

## HOCHSCHILD COHOMOLOGY OF $\text{II}_1$ FACTORS WITH CARTAN MAXIMAL ABELIAN SUBALGEBRAS

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*Abstract* In this paper we prove that, for a type- $\text{II}_1$  factor  $N$  with a Cartan maximal abelian subalgebra, the Hochschild cohomology groups  $H^n(N, N) = 0$  for all  $n \geq 1$ . This generalizes the result of Sinclair and Smith, who proved this for all  $N$  having a separable predual.

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### 1. Introduction

The study of Hochschild cohomology for von Neumann algebras can be traced back to a well-known theorem, due separately to Kadison [6] and Sakai [10], which states that every derivation  $\delta : M \rightarrow M$  on a von Neumann algebra  $M$  is inner, that is, there exists an element  $a \in M$  such that  $\delta(x) = xa - ax$  for all  $x \in M$ . This corresponds to the vanishing of the first continuous Hochschild cohomology group,  $H^1(N, N)$ . It is natural to conjecture that the higher cohomology groups  $H^n(N, N)$  are also trivial. This programme was taken up in the 1970s, in a series of papers by Johnson, Kadison and Ringrose [5, 7, 8], who affirmed the conjecture for type-I algebras and hyperfinite algebras. In the mid 1980s, a parallel theory of completely bounded cohomology was initiated. The groups  $H_{\text{cb}}^n(N, N)$  are computed under the additional assumption that all cocycles and coboundaries, usually assumed to be norm continuous, are completely bounded. Christensen and Sinclair showed that  $H_{\text{cb}}^n(N, N) = 0$  for all von Neumann algebras  $N$  (see [11] for details of the proof). It was proved in [2] that continuous and completely bounded cohomologies coincide for all von Neumann algebras stable under tensoring with the hyperfinite  $\text{II}_1$  factor  $R$ . Thus, it is known that the continuous Hochschild cohomology vanishes for all von Neumann algebras of type I,  $\text{II}_\infty$  and III, and for all type- $\text{II}_1$  algebras stable under tensoring with  $R$ . This leaves open the case of the general type- $\text{II}_1$  von Neumann algebra. By direct integral techniques, it is enough in the separable case to compute the groups  $H^n(N, N)$  when  $N$  is a factor.

As in the Kadison–Sakai theorem, the vanishing of Hochschild cohomology groups gives structural information about certain bounded  $n$ -linear maps on the von Neumann

algebra, in particular, that they can be formed from the  $(n-1)$ -linear maps. The vanishing of certain other higher cohomology groups also yields some nice perturbation results for von Neumann algebras (see [11, Chapter 7]). The Hochschild cohomology groups have been shown to be trivial for a few large classes of type-II<sub>1</sub> factors, including those with property  $\Gamma$  [3], and those which contain a Cartan maximal abelian subalgebra (masa) and have a separable predual [12]. The purpose of this paper is to extend the latter result to include arbitrary II<sub>1</sub> factors with a Cartan masa. The techniques in [12] depended heavily on separability, and so could not be modified to encompass non-separable algebras. The techniques in the present paper originated in [3, 15], and can be viewed as part of a general strategy to relate properties of non-separable type-II<sub>1</sub> factors to their separable subalgebras.

## 2. Notation and background

### 2.1. Finite von Neumann algebras

The setting of this paper is a finite von Neumann algebra  $N$  with a faithful, normal trace  $\tau$ . We will call such an algebra *separable* if one of the following equivalent conditions holds [16]:

- (i) there exists a countable set of projections in  $N$  generating a weakly dense subalgebra of  $N$ ;
- (ii) the Hilbert space  $L^2(N, \tau)$  (on which  $N$  is faithfully represented) is separable in its norm;
- (iii) the predual of  $N$  is a separable Banach space.

The weakly closed unit ball of  $N$  is then separable in this topology, being a compact metric space.

If  $A \subseteq M \subseteq N$  is an inclusion of von Neumann subalgebras, we denote the *normalizer of  $A$  in  $M$*  by

$$\mathcal{N}(A, M) = \{u \in \mathcal{U}(M) : uAu^* = A\},$$

where  $\mathcal{U}(M)$  denotes the unitary group of  $M$ . We will write  $\mathcal{N}(A)$  for  $\mathcal{N}(A, N)$ , and refer to this set as the *normalizer of  $A$* . Dixmier [4] classified the masas  $A$  of a von Neumann algebra  $M$  as singular, Cartan or semi-regular, according to whether  $\mathcal{N}(A, M)'' = A$ ,  $\mathcal{N}(A, M)'' = M$  or  $\mathcal{N}(A, M)''$  is a proper subfactor of  $M$ . (See [14] for a comprehensive exposition of the theory of masas in von Neumann algebras.)

The following facts about finite von Neumann algebras are standard, but are critical to what follows, so we include some discussion for the reader's convenience. Recall that if  $Q$  is a von Neumann subalgebra of a finite von Neumann algebra  $N$  with trace  $\tau$ , then there exists a bounded conditional expectation  $\mathbb{E}_Q : N \rightarrow Q$  which is characterized by the following two properties:

- (i) for all  $x \in N$  and  $q_1, q_2 \in Q$ ,  $\mathbb{E}_Q(q_1 x q_2) = q_1 \mathbb{E}_Q(x) q_2$  ( $Q$ -bimodularity);
- (ii) for all  $x \in N$ ,  $\tau(\mathbb{E}_Q(x)) = \tau(x)$  ( $\mathbb{E}_Q$  is trace preserving).

We emphasize that  $\mathbb{E}_Q$  is unique among all maps from  $N$  into  $Q$  with these two properties, and not just those assumed to be continuous and linear. This observation leads to the following result, which is essentially in [1], and can be found in [14].

**Proposition 2.1.** *Let  $N$  be a finite von Neumann algebra and let  $Q$  be a von Neumann subalgebra of  $N$ , with faithful, normal trace  $\tau$ . For any  $x \in N$ ,  $\mathbb{E}_{Q' \cap N}(x)$  is the unique element of minimal  $\|\cdot\|_2$ -norm in the weak closure of*

$$K_Q(x) = \text{conv}\{uxu^* : u \in \mathcal{U}(Q)\}.$$

It is important to consider whether the weak closure of the set  $K_Q(x)$  is taken in  $N$  or  $L^2(N)$ . The embedding of  $N$  into  $L^2(N)$  is continuous, when both spaces are given their respective weak topologies. Thus, the (compact) weak closure of any ball in  $N$  is weakly closed (hence also  $\|\cdot\|_2$ -closed, by convexity) in  $L^2(N)$ . Conversely, the preimage of a weakly closed (equivalently,  $\|\cdot\|_2$ -closed) ball in  $L^2(N)$  is weakly closed in  $N$ . Thus, the weak closure in  $N$ , the weak closure in  $L^2(N)$  and the  $\|\cdot\|_2$  closure in  $L^2(N)$  of  $K_Q(x)$  all coincide. We will use the common notation  $K_Q^w(x)$  for all three closures.

**Lemma 2.2.** *Let  $N$  be a finite von Neumann algebra with a normal, faithful trace  $\tau$ . Let  $M$  be a separable subalgebra of  $N$ . Suppose there exists a subset  $S \subseteq N$  of unitaries such that  $S'' = N$ . Then there exists a countable subset  $F$  of  $S$  such that  $F'' \supseteq M$ .*

**Proof.** We first claim that  $C^*(S)$  is  $\|\cdot\|_2$ -dense in  $L^2(N)$ . Let  $x \in N$ . Since  $C^*(S)$  is strongly dense in  $N$ , by the Kaplansky density theorem, there exists a net  $\{x_\alpha\}$  in  $C^*(S)$  such that  $\|x_\alpha\| \leq \|x\|$  and  $x_\alpha$  converges to  $x$   $*$ -strongly. Then also  $(x_\alpha - x)^*(x_\alpha - x)$  converges to 0 weakly. Moreover,  $\|(x_\alpha - x)^*(x_\alpha - x)\|$  is uniformly bounded and  $\tau$  is a normal state (hence weakly continuous on bounded subsets of  $N$  by [9, Theorem 7.1.12]); then  $\|x_\alpha - x\|_2^2 = \tau((x_\alpha - x)^*(x_\alpha - x))$  converges to 0. Thus,  $x \in \overline{C^*(S)}^{\|\cdot\|_2}$ , and the claim follows.

Note that  $L^2(M)$  is a separable Hilbert subspace of  $L^2(N)$ . Let  $\{\xi_n\}_{n=1}^\infty$  be a dense subset. For each  $n$ , there is a sequence  $\{s_{nk}\}_{k=1}^\infty$  in  $C^*(S)$  such that  $s_{nk}$  converges in  $\|\cdot\|_2$ -norm to  $\xi_n$ . The operators  $s_{nk}$  lie in the norm closure of  $\text{Alg}(F)$  for some countable subset  $F$  of  $S$ . It follows that  $L^2(M) \subseteq L^2(\text{Alg}(F)) \subseteq L^2(F'')$ . We claim that this implies  $M \subseteq F''$ . Let  $\mathbb{E}_M : N \rightarrow M$  and  $\mathbb{E}_{F''} : N \rightarrow F''$  denote the respective trace-preserving conditional expectations, obtained by restricting the Hilbert space projections  $e_M : L^2(N) \rightarrow L^2(M)$  and  $e_{F''} : L^2(N) \rightarrow L^2(F'')$  to  $N$ . For any  $x \in M$ , we have

$$x = e_{F''}(x) + (1 - e_{F''})(x) = e_{F''}(x),$$

since  $L^2(M) \subseteq L^2(F'')$ . But then  $x = \mathbb{E}_{F''}(x)$ , so  $M \subseteq F''$ . □

### 2.2. Cohomology

In this section we collect the basic definitions and results of Hochschild cohomology that will be used in the next section. The reader may wish to consult [11] for a more detailed exposition. Let  $M$  be a von Neumann algebra and let  $X$  be a Banach  $M$ -bimodule (in the next section, we will restrict our attention to the case when  $X = M$ ). Let  $\mathcal{L}^n(M, X)$

denote the vector space of  $n$ -linear bounded maps  $\phi : M^n \rightarrow X$ . Define the coboundary map  $\partial : \mathcal{L}^n(M, X) \rightarrow \mathcal{L}^{n+1}(M, X)$  by

$$\begin{aligned} \partial\phi(x_1, \dots, x_{n+1}) &= x_1\phi(x_2, \dots, x_{n+1}) + \sum_{j=1}^n (-1)^j \phi(x_1, \dots, x_{j-1}, x_j x_{j+1}, \dots, x_{n+1}) \\ &\quad + (-1)^{n+1} \phi(x_1, \dots, x_n) x_{n+1}. \end{aligned}$$

An algebraic computation shows that  $\partial^2 = 0$ . We thus obtain the Hochschild complex and define the  $n$ th continuous Hochschild cohomology group to be

$$H^n(M, X) = \frac{\ker \partial}{\operatorname{Im} \partial}.$$

The maps  $\phi \in \ker \partial$  are called  $(n + 1)$ -cocycles, and the maps in  $\operatorname{Im} \partial$  are called  $n$ -coboundaries. It is easy to check that the 2-cocycles are precisely the bounded derivations from  $M$  into  $X$ .

Various extension and averaging results are of central importance in computing cohomology groups. The most basic extension theorem states that normal cohomology and continuous cohomology are equal when the space  $X$  is assumed to be a dual normal  $M$ -module (see [13, Theorem 3.3]). In that result, it is shown that a continuous cocycle can be modified by a coboundary so that the resulting cocycle is separately normal in each variable. Thus, we will assume in what follows that all  $n$ -cocycles are separately normal in each variable. We denote the vector space of bounded, separately normal  $n$ -linear maps from  $M$  to  $X$  by  $\mathcal{L}_w^n(M, X)$ . The general strategy of averaging arguments in cohomology is to replace a continuous  $n$ -cocycle  $\phi$  with a modified cocycle  $\phi + \partial\psi$  which has more desirable continuity and modularity properties. The coboundary  $\partial\psi$  is obtained by an averaging process over a suitable group of unitaries in the underlying von Neumann algebra. In the present work, it will be essential that modifications to a given cocycle are made while preserving certain norm estimates on the original cocycle. The averaging result we need is the following, which can be found in [11].

**Lemma 2.3.** *Let  $R$  be a hyperfinite von Neumann subalgebra of a von Neumann algebra  $M$ . Then there is a bounded linear map  $L_n : \mathcal{L}_w^n(M, M) \rightarrow \mathcal{L}_w^{n-1}(M, M)$  such that  $\phi + \partial L_n \phi$  is a separately normal  $R$ -module map for any  $n$ -cocycle  $\phi$ . Moreover,  $\|L_n\| \leq \ell(n)$ , a constant depending only on  $n$ .*

We will also need two extension results, the first of which is essentially [11, Lemma 3.3.3].

**Lemma 2.4.** *Let  $A$  be a  $C^*$ -algebra on a Hilbert space  $\mathcal{H}$  with weak closure  $\bar{A}$  and let  $X$  be a dual Banach space. If  $\phi : A^n \rightarrow X$  is bounded,  $n$ -linear and separately normal in each variable, then  $\phi$  extends uniquely, without changing the norm, to a bounded, separately normal  $n$ -linear map  $\bar{\phi} : (\bar{A})^n \rightarrow X$ .*

**Lemma 2.5.** *Let  $A$  be a  $C^*$ -algebra, and denote its weak closure by  $\bar{A}$ . Then for each  $n$  there exists a linear map  $V_n : \mathcal{L}_w^n(A, X) \rightarrow \mathcal{L}_w^n(\bar{A}, X)$  such that, when  $\phi$  is as in Lemma 2.4,  $V_n \phi = \bar{\phi}$ . Moreover,  $\|V_n\| \leq 1$  and  $\partial V_n = V_{n+1} \partial$  for all  $n \geq 1$ .*

**Proof.** For  $\phi$  in  $\mathcal{L}_w^n(A, X)$ , we define  $V_n\phi = \bar{\phi}$ . This is well defined and linear by the uniqueness in Lemma 2.4. Moreover, since the extension in Lemma 2.4 is norm preserving, we have  $\|V_n\| \leq 1$ . Now, for any  $x_1, x_2, \dots, x_{n+1} \in A$ , we have

$$\partial V_n\phi(x_1, \dots, x_{n+1}) = \partial\phi(x_1, \dots, x_{n+1}) = V_{n+1}\partial\phi(x_1, \dots, x_{n+1}).$$

Both of the maps  $\partial V_n\phi$  and  $V_{n+1}\partial\phi$  are separately normal, so are uniquely defined on  $(\bar{A})^n$ . Thus, we have

$$\partial V_n\phi(x_1, \dots, x_{n+1}) = V_{n+1}\partial\phi(x_1, \dots, x_{n+1})$$

for all  $x_1, \dots, x_{n+1}$  in  $\bar{A}$ . This completes the proof. □

### 3. Main results

A corollary of the following result is that the study of Cartan masas can, in many instances, be reduced to the separable case. The techniques of the proof come from [15, Theorem 2.5], in which a similar result is proved for singular masas in  $\text{II}_1$  factors.

**Proposition 3.1.** *Let  $N$  be a  $\text{II}_1$  factor with Cartan masa  $A$ , and let  $M_0$  be a separable von Neumann subalgebra of  $N$ . Let  $\phi : N^n \rightarrow N$  be a separately normal  $n$ -cocycle. Then there exists a separable subfactor  $M$  such that  $M_0 \subseteq M \subseteq N$ ,  $M \cap A$  is a Cartan masa in  $M$  and  $\phi$  maps  $M^n$  into  $M$ .*

**Proof.** For a von Neumann algebra  $Q$  and  $x \in Q$ , denote by  $K_Q^n(x)$  and  $K_Q^w(x)$ , respectively, the operator and  $\|\cdot\|_2$ -norm closures of the set  $K_Q(x)$ . Recall that when  $Q$  is a von Neumann subalgebra of  $M$ , for any  $x \in M$ ,  $\mathbb{E}_{Q' \cap M}(x)$  picks out the element of minimal  $\|\cdot\|_2$ -norm in  $K_Q^w(x)$ . We will construct, inductively, a sequence of separable von Neumann algebras

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots \subseteq N$$

and abelian subalgebras  $B_k \subseteq M_k$  so that  $M = (\bigcup_{k=1}^\infty M_k)''$  has the required properties. The von Neumann algebra  $B = (\bigcup_{n=0}^\infty B_n)''$  will be a masa in  $M$ , and thus equal to  $M \cap A$ . The inductive hypothesis is as follows. We assume that, for  $1 \leq j \leq k - 1$  and fixed sequences  $\{y_{j,r}\}_{r=1}^\infty$ ,  $\|\cdot\|_2$ -norm dense in the  $\|\cdot\|_2$ -closed unit ball of the separable von Neumann algebra  $M_j$ , sets  $M_1, \dots, M_k$  and  $B_j = M_j \cap A$  satisfy the following:

- (i)  $\mathbb{E}_A(y_{j,r}) \in B_j \cap K_{B_{j+1}}^w(y_{j,r})$  for  $r \geq 1$ ;
- (ii)  $K_{M_{j+1}}^n(y_{j,r}) \cap \mathbb{C}1$  is non-empty for all  $r \geq 1$ ;
- (iii) for each  $j$ , there is a countable set of unitaries  $\mathcal{U}_{j+1} \subseteq N(A) \cap M_{j+1}$  such that  $\mathcal{U}_{j+1}'' \supseteq M_j$ ;
- (iv) the cocycle  $\phi$  maps  $(M_j)^n$  into  $M_{j+1}$ .

In the inductive step we will construct  $M_{k+1}$  so that the sets  $M_1, \dots, M_{k+1}$  and  $B_1, \dots, B_k, B_{k+1} = M_{k+1} \cap A$  satisfy properties (i)–(iv).

We first prove that this sequence of algebras gives the desired result. The von Neumann algebra  $M = (\bigcup_{n=0}^\infty M_n)''$  will be a separable subalgebra of  $N$ , by our construction. We show that  $M$  is a factor.

For any  $x \in M$ ,  $K_M^w(x) \cap \mathbb{C}1$  is non-empty, by the following approximation argument. First suppose that  $x \in M_k$ , for some  $k \geq 1$  and  $\|x\| \leq 1$ . Let  $\varepsilon > 0$  be given. Choose an element  $y_{k,r} \in M_k$  as above with  $\|y_{k,r} - x\|_2 < \varepsilon$ . By condition (ii) we can choose an element

$$a_{k,r} = \sum_{i=1}^m \lambda_i u_i y_{k,r} u_i^* \in K_{M_{k+1}}(y_{k,r}),$$

whose  $\|\cdot\|_2$ -norm distance to  $\mathbb{C}1$ , which we denote  $\text{dist}_2(a_{k,r}, \mathbb{C}1)$ , is less than  $\varepsilon$ . Then  $a = \sum_{i=1}^m \lambda_i u_i x u_i^*$  is an element of  $K_{M_{k+1}}(x)$  with  $\text{dist}_2(a, \mathbb{C}1) < 2\varepsilon$ . It follows that  $K_{M_{k+1}}^w(x) \cap \mathbb{C}1$  is non-empty.

Now let  $x \in M$ ,  $\|x\| \leq 1$ , and fix  $\varepsilon > 0$ . By the Kaplansky density theorem, there exists a  $k \geq 1$ , and an element  $x_\alpha \in M_k$  of norm at most 1 such that  $\|x_\alpha - x\|_2 < \varepsilon$ . Then by the argument above,  $K_{M_{k+1}}^w(x_\alpha) \cap \mathbb{C}1$  is non-empty. It follows, by a similar argument to that above, that there exists an element  $a \in K_M(x)$  with  $\text{dist}_2(a, \mathbb{C}1) < 2\varepsilon$ . Thus,  $K_M^w(x)$  has non-empty intersection with  $\mathbb{C}1$ . By scaling, this result is true for  $x \in M$  of arbitrary norm. To see that  $M$  is a factor, note that if  $x$  is central in  $M$ , then  $K_M^w(x) = \{x\}$  and, since this set meets  $\mathbb{C}1$ ,  $x$  must be a multiple of the identity.

We now prove that  $B = (\bigcup_{n=0}^\infty B_n)''$  is a masa in  $M$ . First, condition (i) implies that  $\mathbb{E}_A(x) \in B \cap K_B^w(x) \subseteq B \cap K_A^w(x)$  for all  $x \in M$ . This follows from an approximation argument similar to the one above, in which we first prove the claim for all  $x \in M_k$  with  $\|x\| \leq 1$  and then extend to all of  $M$ . Since  $\mathbb{E}_A(x)$  is the element of minimal  $\|\cdot\|_2$ -norm in  $K_A^w(x)$ , it also has this property in  $K_B^w(x)$ . But then  $\mathbb{E}_A(x) = \mathbb{E}_{B' \cap M}(x)$ . Since, for all  $x \in M$ ,  $\mathbb{E}_A(x) = \mathbb{E}_B(x)$ , one has

$$x = \mathbb{E}_{B' \cap M}(x) = \mathbb{E}_A(x) = \mathbb{E}_B(x)$$

for all  $x \in B' \cap M$ . Thus,  $B' \cap M \subseteq B$ . Since  $B$  is abelian, the opposite inclusion also holds. Thus,  $B$  is a masa in  $M$ , and  $B = M \cap A$ . We show that  $B$  is Cartan. By condition (iii), we will have  $M = (\bigcup_{k=0}^\infty \mathcal{U}_{k+1}'')''$ . We claim that the latter set is precisely  $\mathcal{N}(B, M)''$ , and hence that  $B$  is Cartan in  $M$ . Fix  $k \geq 0$  and let  $u \in \mathcal{U}_{k+1}$ . Then, since  $u \in \mathcal{N}(A) \cap M_{k+1}$ , for any  $j \leq k+1$  we have

$$uB_j u^* = u(A \cap M_j)u^* \subseteq A \cap M_{k+1} = B_{k+1} \subseteq B.$$

If  $j > k+1$ , since  $u \in M_{k+1} \subseteq M_j$ , we have

$$uB_j u^* = u(A \cap M_j)u^* = A \cap M_j = B_j \subseteq B.$$

Then  $u(\bigcup_{j=0}^\infty B_j)u^* \subseteq B$  and  $u \in \mathcal{N}(B, M)$ . Then also  $\mathcal{U}_{k+1}' \subseteq \mathcal{N}(B, M)''$ . The claim follows. Then  $M \subseteq \mathcal{N}(B, M)''$  and, since the other containment holds trivially,  $M = \mathcal{N}(B, M)''$ . That is,  $B$  is a Cartan masa in  $M$ . Finally, by our construction of  $M$ ,

condition (iv) will imply that  $\phi$  maps  $M^n$  into  $M$ , since  $\phi$  is separately normal in each variable.

We proceed to construct the algebras  $M_k, B_k$ . Set  $B_0 = M_0 \cap A$ . Assume that  $M_1, \dots, M_k$  and  $B_1, \dots, B_k$  have been constructed such that they satisfy conditions (i)–(iv) specified above. By Dixmier’s approximation theorem [9, Theorem 8.3.5] each sequence  $\{y_{k,r}\}_{r=1}^\infty$  is inside a von Neumann algebra  $Q_0 \subseteq N$  (generated by a countable set of unitaries) such that  $K_{Q_0}^n(y_{k,r}) \cap \mathbb{C}1$  is non-empty. To get condition (iii), observe that, since  $\mathcal{N}(A)'' = N$ , Lemma 2.2 applies and there is a countable set of unitaries  $\mathcal{U}_{k+1}$  satisfying the desired condition. Since  $\phi$  is normal in each variable, by [13, Theorem 4.4],  $\phi$  is jointly  $\|\cdot\|_2$ -norm continuous when restricted to bounded balls in  $(M_k)^n$ . Since  $M_k$  is separable, it follows that  $\phi((M_k)^n)$  generates a separable von Neumann algebra  $Q_1$ . Finally, since for all  $r \in \mathbb{N}$ ,  $\mathbb{E}_A(y_{k,r}) \in K_A^w(y_{k,r}) \cap B_k$ , there exists a set of unitaries  $\{u_m\}_{m=1}^\infty$  generating a von Neumann algebra  $Q_2 \subseteq A$  such that  $\mathbb{E}_A(y_{k,r}) \in K_{Q_2}^w(y_{k,r})$ , for all  $r$ . This will give condition (i). We complete the construction by letting  $M_{k+1}$  be the von Neumann algebra generated by  $Q_0, Q_1, Q_2, \mathcal{U}_{k+1}, M_k$  and  $\mathbb{E}_A(M_k)$ .  $\square$

We are now in a position to compute the cohomology groups  $H^n(N, N)$ , where  $N$  is a general type- $\text{II}_1$  factor with a Cartan subalgebra. Note that, for any such  $N$ , associated to each finite set  $F \subseteq N$  is the separable von Neumann algebra  $M_F \subseteq N$  which it generates. Thus, every such  $\text{II}_1$  factor  $N$  satisfies the hypothesis of Proposition 3.1. This leads to our main theorem.

**Theorem 3.2.** *Let  $N$  be a type- $\text{II}_1$  factor with a Cartan subalgebra  $A$ . Then  $H^n(N, N) = 0$  for all  $n \geq 1$ .*

**Proof.** Since the case when  $n = 1$  is the Kadison–Sakai result, we assume that  $n \geq 2$ . We refer the reader to the proof of the separable case in [12]. Let  $\phi : N^n \rightarrow N$  be a cocycle, which we may assume to be separately normal. By Proposition 3.1, for each finite set  $F \subseteq N$ , there exists a separable subfactor  $N_F$  such that  $M_F \subseteq N_F \subseteq N$ ,  $N_F \cap A$  is Cartan in  $N_F$  and  $\phi$  maps  $(N_F)^n$  into  $N_F$ . Denote the restriction of  $\phi$  to  $N_F$  by  $\phi_F$ . By the separable case,  $\phi_F$  is a coboundary, i.e. there exists an  $(n - 1)$ -linear and bounded map  $\psi : (N_F)^{n-1} \rightarrow N_F$  such that  $\phi_F = \partial\psi_F$ . Moreover, there is a uniform bound on  $\|\psi_F\|$ ; we confine this argument to the end of the proof. Let  $\mathbb{E}_F : N \rightarrow N_F$  be the conditional expectation. Define  $\theta_F : N^{n-1} \rightarrow N$  by  $\theta_F = \psi_F \circ (\mathbb{E}_F)^{n-1}$ . Order the finite subsets of  $M$  by inclusion. Because  $\|\psi_F\|$  is uniformly bounded, so is  $\|\theta_F\|$ . Now, for any  $(n - 1)$ -tuple  $(x_1, \dots, x_{n-1}) \in N^{n-1}$ ,  $\{\theta_F(x_1, \dots, x_{n-1})\}_F$  is a bounded net, as  $F$  ranges over all finite sets containing  $x_1, \dots, x_{n-1}$ . This has an ultraweakly convergent subnet (which we also denote by  $\{\theta_F(x_1, \dots, x_{n-1})\}_F$ ), by ultraweak compactness of bounded subsets of  $N$  [9, Chapter 7]. Define  $\theta : N^{n-1} \rightarrow N$  by

$$\theta(x_1, \dots, x_{n-1}) = \lim_F \theta_F(x_1, \dots, x_{n-1}).$$

Then  $\theta$  is clearly  $(n - 1)$ -linear, and bounded by the uniform bound on  $\|\theta_F\|$ . We claim that  $\phi = \partial\theta$ . Let  $(x_1, \dots, x_n) \in M^n$ . Then

$$\phi(x_1, \dots, x_n) = \phi_F(x_1, \dots, x_n) = \partial\theta_F(x_1, \dots, x_n),$$

for all finite sets  $F$  containing  $x_1, \dots, x_n$ . Ordering these sets by inclusion, we will obtain a subnet  $\{\theta_F\}$  such that  $\partial\theta_F(x_1, \dots, x_n)$  converges weakly to  $\partial\theta(x_1, \dots, x_n)$ . Passing to limits in the above equality gives  $\phi(x_1, \dots, x_n) = \partial\theta(x_1, \dots, x_n)$ . This proves the claim, and the result follows.

It remains to be shown that there is a uniform bound on the norms of the maps  $\psi_F$ , constructed above. It suffices to obtain an estimate for each of these maps in terms of  $n$  and  $\|\phi_F\|$ , since the latter quantity is dominated by  $\|\phi\|$  for all  $F$ . We drop the index  $F$ , since it plays no further role in the proof.

Now  $M$  will denote a separable  $\text{II}_1$  factor with Cartan masa  $A$ . By [12, Theorem 2.2], there is a hyperfinite factor  $R$  such that

$$A \subseteq R \subseteq M \quad \text{and} \quad R' \cap M = \mathbb{C}1.$$

Let  $\phi : M^n \rightarrow M$  be a separately normal cocycle. We will show that  $\phi$  is the image under  $\partial$  of an  $(n-1)$ -linear map whose norm is at most  $K\|\phi\|$ , where  $K$  is a constant depending only on  $n$ . By Lemma 2.3, there exists a map  $L_n$  such that  $\theta = \phi - \partial L_n \phi$  is a separately normal  $R$ -module map, and  $\|\theta\| \leq l(n)\|\phi\|$ , where  $l(n)$  is a constant depending only on  $n$ . Let  $\mathcal{U}$  be a generating set of unitaries for  $M$ . Then the weak closure of  $C^*(\mathcal{U})$  is  $M$ . Let  $\phi_C : C^*(\mathcal{U})^n \rightarrow M$  denote the restriction of  $\phi$ . The proof of the separable case gives  $\theta = \partial\alpha$ , where  $\alpha : C^*(\mathcal{U})^{n-1} \rightarrow M$  is  $(n-1)$ -linear and has norm at most  $\sqrt{2}\|\theta\|$ . Then

$$\phi_C = \theta + \partial\zeta = \partial(\alpha + \zeta),$$

where  $\zeta$  denotes the restriction of  $L_n \phi$  to  $C^*(\mathcal{U})^{n-1}$ . Write  $\psi$  for  $\alpha + \zeta$ . Then, by the method above, there exists a constant  $K(n)$  such that

$$\|\psi\| \leq K(n)\|\phi\|.$$

We now wish to extend the equality  $\phi_C = \partial\psi$  to one involving maps defined on  $M$ , while preserving the latter norm estimate. Applying the map  $V_n$  from Lemma 2.5 to both sides of the above equality, we get

$$V_n \phi_C = V_n(\partial\psi) = \partial(V_{n-1}\psi).$$

Because the separately normal map  $\phi_C$  extends uniquely to  $\phi$  on  $M$ , this yields  $\phi = \partial(V_{n-1}\psi)$ . The map  $V_{n-1}\psi$  is in  $\mathcal{L}_w^{n-1}(M, M)$ , and  $\|V_{n-1}\psi\| \leq K(n)\|\phi\|$ . This gives the required norm estimate, completing the proof of the theorem.  $\square$

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