt. \tag{36}$$

We need to know the asymptotic behavior of this integral as $|a| \to 0$, then compare with the known asymptotic behavior of $J(a, f')$ [J-Y2].

Drop the reference of v. Let P_E be the prime ideal of E. Assume E is generated over F by a root ν of $4x^3 - \alpha x - \beta = 0$. Let u be a representative of $E^\times/E^{\times 2}$, we consider the behavior of $I(u^{-1}\nu^{-2}, \phi)$ as $|\nu| \to 0$.

Make changes of variables $t \to t\nu^{-1}$ and $z \to a\nu$ in (36), we get:

$$\iint \phi(u^{-1}z^2, z\nu, zt)\psi\left(\frac{uN(t) - T(t)}{\nu}\right)|z|^3\lambda(\nu z)\, d^\times z\, dt$$

which is

$$\iint \sigma_\psi(z)\phi(u^{-1}, \nu, t)\psi\left(\frac{uN(t) - T(t)}{\nu}\right)\lambda(\nu z)\, d^\times z\, dt. \tag{37}$$

One can apply the stationary phase to the integral over t. Write $t \in E$ as $b + c\nu + d\nu^2$. The function $h(t) = uN(t) - T(t)$ is a cubic polynomial in (b, c, d). It has a critical point only when u is a square. Thus when u is not a square, $I(u^{-1}\nu^{-2}, \phi) = 0$ for $|\nu|$ sufficiently small.

From now on, we consider the case $u = 1$. The critical points of $h(t)$ are $(\pm 1, 0, 0)$ and they are regular critical points (Hessian is nonsingular at the critical point). When $|\nu| \to 0$, the theory of stationary phase says the above integral equals:

$$\sum_C \iint_{P_E} \sigma_\psi(z)\phi(1, 0, C)\psi\left(\frac{h_C(C+V)}{\nu}\right)\lambda(\nu z)\, d^\times z\, dV$$

where the sum is over the critical points $C = \pm 1$, and h_C is the degree 2 Taylor polynomial of h around C; $h_C(C+V) = -2C + T\big(\theta(V)\big)$. For $V = b + c\nu + d\nu^2$, the quadratic form $T\big(\theta(V)\big)$ is:

$$3b^2 - \frac{\alpha}{4}c^2 + \frac{\alpha^2}{16}d^2 - \frac{3\beta}{4}cd + \alpha bd.$$

If $\alpha \neq 0$, then the quadratic form becomes:

$$3\left(b + \frac{\alpha d}{6}\right)^2 - \frac{\alpha}{4}\left(c + \frac{3\beta d}{2\alpha}\right)^2 + \frac{-\alpha^3 + 27\beta^2}{48\alpha}d^2.$$

One can then use the Weil's formula to evaluate the integral. Note that $\alpha^3 - 27\beta^2$ is the discriminant of the equation $4x^3 - \alpha x - \beta = 0$. We get:

$$\int \sum_{C=\pm 1} \sigma_\psi(z)\phi(1,0,C)\psi\left(-\frac{2C}{v}\right)\left[\gamma(-C\frac{v}{6},\psi)\gamma(C\frac{2v}{\alpha},\psi)\gamma(C\frac{24\alpha v}{\alpha^3-27\beta^2},\psi)\right]^{-1}$$
$$|\Delta_{E/F}|^{-1/2}\left|\frac{v^3}{2}\right|^{1/2}\lambda(vz)\,d^\times z.$$

By the proof of Lemma 2, we have $\alpha^3 - 27\beta^2$ is a square. One can use this fact and the formulas (27)–(31) in [J1] to simplify the above product of Weil constants. We skip the computation, the product equals

$$\gamma(1,\psi)^3\mu(-2vC,\psi)[\alpha,-3][-1,-1].$$

Again use the fact $\alpha^3 - 27\beta^2$ is a square, we see that by definition, $[\alpha,-3] = 1$. In conclusion, when $|v| \to 0$, $I(v^{-2},\phi)$ equals:

$$\int \sum_{C=\pm 1} \sigma_\psi(z)\phi(1,0,C)\psi\left(-\frac{2C}{v}\right)|\Delta_{E/F}|^{-1/2}\left|\frac{v^3}{2}\right|^{1/2}\gamma(1,\psi)^{-3}$$
$$\cdot\,\mu(-2vC,\psi)^{-1}[-1,-1]\lambda(vz)\,d^\times z$$

which is

(38)
$$I^s(\phi)|\Delta_{E/F}|^{-1/2}[-1,-1]\lambda(v)\left|\frac{v^3}{2}\right|^{1/2}\gamma(1,\psi)^{-3}\sum_{C=\pm 1}\psi\left(-\frac{2C}{v}\right)\mu(-2vC,\psi)^{-1}\lambda(C).$$

When $\alpha = 0$, the quadratic form $T\big(\theta(V)\big)$ is $3b^2 - \frac{3\beta}{4}cd$. Let $c' = \frac{c+d}{2}$ and $d' = \frac{c-d}{2}$, then it becomes $3b^2 - \frac{3\beta}{4}(c'^2 - d'^2)$. We can use Weil's formula to integrate over b, c', d'. We arrive at the same result, noting that -3 is a square in this case.

Meanwhile, using a computation as in [J-Y2], we see when $a = uv^2$ and $|v| \to 0$, $J(a, f')$ equals 0 when u is not a square, and when $u = 1$, $J(v^2, f')$ equals:

$$J^s(f')\lambda^{-1}(v)|2v|^{-1/2}\gamma(1,\psi)^{-3}[-1,-1]\sum_{C=\pm 1}\psi\left(-\frac{2C}{v}\right)\mu(-2vC,\psi)^{-1}\lambda(C).$$

Our assertion follows then from the standard argument in [J-Y2]. ■

5 Local Integral: Unramified Case—Cubic Extension

In the next three sections, we assume F is a local nonarchimedean field, with odd residue characteristic q. Assume ψ is an additive character of F of order 0, and λ is unramified. We will consider the cases when the extension over F is unramified. With these assumptions, the matching condition in Section 4 becomes:

$$I(a^{-1}, f * \phi) = [a, \delta_{E/F}]|a|\lambda(a)J(a, f')$$

and $I^s(f * \phi) = J^s(f')$.

In this section, we assume E is an unramified cubic extension of F. Let ϕ_0 be the characteristic function of the lattice $R_F^\times \oplus R_F \oplus R_E$. We prove that if f' is the image of f under the Hecke algebra homomorphism between $\mathrm{GL}(2, E)$ and $\mathrm{GL}(2, F)$, then the pair (f, ϕ_0) match the function f'.

5.1 Homomorphism Between Hecke Algebras

Let K_F, K_E be the maximal compact subgroups $\mathrm{GL}(2, R_F)$ and $\mathrm{GL}(2, R_E)$. The Hecke algebra $\mathcal{H}\big(\mathrm{GL}(2, F)//K_F, \lambda\big)$ of $\mathrm{GL}(2, F)$ consist of the smooth functions of compact support modulo center that are biinvariant under K_F, and satisfy $f(zg) = \lambda^{-1}(z)f(g)$. The Hecke algebra $\mathcal{H}\big(\mathrm{GL}(2, E)//K_E, \lambda \circ N\big)$ is defined similarly.

Denote by $\rho(\chi)$ the representation of $\mathrm{GL}(2, F)$ induced by the character:

$$z\begin{bmatrix} a & x \\ 0 & 1 \end{bmatrix} \rightarrow \lambda(z)\chi(a).$$

If $\chi = |\cdot|^s$ is unramified, $\rho(\chi)$ contains a vector fixed under K_F. Call such a vector v_0. Then for $f' \in \mathcal{H}\big(\mathrm{GL}(2, F)//K_F, \lambda\big)$,

$$\rho(\chi)(f')v_0 = f'^{\wedge}(s)v_0.$$

The map $f' \rightarrow f'^{\wedge}(s)$ is an algebra homomorphism from $\mathcal{H}\big(\mathrm{GL}(2, F)//K_F, \lambda\big)$ to $\mathbf{C}^\times$.

Let S_m be the set $\{g \in \mathrm{GL}(2, F) \mid |\det(g)| = q^m, \|g\| \leq q^m\}$. Here $\|g\| = \max\{|g_{ij}|\}$ where g_{ij} are the entries of g. Define f'_m the Hecke function by:

$$f'_m(zg) = \lambda(z)^{-1}, \quad g \in S_m; \text{ otherwise } f'_m(zg) = 0.$$

Then f'_m, $m = 0, 1 \ldots$ is a basis of $\mathcal{H}\big(\mathrm{GL}(2, F)//K_F, \lambda\big)$. Similarly, we define f_m as above a basis of $\mathcal{H}\big(\mathrm{GL}(2, E)//K_E, \lambda \circ N\big)$. The algebra homomorphism $f_m \rightarrow f_m^{\wedge}(s)$ is defined similarly with χ replaced by $\chi \circ N$.

Let ϖ be the uniformizer of P_F. Let $f_{-1} = f_{-2} = f'_{-1} = f'_{-2} \equiv 0$. Then

Proposition 3 *For any s a complex number, any $m > 0$,*

$$\big(f_m - q^3\lambda(\varpi^{-3})f_{m-2}\big)^{\wedge}(s) = \big(f'_{3m} - q\lambda(\varpi^{-1})f'_{3m-2}\big)^{\wedge}(s). \tag{39}$$

The map

$$f_m - q^3\lambda(\varpi^{-3})f_{m-2} \rightarrow f'_{3m} - q\lambda(\varpi^{-1})f'_{3m-2}$$

determines an injective homomorphism from $\mathcal{H}\big(\mathrm{GL}(2, E)//K_E, \lambda \circ N\big)$ *to* $\mathcal{H}\big(\mathrm{GL}(2, F)//K_F, \lambda\big)$.

Proof It follows from the formula

$$f_m^{\wedge}(s) = \sum_{i=0}^{m} q^{3(2i-m)s} q^{3m/2} \lambda(\varpi^{3(i-m)}). \qquad \blacksquare$$

We show that if f and f' correspond under the above homomorphism, then (f, ϕ_0) and f' match in the sense of Section 4.

5.2 Computation of $I\Big(a^{-1}, \big(f_m - q^3\lambda(\varpi^{-3}) f_{m-2}\big) * \phi_0\Big)$

Let $\Phi_i(x)$ be the characteristic function of the set $\{x \in F : |x| = q^i\}$. Define

$$I_m(a) = \int_{t\in E, |t|\le q^{3m}} \psi\big(aN(t) - T(t)\big)\, dt. \tag{40}$$

Recall from Section 4 (36) that $I(a^{-1}, f_m * \phi_0)$ equals

$$\iiint_{\mathrm{GL}(2,E)/E^\times} f_m(g^{-1})\sigma_\psi(zg)\phi_0(a^{-1},1,t)\psi\big(aN(t) - T(t)\big)\, dt\, \lambda(z)\, d^\times z\, dg. \tag{41}$$

Lemma 5 *When $m \ge 0$,*

$$I(a^{-1}, f_m * \phi_0) = \sum_{j=0}^{\infty} \Phi_{3m-2j}(a) I_{j-m}(a) q^{6m-3j} \lambda(\varpi^{j-3m}). \tag{42}$$

Proof Note that ϕ_0 is fixed under the action of $k \in K_E$. From the Iwasawa decomposition, we see S_m is the disjoint union:

$$\bigcup_{i=0}^{m} \bigcup_{w\in P_E^{-m+i}/P_E^{2i-m}} K_E \begin{bmatrix} \varpi^{-i} & \\ & \varpi^{i-m} \end{bmatrix} \begin{bmatrix} 1 & w \\ & 1 \end{bmatrix}. \tag{43}$$

From the equivariance of f under center, and Lemma 1, $I(a^{-1}, f_m * \phi_0)$ equals

$$\int_{g\in S_m} \sigma_\psi\big(n(t)zg^{-1}\big)\phi_0(a^{-1},1,0)\psi\big(-T(t)\big)\lambda(z)\, dg\, d^\times z\, dt. \tag{44}$$

We can separate the domain for g according to (43). Let i be as in (43), the contribution to (44) from the subset with index i gives

$$q^{6i-3m} \iiint_{|w|\le q^{3(m-i)}} \sigma_\psi \left(n(t)zn(w) \begin{bmatrix} \varpi^i & \\ & \varpi^{m-i} \end{bmatrix} \right) \phi_0(a^{-1},1,0)\psi\big(-T(t)\big)\lambda(z)\, d^\times z\, dt\, dw.$$

If $i \ne m$, a change of variable $t \to t - w$ shows the above integral is 0. When $i = m$, integrating over w, we get:

$$q^{3m} \iint \sigma_\psi \left(n(t)z \begin{bmatrix} \varpi^m & \\ & 1 \end{bmatrix} \right) \phi_0(a^{-1},1,0)\psi\big(-T(t)\big)\lambda(z)\, d^\times z\, dt. \tag{45}$$

The formulas in Lemma 1 then gives the expression

$$q^{6m}\lambda(\varpi^{-3m})\iint \phi_0(a^{-1}z^2\varpi^{-3m}, z, z\varpi^{-m}t)\psi\big(aN(t)-T(t)\big)\lambda(z)|z|^3\, d^\times z\, dt.$$

It is clear this is the expression in the Lemma. ■

Lemma 6 *When $m \geq 1$:*

$$I\Big(a, \big(f_m - q^3\lambda(\varpi^{-3})f_{m-2}\big) * \phi_0\Big) = \sum_{j=0}^{2} q^{6m-3j}\lambda(\varpi^{j-3m})\Phi_{3m-2j}(a)I_{j-m}(a).$$

Proof From Lemma 5, we only need to show that $\Phi_{3m-2j}(a)[I_{j-m}(a) - I_{j-m-1}(a)] = 0$ when $j \geq 3$. Since $I_{j-m}(a) - I_{j-m-1}(a)$ equals:

$$\int_{|t|=q^{3j-3m}} \psi\big(aN(t) - T(t)\big)\, dt$$
$$= q^3 \int_{|v|\leq q^{-3}} \int_{|t|=q^{3j-3m}} \psi\big(aN[t(1+v)] - T(t) - T(tv)\big)\, dt\, dv.$$

It is easy to check when $|a| = q^{3m-2j}$, the above integral over v equals 0 for $j \geq 2$. ■

Lemma 7 *When $m \geq 1$,*

$$I\Big(a, \big(f_m - q^3\lambda(\varpi^3)f_{m-2}\big) * \phi_0\Big) = \begin{cases} q^{3m}\lambda(\varpi^{-3m}) & |a| = q^{3m} \\ (-q^{3m-1} - q^{3m-2})\lambda(\varpi^{1-3m}) & |a| = q^{3m-2} \\ q^{3m-3}\lambda(\varpi^{2-3m}) & |a| = q^{3m-4}. \end{cases}$$

It equals 0 in other cases.

Proof We only need to compute $I_{j-m}(a)$ for $|a| = q^{3m-2j}$ and $j = 0, 1, 2$. When $|a| = q^{3m}$, $\psi\big(aN(t) - T(t)\big) = 1$ when $|t| \leq q^{-3m}$, thus $I_{-m}(a) = q^{-3m}$.

When $|a| = q^{3m-4}$, the argument in Lemma 6 shows the integration in $I_{2-m}(a)$ over $|t| = q^{6-3m}$ gives 0, and the integration over $|t| \leq q^{3-3m}$ gives q^{3-3m}.

When $|a| = q^{3m-2}$, the integral in $I_{1-m}(a)$ reduces to a finite field situation; it equals:

$$q^{-3m} \sum_{t\in F_{q^3}} \psi\big(N(t)\big).$$

As for $r \in F_q^\times$, there are $q^2 + q + 1$ solutions to $N(t) = r$, and $N(t) = 0$ has one solution, the above sum equals $-q^{-3m+1} - q^{-3m+2}$. ■

5.3 Computation of $J(a, f'_m)$

Recall

$$J(a, f') = \int f'\left(\begin{bmatrix}1 & x\\ & 1\end{bmatrix}\begin{bmatrix} & -1\\ a & \end{bmatrix}\begin{bmatrix}1 & y\\ & 1\end{bmatrix}\right)\psi(x+y)\,dx\,dy.$$

Lemma 8 *When $m \geq 1$,*

(46) $$J(a, f'_m) = \Phi_m(a) - \Phi_{m-2}(a)\lambda(\varpi^{-1}).$$

Proof By definition of f'_m, $J(a, f'_m)$ equals

$$\int_{|ab|,|abx|,|aby|,|abxy-b|\leq q^m,|ab^2|=q^m} \psi(x+y)\lambda(b)\,d^\times b\,dx\,dy.$$

First consider the contribution from the part with $|ab| = q^m$, then $|b| = 1$, above integral is $\Phi_m(a)\int_{x,y\in R_F} 1\,dx\,dy$, which equals $\Phi_m(a)$. Over the subset $|ab| < q^m$, then the subset $|abx| < q^m$ contributes 0, (as the integration over y will then equal 0). Thus we may impose a condition $|abx| = q^m$, the integral becomes:

$$\int_{|ab|<q^m,|ab^2|=q^m,|abx|=q^m} \psi(x + a^{-1}x^{-1})\lambda(b)\,d^\times b\,dx.$$

By a change of variable, it becomes

$$\int_{|ab|<q^m,|ab^2|=q^m,|abx|=q^m} \psi(x)\lambda(b)\,d^\times b\,dx.$$

Clearly the above integral equals $-\Phi_{m-2}(a)\lambda(\varpi^{-1})$. ■

5.4 Comparison in the case $m \geq 1$

Compare the Lemmas 7 and 8, we see

Proposition 4 *When $m \geq 1$,*

(47) $$I\Big(a, \big(f_m - q^3\lambda(\varpi^{-3})f_{m-2}\big) * \phi_0\Big) = |a|\lambda(a)J\big(a, f'_{3m} - q\lambda(\varpi^{-1})f'_{3m-2}\big).$$

Since $\delta_{E/F}$ is a square in the case at hand, this is the equation stated at the beginning of the section.

5.5 Comparison of $I(a, f_0 * \phi_0)$ and $J(a, f_0')$

It is clear that $I(a, f_0 * \phi_0) = I(a, \phi_0)$. Thus

$$I(a, f_0 * \phi_0) = \iint |z|^3 \phi_0(a^{-1}z^2, z, zt)\psi\big(aN(t) - T(t)\big)\, dt\lambda(z)\, d^\times z. \tag{48}$$

This integral is compared with the Kloosterman integral:

$$J(a, f_0') = \int_{|ab|\leq 1, |ab^2|=1, |abx|\leq 1, |aby|\leq 1, |abxy-b|\leq 1} \psi(x+y)\lambda(b)\, d^\times b\, dx\, dy. \tag{49}$$

Proposition 5

$$I(a, f_0 * \phi_0) = |a|\lambda(a) J(a, f_0'). \tag{50}$$

Proof Assume $|a| = q^l$. If l is odd, then both sides equal 0. Assume $l = 2n$, the $I(a, f_0 * \phi_0)$ equals:

$$\lambda(\varpi^{-n})q^{-3n} \int_{|a|\leq 1, |t|\leq q^{3n}} \psi\big(aN(t) - T(t)\big)\, dt \tag{51}$$

while $J(a, f_0')$ equals:

$$\lambda(\varpi^{-n}) \int_{|a|\leq 1, |x|\leq q^{-n}, |y|\leq q^{-n}, |axy-1|\leq q^n} \psi(x+y)\, dx\, dy. \tag{52}$$

The identity is trivial when $n > 0$ as both sides equal 0, and when $n = 0$ as both sides equal 1. When $|a| = q^{-2}$, we get the finite field case. We need to show for $b \in F_q^\times$:

$$\sum_{t\in F_{q^3}} \psi\big(b^{-1}N(t) - T(t)\big) = q \sum_{x\in F_q^\times} \psi(bx + x^{-1}). \tag{53}$$

This follows from the identities:

$$\sum_{b\in F_q^\times} \sum_{t\in F_{q^3}} \psi\big(b^{-1}N(t) - T(t)\big)\chi(b) = q \sum_{b\in F_q^\times} \sum_{x\in F_q^\times} \psi(bx + x^{-1})\chi(b) \tag{54}$$

for all χ characters of $F_q^\times$. To prove (54), change $b \to bN(t)$ on the left and $b \to bx^{-1}$ on the right. Then both sides of (54) are given by products of Gaussian sums. The identity follows from the Hasse-Davenport relation. We have shown (50) in this case.

When $|a| < q^{-2}$, we use the argument in the proof of Theorem 4. Here $|a|$ is small enough so that the stationary arguments used in Theorem 4 work. We get the equation (50) in this case as well. ∎

5.6 Singular Orbits

It is easy to compute $I^s(f_m * \phi_0)$ and $J^s(f'_m)$. Using the argument in Lemma 5, we see $I^s(f_m * \phi_0) = q^{3n}\lambda(\varpi^{-3n})$ if $m = 2n$, and 0 if m is odd. Meanwhile $J^s(f'_m) = 0$ unless $m = 0$ where $J^s(f'_0) = 1$. This implies

$$I^s\Big(\big(f_m - q^3\lambda(\varpi^{-3})f_{m-2}\big) * \phi_0\Big) = J^s\big(f'_{3m} - q\lambda(\varpi^{-1})f'_{3m-2}\big), \quad m \geq 0.$$

5.7 Conclusion

We have proved that when E is an unramified cubic extension of F, if $f \to f'$ is the Hecke algebra homomorphism defined in Proposition 3, then (f, ϕ_0) and f' match.

6 Local Integral: Unramified Case—Split Case

In this section, we assume $E = F \oplus F \oplus F$. Let ϕ_0 be the characteristic function of the lattice $R_F^\times \oplus R_F^4$. We prove that if f' is the image of f under the Hecke algebra homomorphism between $\mathrm{GL}(2, E)$ and $\mathrm{GL}(2, F)$, then the pair (f, ϕ_0) match the function f'.

In this case, f is a linear combination of functions $f_1 \otimes f_2 \otimes f_3$, where $f_1, f_2, f_3 \in \mathcal{H}\big(\mathrm{GL}(2, F)//K_F, \lambda\big)$. The homomorphism between the Hecke algebras is given by $f_1 \otimes f_2 \otimes f_3 \to f_1 *' f_2 *' f_3$ [L], here $*'$ is just the usual convolution. We prove:

Proposition 6 *For all* $f_1, f_2, f_3 \in \mathcal{H}\big(\mathrm{GL}(2, F)//K_F, \lambda\big)$,

$$I(a^{-1}, f_1 \otimes f_2 \otimes f_3 * \phi_0) = J(a, f_1 *' f_2 *' f_3)|a|\lambda(a) \tag{55}$$

and $I^s(f_1 \otimes f_2 \otimes f_3 * \phi_0) = J^s(f_1 *' f_2 *' f_3)$.

Proof Denote by 1_F the unit element of $\mathcal{H}\big(\mathrm{GL}(2, F)//K_F, \lambda\big)$. We use a method in [M-R2].

Lemma 9 *If f is a Hecke function on* $\mathrm{GL}(2, F)$*, we have*

$$\int_{F^\times \backslash \mathrm{GL}(2,F)} f(h^{-1}) 1_F\big(n(x)gh\big)\psi(-x)\, dx\, dh \tag{56}$$

$$= \int_{F^\times \backslash \mathrm{GL}(2,F)} \int f(h^{-1})\sigma_\psi\big(z\iota(g, h)\big)\phi_0(1, 0, -1, -1, -1)\lambda(z)\, d^\times z\, dh. \tag{57}$$

where $\iota(g, h)$ is either $gh \otimes 1 \otimes 1$, or $g \otimes h \otimes 1$ or $g \otimes 1 \otimes h$.

Proof of the Lemma Let $F_1(g)$ and $F_2(g)$ be the expressions (56) and (57) respectively. Clearly

$$F_i\big(n(t)zgk\big) = \psi(t)\lambda^{-1}(z)F_i(g), \quad k \in K_F, i = 1, 2. \tag{58}$$

Thus one only needs to show that $F_1(d_a) = F_2(d_a)$.

Case 1: $\iota(g,h) = gh \otimes 1 \otimes 1$**.** In this case, we only need to show the identity for the case $f(h) = 1_F(h)$. It is clear then $F_1(d_a) = F_2(d_a) = \Phi_0(a)$.

Case 2: $\iota(g,h) = g \otimes h \otimes 1$**.** Let $f = f_m$. Use the decomposition given in (43), we can compute $F_2(d_a)$. Only the coset with $i = m$ in (43) gives a nonzero contribution to $F_2(d_a)$. We get $F_2(d_a)$ equals:

$$q^m \int \sigma_\psi(z)\sigma_\psi(d_a \otimes d_{\varpi^m} \otimes 1)\phi_0(1,0,-1,-1,-1)\lambda(z)\, d^\times z. \tag{59}$$

This expression equals $\lambda(\varpi^{-m})\Phi_{-m}(a)$. On the other hand $F_1(d_a)$ equals:

$$\int_{|az|,|z|,|zx| \le q^m} \psi(x)\lambda(z)\, d^\times z\, dx \tag{60}$$

which equals $\lambda(\varpi^{-m})\Phi_{-m}(a)$. We have proved the lemma in this case.

Case 3: $\iota(g,h) = g \otimes 1 \otimes h$**.** Proved in the same way as in Case 2. ■

With the Lemma, we have $J(a, f_1 *' f_2 *' f_3)$ equals

$$\int f_1(h_1^{-1})f_2(h_2^{-1})f_3(h_3^{-1})\sigma_\psi(z)$$
$$\sigma_\psi\big(wd_a n(y)h_1h_2h_3 \otimes 1 \otimes 1\big)\phi_0(1,0,-1,-1,-1)\psi(-y)\lambda(z)\, d^\times z\, dh_1\, dh_2\, dh_3\, dy.$$

(Here we change ψ in the orbital integral to ψ^{-1}, which does not affect the value of the orbital integral.) Use the equality in Lemma 9 between different $\iota(g,h)$, the above integration becomes:

$$\int \sigma_\psi\big(zwd_a n(y)\big)\, f_1 \otimes f_2 \otimes f_3 * \phi_0(1,0,-1,-1,-1)\psi(-y)\lambda(z)\, d^\times z\, dy. \tag{61}$$

Use the formula for the representation of $\widetilde{\mathrm{GSp}}_4(F)$, the above integral equals

$$\int f_1 \otimes f_2 \otimes f_3 * \phi_0(az^2, az, azy, azt_2, azt_3)$$
$$\cdot\, \psi(ayt_2t_3 - y - t_2 - t_3)|a^2z^3|\lambda(z)\, d^\times z\, dy\, dt_2\, dt_3.$$

Compare with (16), a change of variable $z \to a^{-1}z$ gives the identity (55).

A similar argument works for the singular orbit integral $J^s(f_1 *' f_2 *' f_3)$. We will skip the proof. ■

7 Local Integrals: Unramified Case—Quadratic Case

In this section, we assume $E = F \oplus K$, where K is a quadratic extension of F. Let ϕ_0 be the characteristic function of the lattice $R_F^\times \oplus R_F^2 \oplus R_K$. We prove that if f' is the image of f under the Hecke algebra homomorphism between $\mathrm{GL}(2,E)$ and $\mathrm{GL}(2,F)$, then the pair (f,ϕ_0) match the function f'.

In this case, f is a linear combination of functions $f_1 \otimes f_2$, where $f_1 \in \mathcal{H}\big(\mathrm{GL}(2,F)//K_F,\lambda\big)$ and $f_2 \in \mathcal{H}\big(\mathrm{GL}(2,K)//K_K,\lambda\circ N_{K/F}\big)$. The homomorphism between the Hecke algebras is given by $f_1\otimes f_2 \to f_1 *' \nu(f_2)$, where $f_2 \to \nu(f_2)$ is the Hecke algebra homomorphism corresponding to the quadratic base change [J-Y]. Explicitly, let f''_m be the basis of Hecke algebra of $\mathrm{GL}(2,K)$ as defined in Section 5, the map ν is defined by

$$\nu\big(f''_m - q^2\lambda(\varpi^{-2})f''_{m-2}\big) = f'_{2m} - q\lambda(\varpi^{-1})f'_{2m-2}. \tag{62}$$

We proceed as in Section 6:

Lemma 10 *With above notations,*

$$\int_{F^\times\backslash \mathrm{GL}(2,F)} f_1(h^{-1})1_F\big(n(x)gh\big)\psi(-x)\,dx\,dh \tag{63}$$

$$= \int_{F^\times\backslash \mathrm{GL}(2,F)} f_1(h^{-1})\sigma_\psi(gh\otimes 1)\phi_0(1,0,-1,-1)\lambda(z)\,d^\times z\,dh \tag{64}$$

$$= \int_{K^\times\backslash \mathrm{GL}(2,K)} f_2(h^{-1})\sigma_\psi(g\otimes h)\phi_0(1,0,-1,-1)\lambda(z)\,d^\times z\,dh \tag{65}$$

where $f_1 = \nu(f_2)$.

Proof The first equality follows in the same way as the case 1 of Lemma 9. To show the second equality, we again compute $F_2(d_a)$ where F_2 is the expression (65). When $f_2 = f''_m$, we get $F_2(d_a)$ equal to:

$$q^{2m}\int \sigma_\psi(z)\sigma_\psi(d_a\otimes d_{\varpi^m})\phi_0(1,0,-1,-1)\lambda(z)\,d^\times z$$

which is

$$q^{2m}\int \zeta(z)|z^3a^2\varpi^{4m}|\phi_0(a\varpi^{2m}z^2,0,-z\varpi^{2m},-za\varpi^m)\lambda(z)\,d^\times z. \tag{66}$$

Recall $\zeta(z)$ is the quadratic character on $F^\times$ associated to K. The expression (66) equals $\sum_{j=0}^m(-1)^{m+j}\Phi_{-2j}q^{m-j}\lambda(\varpi^{-m-j})$. Thus if $f_2 = f''_m - q^2\lambda(\varpi^{-2})f''_{m-2}$, we get

$$F_2(d_a) = \Phi_{-2m}\lambda(\varpi^{-2m}) - q\Phi_{2-2m}\lambda(\varpi^{1-2m}).$$

The formula for the expression (63) evaluated at $f_1 = \nu(f_2)$ and $g = d_a$ can be found easily as in Lemma 9. The comparison gives the equality in the Lemma. ∎

From Lemma 10 and the argument in the proof of Proposition 6, we get:

Proposition 7 *Let f_1, f_2 be Hecke functions of* $\mathrm{GL}(2,F)$ *and* $\mathrm{GL}(2,K)$ *respectively, then*

$$I(a^{-1}, f_1 \otimes f_2 * \phi) = \zeta(a) J\big(a, f_1 *' \nu(f_2)\big) |a| \lambda(a) \tag{67}$$

and $I^s(f_1 \otimes f_2 * \phi) = J^s\big(f_1 *' \nu(f_2)\big)$.

8 The Comparison

From Sections 5, 6 and 7, we get (under the assumption at the beginning of Section 5):

Theorem 5 *If f and f' are Hecke functions of* $\mathrm{GL}(2,E)$ *and* $\mathrm{GL}(2,F)$, *and* $f^{\wedge}(s) = f'^{\wedge}(s)$, *then f and (f, ϕ_0) match.*

Back to the global situation. From the local results in Theorems 4, 5, we get

Theorem 6 *Let $f = \bigotimes f_v$, $\phi = \bigotimes \phi_v$ and $f' = \bigotimes f'_v$ be functions smooth of compact support modulo center. If for any v, (f_v, ϕ_v) and f'_v match, then*

$$I(f, \phi) = J(f'). \tag{68}$$

For any f' as above there exists (f, ϕ) that matches f' over all places, and the converse holds also. Moreover, at almost all finite place v, if f'_v is the image of f_v under the Hecke algebra homomorphism, let ϕ_v be the characteristic function of $R_v^{\times} \oplus \Lambda(R_v)$, then (f_v, ϕ_v) and f'_v match.

Proof The first assertion follows from Proposition 1 and (32), the matching condition and the fact $\prod \mu(\delta_{E_v/F_v}, \psi) = 1$, $\prod |a|_v = 1$ and $\prod [2a_v, \delta_{E_v/F_v}] \lambda(a) = 1$ when $a \in F^{\times}$. The other assertions are given by Theorems 4 and 5. ■

From Theorem 6, we can apply the standard arguments (see [M-R]) to prove the Theorems 1, 2 stated in the introduction. We can apply the strong multiplicity one theorem for GL(2). One gets when (f, ϕ) and f' match, for each cuspidal representation π' of $\mathrm{GL}(2,F)$ with central character λ, either

$$J_{\pi'}(f') = I'_{\mu}(f, \phi) \tag{69}$$

for a unique μ with $\mu^2 = \zeta\lambda \circ N$ on $\mathbf{A}_E^1$, or

$$J_{\pi'}(f') = I_{\pi}(f, \phi) \tag{70}$$

for a unique cuspidal representation π. In the first case, E/F is nonnormal, the cuspidal representation π' has a base change that is not cuspidal. Its local components are of the form $\pi\big(I(\xi)\big) \otimes \nu$ for almost all places, thus π' must be of the form $\pi\big(I(\xi)\big) \otimes \nu$. The conclusions in Theorems 1, 2 follow immediately from the equation (70) and the fact that all cuspidal representations of GL(2) have nontrivial Whittaker models.

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