

## TRIPLE COVERS OF K3 SURFACES

ALICE GARBAGNATI<sup>1</sup> AND MATTEO PENEGINI<sup>2\*</sup>

**Abstract.** We study triple covers of K3 surfaces, following Miranda (1985, *American Journal of Mathematics* 107, 1123–1158). We relate the geometry of the covering surfaces with the properties of both the branch locus and the Tschirnhausen vector bundle. In particular, we classify Galois triple covers computing numerical invariants of the covering surface and of its minimal model. We provide examples of non-Galois triple covers, both in the case in which the Tschirnhausen bundle splits into the sum of two line bundles and in the case in which it is an indecomposable rank 2 vector bundle. We provide a criterion to construct rank 2 vector bundles on a K3 surface  $S$  which determine a non-Galois triple cover of  $S$ . The examples presented are in any admissible Kodaira dimension, and in particular, we provide the constructions of irregular covers of K3 surfaces and of surfaces with geometrical genus equal to 2 whose transcendental Hodge structure splits in the sum of two Hodge structures of K3 type.

### §1. Introduction

The Galois covers of K3 surfaces are a quite classical and interesting argument of research: for example, the K3 surfaces which are Galois covers of other K3 surfaces are classified in [X], and the abelian surfaces which are Galois covers of K3 surfaces are classified in [Fu]. The study of surfaces with higher Kodaira dimension which are covers of K3 surfaces is less systematic, and sporadic examples appear in order to construct specific surfaces (see, e.g., [CD], [L1], [L2], [PZ], [RRS], [Sa]. Nevertheless, a systematic approach to the study of the double covers of K3 surfaces is presented in [G3], where smooth double covers are classified and their birational invariants are given. In the same paper, certain cover surfaces with  $p_g = 2$  are described with more details, since the geometry of these surfaces is quite interesting (see, e.g., [L1], [L2]).

The aim of this paper is to analyze triple covers of K3 surfaces. One of the main differences between covers of degree 2 and the ones of degree 3 is that the latter are not necessarily Galois.

In §2, we present the general theory of the triple covers of surfaces following [Mil], [T]. We consider a smooth surface  $S$  and a triple cover  $f: X \rightarrow S$  which is naturally associated with a rank 2 vector bundle  $\mathcal{E}$ , with the property

$$f_*\mathcal{O}_X = \mathcal{O}_S \oplus \mathcal{E}.$$

The vector bundle  $\mathcal{E}$  is called the Tschirnhausen vector bundle of the cover. There are three possibilities:

---

Received October 8, 2021. Revised February 21, 2022. Accepted April 30, 2022.

2020 Mathematics subject classification: Primary 14E20, 14J28; Secondary 14J27, 14J29.

Alice Garbagnati and Matteo Penegini were partially supported by the Italian National Group for Algebraic and Geometric Structures and their Applications.

© (2022) The Authors. The publishing rights in this article are licensed to Foundation Nagoya Mathematical Journal. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (<https://creativecommons.org/licenses/by/4.0/>), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.



- the cover is Galois (in particular,  $\mathbb{Z}/3\mathbb{Z}$ -cyclic); this happens if  $\mathcal{E}$  splits in the direct sum of two line bundles  $\mathcal{L}$  and  $\mathcal{M}$  which are determined by the nontrivial characters of  $\mathbb{Z}/3\mathbb{Z}$  (see Paragraph 2.12). In this case, the triple cover is totally ramified and the singularities of  $X$  are due only to singularities of the branch locus;
- $\mathcal{E}$  splits into the direct sum of two line bundles  $\mathcal{L}$  and  $\mathcal{M}$ , but the cover is not Galois. In this case, there are components in the branch locus which are of simple, but not total, ramification, and we refer to this situation as a *split nonGalois* triple cover; and
- $\mathcal{E}$  is indecomposable; moreover, in this case, the cover is not Galois and we refer to this case as the *nonsplit* triple cover.

We are interested in calculating the numerical invariants of the covering surface  $X$ :  $p_g(X)$ ,  $q(X)$ ,  $c_1(X)^2$ ,  $c_2(X)$ , and  $\kappa(X)$  (which are, resp., the geometric genus, the irregularity, the square of the first Chern class, the second Chern class, and the Kodaira dimension of the surface  $X$ ). We relate them with the properties of the surface  $S$  and of the bundle  $\mathcal{E}$ . Some of these numbers are not birational invariants; hence, if  $X$  is singular, we need to find the minimal model of  $X$  to determine the numerical invariants for this model. This aspect is highly nontrivial, even if one restricts itself to the Galois triple covers; indeed, it requires a careful analysis not just of the singularities of  $X$ , but also of the configurations of the  $(-1)$ -curves appearing in its minimal resolution.

We restrict to the situation where  $S$  is a K3 surface. We provide a criterion to determine the Kodaira dimension of the cover surface  $X$ , and due to the example constructed in the paper, we prove the following theorem.

**THEOREM 1.1.** *There exist pairs  $(S, f)$  such that  $f : X \rightarrow S$  is a triple cover either Galois, or split non-Galois or nonsplit and with either  $\kappa(X) = 1$  or  $\kappa(X) = 2$ . There exist pairs  $(S, f)$  such that  $f : X \rightarrow S$  is a triple cover either Galois or split non-Galois with  $\kappa(X) = 0$  and  $X$  is necessarily (a possibly singular model of) either a K3 surface or an abelian surface.*

Since  $X$  is a cover of  $S$ , it is not possible that  $\kappa(X) = -\infty$ . We do not know if there exists a nonsplit triple cover  $f : X \rightarrow S$  with  $\kappa(X) = 0$ .

We consider the Galois triple covers of K3 surfaces, obtaining both general results (in §3) and constructing explicitly families of examples (in §4). First, we discuss the singularities of  $X$ , all coming from the singularities in the branch locus of the cover  $f : X \rightarrow S$ . There are two different strategies to resolve the singularities of the triple cover: one can blow up  $S$  in the singularities of the branch locus until one obtains a birational model of  $S$  such that the strict transform of the branch locus is smooth, then one can construct the smooth triple cover of this surface. The surface obtained is birational to  $X$ , and it is called the canonical resolution of  $X$  (cf. [T]). However, one can also consider the possibly singular surface  $X$  and then resolve its singularities obtaining a resolution which is called minimal resolution of  $X$ . Note that neither the canonical resolution nor the minimal resolution is necessarily minimal surface. In §§3.3 and 3.4, we construct both these resolutions if the singularities of the branch locus are ordinary, and we observe that they are *negligible* (see Definition 2.15 and Proposition 2.16), which allows us to compute the numerical invariants not only of  $X$ , see Proposition 3.12. Some other singularities in the branch locus are considered in Theorem 4.12, and they are proved to be negligible too.

Under mild conditions on the smoothness of some components of the branch locus, we are often able to identify all the  $(-1)$ -curves that appear in the resolutions considered and therefore to compute the numerical invariants of the minimal model of  $X$  (see Propositions 4.1, 4.11, and 4.6).

The main result of this part is a systematic classification of the Galois triple covers of  $S$ , which can be summarized in the following theorem.

**THEOREM 1.2.** *Let  $f: X \rightarrow S$  be a normal Galois triple cover of a K3 surface, whose branch locus has  $n \geq 1$  connected components,  $D_1, \dots, D_n$ , and let  $\Lambda_{D_i}$  be the lattice generated by the irreducible components of  $D_i$ :*

- $k(X) = 0$  if and only if all the lattices  $\Lambda_{D_i}$  are negative definite; in this case,  $\Lambda_{D_i} \simeq A_2(-1)$ ,  $n = 6$  or  $n = 9$ , and  $X$  has a trivial canonical bundle.
- $k(X) = 2$  if and only if there exists a lattice  $\Lambda_{D_i}$  whose signature is  $\text{sgn}(\Lambda_{D_i}) = (1, \text{rank}(\Lambda_{D_i}) - 1)$ ; in this case, all the others  $\Lambda_{D_j}$  are isometric to  $A_2(-1)$ .
- $k(X) = 1$  if and only if there are no lattices  $\Lambda_{D_i}$  which are indefinite and there exists at least a lattice  $\Lambda_{D_i}$  which is degenerate. Since  $k(X) = 1$  it admits an elliptic fibration (whose fibre class is  $F_X$ ), there exists an elliptic fibration on  $S$  whose fibre class is  $F_S$ , such that  $f^*(F_S)$  is a multiple of  $F_X$ .

*In particular, if there is a component  $D_1$  in the branch locus such that  $D_1$  is an irreducible curve, then it holds:*

- if  $D_1^2 = 0$ , then  $k(X) = 1$ ; and
- if  $D_1^2 > 0$ , then  $k(X) = 2$ ,  $n \leq 10$ ,  $D_1^2 = 6d$ , for an integer  $d > 0$ , and there exists an integer  $k$  such that  $d = n - 1 + 3k$  and  $k \geq -2$ . If  $D_1$  is smooth and  $X^\circ$  is the minimal model of  $X$ , then

$$\chi(X^\circ) = 5 + n + 5k, \quad K_{X^\circ}^2 = 8n - 8 + 24k, \quad e(X^\circ) = 67 + 5n + 36k.$$

We construct two interesting kinds of examples: in Corollary 4.3, we construct Galois triple covers  $f: X \rightarrow S$  of K3 surfaces  $S$  which have  $p_g(X) = 2$ . Hence, the transcendental Hodge structure of the  $X$  is of type  $(2, \star, 2)$ . Since the pullback of the transcendental Hodge structure of  $S$  is of K3 type, that is, of type  $(1, \star', 1)$ , there is a splitting of the transcendental Hodge structure of  $X$  in the sum of two Hodge structures of K3 type. One of them is of course geometrically associated with a K3 surface, that is, to  $S$ . It would be interesting to find another K3 surface associated with the other Hodge structure of K3 type.

The second example of geometric interest is the construction of irregular triple covers of a regular surface (see §4.10). If the Kodaira dimension of the surface  $X$  is 0, or 1, the construction is well known: there are triple covers of K3 surfaces with abelian surfaces (which are irregular and with Kodaira dimension 0); the base change on an elliptically fibered K3 surface often produces elliptic fibrations (with Kodaira dimension equal to 1) with a nonrational base curve (which forces the surface to be irregular). The situation is more complicated if one requires that  $X$  is a surface of general type: such covers exist, but are not very frequent. (In the case of double covers classified in [G2], there are very few examples). Here, we provide an explicit construction in Theorems 4.16 and 4.17.

In §5, we briefly discuss the case of split non-Galois triple covers of K3 surfaces, and we provide an example in any admissible Kodaira dimension. The construction are based on the study of the Galois closure.

In §6, we consider the most general and complicated case, that is, the case where the vector bundle  $\mathcal{E}$  is indecomposable. In this case, we consider a vector bundle  $\mathcal{E}$  defined by the sequence

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{M} \otimes \mathcal{I}_Z \rightarrow 0,$$

where  $\mathcal{L}$  and  $\mathcal{M}$  are line bundles on  $S$ , and  $Z$  is a nonempty zero-dimensional scheme. Our goal is to list reasonable conditions on  $\mathcal{L}$  and  $\mathcal{M}$  which assure the existence of the vector bundle  $\mathcal{E}$  and of a triple cover  $X \rightarrow S$  whose Tschirnhausen is  $\mathcal{E}$ .

**THEOREM 1.3.** *Let  $\mathcal{L}$  and  $\mathcal{M}$  be two line bundles on a K3 surface  $S$  such that*

$$h^0(S, \mathcal{L}^\vee \otimes \mathcal{M}) = 0, \quad h^1(S, \mathcal{L}^\vee \otimes \mathcal{M}) \geq 1, \quad h^0(S, \mathcal{L}^{\otimes 2} \otimes \mathcal{M}^\vee) \geq 1.$$

*Let  $Z$  be a nonempty zero-dimensional scheme on  $S$ . Then there exists the triple cover  $f: X \rightarrow S$  whose Tschirnhausen bundle is any rank 2 indecomposable vector bundle  $\mathcal{E}$  obtained by a nonsplit extension*

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{M} \otimes \mathcal{I}_Z \rightarrow 0.$$

Thanks to this theorem, the problem of finding a vector bundle  $\mathcal{E}$ , which defines a nonsplit triple cover, is reduced to the problem of finding certain line bundles on  $S$ , with required properties. We apply this theorem to construct a nonsplit triple cover with positive Kodaira dimension, and in particular, we perform all the computations in one case, obtaining a surface of Kodaira dimension 1,  $p_g = 6$ , and  $q = 3$ .

### 1.1 Notation and conventions.

We work over the field  $\mathbb{C}$  of complex numbers.

For  $a, b \in \mathbb{Z}$ ,  $a \equiv_n b$  means  $a \equiv b \pmod{n}$ .

By *surface*, we mean a projective, nonsingular surface  $S$ , and for such a surface,  $\omega_S = \mathcal{O}_S(K_S)$  denotes the canonical class,  $p_g(S) = h^0(S, \omega_S)$  is the *geometric genus*,  $q(S) = h^1(S, \omega_S)$  is the *irregularity*, and  $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$  is the *Euler–Poincaré characteristic*. If  $q(S) > 0$ , we call  $S$  an *irregular surface*. The minimal model of a surface  $S$  is denoted by  $S^\circ$ ; the minimal resolution of a singular surface  $S$  is denoted by  $S'$ .

Throughout the paper, we denote Cartier (or Weil) divisors on a variety by capital letters and the corresponding line bundles by italic letters, so we write, for instance,  $\mathcal{L} = \mathcal{O}_S(L)$ . Moreover, if  $d \in H^0(\mathcal{L})$ , the corresponding Weil divisor is denoted by  $D$ .

Given  $Z$  a purely zero-dimensional subscheme of a variety, we often call  $Z$  a zero-cycle and we denote by  $\ell(Z)$  its length.

For a locally free sheaf  $\mathcal{F}$ , we denote its total Chern class by  $c(\mathcal{F})$  and its Chen Character by  $ch(\mathcal{F})$ .

## §2. Triple covers in algebraic geometry

The case of triple covers of algebraic varieties differs sensibly from the double covers case, above of all, because the cover might be not Galois. Therefore, a different approach is needed. This theory of triple covers in algebraic geometry was started by Miranda in his seminal paper [Mi1], and developed further by Casnati and Ekedahl in [CE] and Tan in [T] (see also [FPV], [Pa2]). The main result of this theory is the following.

**THEOREM 2.1** [Mi1, Th. 1.1]. *A triple cover  $f: X \rightarrow Y$  of an algebraic variety  $Y$  is determined by a rank 2 vector bundle  $\mathcal{E}$  on  $Y$  and by a global section  $\eta \in H^0(Y, S^3\mathcal{E}^\vee \otimes \wedge^2 \mathcal{E})$ , and conversely.*

The vector bundle  $\mathcal{E}$  is called the *Tschirnhausen bundle* of the cover, and it satisfies

$$f_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}. \tag{2.1}$$

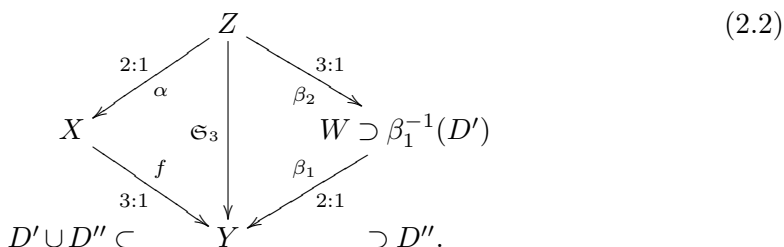
By [CE, Th. 1.5], if  $Y$  is smooth and the section  $\eta \in H^0(Y, S^3\mathcal{E}^\vee \otimes \wedge^2 \mathcal{E})$  is generic, then  $X$  is Gorenstein.

Let  $D$  be a divisor such that  $\mathcal{O}_Y(D) = \wedge^2 \mathcal{E}^{-2}$ .

**PROPOSITION 2.2** [T, Th. 1.3]. *Let  $f: X \rightarrow Y$  be a triple cover and  $Y$  be a normal variety. There exist two divisors  $D'$  and  $D''$  such that  $D = 2D' + D''$  and if  $f$  is totally ramified, then  $D'' = 0$  and  $D'$  is the branch divisor; otherwise,  $D$  is the branch divisor, and  $D'$  is the divisor over which  $f$  is totally ramified.*

**2.3**

We observe that there exists a divisor  $D/2$  such that  $\mathcal{O}_Y(D/2) = \wedge^2 \mathcal{E}^{-1}$ . By the previous proposition,  $D'' = D - 2D'$  is effective, and it is 2-divisible (i.e.,  $D'' = 2(D/2 - D') \in \text{Pic}(Y)$ ). Hence, there exists a double cover of  $Y$  branched on  $D''$ . This double cover is used to get the Galois closure of the triple cover whose Galois group is  $\mathfrak{S}_3$  (see [CP], [T]). We have the following diagram:



We notice that the branch locus of  $\beta_1$  is  $D''$ ;  $\beta_2$  is a Galois triple cover branched along  $\beta_1^{-1}(D')$ ; and the triple cover  $f$  is totally branched on  $D'$  and simply on  $D''$ . Finally, it is worth to notice that this is a special case of dihedral cover studied in [CP, T91, T94].

If  $Y$  is smooth, then  $f$  is smooth over  $Y - D$ ; in other words, all the singularities of  $X$  come from the singularities of the branch locus. More precisely, we have the following proposition.

**PROPOSITION 2.4** [Pa1, Prop. 5.4], [T, Th. 3.2]. *Let  $Y$  be a smooth variety. Let  $y \in Y$ ,  $f^{-1}(y)$  is a singular point of  $X$  if and only if  $y \in \text{Sing}(D)$ , and one of the following conditions holds:*

- (i)  $f$  is not totally ramified over  $y$  or
- (ii)  $f$  is totally ramified over  $y$  and  $\text{mult}_y(D) \geq 3$ .

So—using the notation of Proposition 2.2— $X$  is smooth if and only if:

- (1)  $D'$  is smooth;
- (2)  $D''$  and  $D'$  have no common points; and
- (3)  $D''$  has only cusps as singular points where  $f$  is totally ramified.

We observe that (1) is due to the multiplicity 2 of the divisor  $D'$  in the branch divisor  $D$  and that even if  $D''$  is the divisor where  $f$  is simply branched, it could contain isolated points of total branch.

PROPOSITION 2.5 [T, Th. 4.1]. *Let  $f : X \rightarrow Y$  be a triple cover of a smooth surface  $Y$ , with  $X$  normal. Then there are a finite number of blowups  $\sigma : \tilde{Y} \rightarrow Y$  of  $Y$  and a commutative diagram*

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow{\tilde{\sigma}} & X \\
 \tilde{f} \downarrow & & \downarrow f \\
 \tilde{Y} & \xrightarrow{\sigma} & Y,
 \end{array} \tag{2.3}$$

where  $\tilde{X}$  is the normalization of  $\tilde{Y} \times_Y X$ , such that  $\tilde{f}$  is a triple cover with smooth branch locus. In particular,  $\tilde{X}$  is a resolution of the singularities of  $X$  (in general, this resolution is neither the minimal resolution nor it gives a minimal model of  $X$ ).

Following [T, Para. 4], we call  $\tilde{X}$  the *canonical resolution* of  $X$ . In the case of smooth surfaces, one has the following formulae.

PROPOSITION 2.6 [Mi1, Props. 4.7 and 10.3]. *Let  $f : X \rightarrow Y$  be a triple cover of smooth surfaces with Tschirnhausen bundle  $\mathcal{E}$ . Then:*

- (i)  $h^i(X, \mathcal{O}_X) = h^i(Y, \mathcal{O}_Y) + h^i(Y, \mathcal{E})$ , for all  $i \geq 0$ .
- (ii)  $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) + \chi(\mathcal{E}) = 3\chi(\mathcal{O}_Y) + \frac{1}{2}c_1^2(\mathcal{E}) - \frac{1}{2}c_1(\mathcal{E})K_Y - c_2(\mathcal{E})$ .
- (iii)  $K_X^2 = 3K_Y^2 - 4c_1(\mathcal{E})K_Y + 2c_1^2(\mathcal{E}) - 3c_2(\mathcal{E})$ .
- (iv)  $e(X) = 3e(Y) - 2c_1(\mathcal{E})K_Y + 4c_1^2(\mathcal{E}) - 9c_2(\mathcal{E})$ .

**2.7**

Here, we analyze shortly the relation between the canonical bundle of the covering surface and the base one. Let  $f : X \rightarrow Y$  be a triple cover with Tschirnhausen bundle  $\mathcal{E}$ . We assume that  $Y$  is smooth and  $X$  normal. Following [CE], we observe that, to each cover  $f : X \rightarrow Y$  of degree  $d$ , it is associated an exact sequence

$$0 \rightarrow \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X \rightarrow \mathcal{E}^\vee \rightarrow 0$$

whose dual sequence is

$$0 \rightarrow \mathcal{E} \rightarrow f_*\omega_{X|Y} \rightarrow \mathcal{O}_Y \rightarrow 0,$$

defining the relative dualizing sheaf  $\omega_{X|Y}$ .

If we assume that  $X$  is Gorenstein, by [CE, Th. 2.1], the ramification divisor  $R$  satisfies  $\mathcal{O}_X(R) = \omega_{X|Y}$  where  $R$  is the set of the critical points of the map  $f : X \rightarrow Y$ . Being  $X$  normal, following [R], we define the canonical divisor  $K_X$  of  $X$  as the Weil divisor whose restriction to the smooth locus is the canonical divisor. Since  $X$  is assumed to be Gorenstein,  $K_X$  is a Cartier divisor as well.

The restriction of  $f$  to  $X_0$ , the smooth locus of  $X$ , is a triple cover  $f_0 : X_0 \rightarrow f(X_0)$ . By the Hurwitz ramification formula,

$$K_{X_0} = f_0^*(K_{f(X_0)}) + (\omega_{X|Y})|_{X_0}.$$

Indeed, by the definition of the canonical divisor, we obtain the following equality:

$$K_X = f^*(K_Y) + R. \quad (2.4)$$

In the sequel, we shall be particularly interested in the case when  $Y$  is a K3 surface—or, in general, when  $Y$  is a surface with a trivial canonical bundle. Then (2.4) simplifies to

$$K_X = f^*(K_Y) + R = R. \quad (2.5)$$

In addition, by (2.1) and by duality for finite flat morphisms, we obtain

$$f_*\mathcal{O}_X(K_X) \cong (f_*\mathcal{O}_X)^\vee \otimes \mathcal{O}_Y(K_Y) \cong \mathcal{O}_Y(K_Y) \oplus (\mathcal{E}^\vee \otimes \mathcal{O}_Y(K_Y)),$$

which yields at once

$$h^0(\mathcal{O}_X(K_X)) \geq h^0(\mathcal{O}_Y(K_Y)). \quad (2.6)$$

The formula (2.5) is particularly useful to determine the Kodaira dimension of the covering surface  $X$ ; indeed, it holds the following proposition.

**PROPOSITION 2.8.** *Let  $f: X \rightarrow Y$  be a triple cover with  $Y$  smooth and  $X$  Gorenstein, and let  $R$  be the ramification divisor. Write  $|R| = |M| + F$  with  $|M|$  the moving part and  $F$  the fixed part of  $R$ . Suppose that  $\mathcal{O}(K_Y) \cong \mathcal{O}_Y$ , then the Kodaira dimension  $\kappa(X)$  of  $X$  is greater than or equal to 0. Moreover, it holds:*

$\kappa(\mathbf{X}) = \mathbf{0}$  if and only if  $|M| = \emptyset$  and either  $F$  is supported on rational curves or  $F = \emptyset$ .

Moreover,  $X$  cannot be neither an Enriques surface nor a bielliptic one.

$\kappa(\mathbf{X}) = \mathbf{1}$  if and only if  $|M| \neq \emptyset$  and the general member of  $|M|$  is supported on elliptic curves.

$\kappa(\mathbf{X}) = \mathbf{2}$  if and only if  $|M| \neq \emptyset$  and the general member of  $|M|$  is supported on curves of genus  $g \geq 2$ .

*Proof.* Suppose, first, that  $f$  is étale of degree  $d$ , then  $\kappa(X) = d \cdot \kappa(Y) = 0$ , and hence we can assume that  $f$  is ramified.

Being  $f$  a triple cover of a surface  $Y$  with trivial canonical bundle by (2.6), we have  $h^0(\mathcal{O}(K_X)) \geq 1$ , and hence the Kodaira dimension of  $X$  satisfies  $\kappa(X) \geq 0$ . Moreover, if  $\kappa(X) = 0$ , then  $X$  cannot be neither an Enriques surface nor a bielliptic one.

By the Hurwitz formula (2.5), the canonical divisor is  $K_X = R$ . If the divisor  $R$  is supported at least on a moving positive genus curve, then (2.5) implies that  $K_X$  is nontrivial, and thus  $\kappa(X) > 0$ . Moreover,  $X$  is a properly elliptic surface if and only if its canonical bundle is supported only on elliptic curves (see [Mi2, §III]), and thus, again by (2.5), we have the second case of the proposition, whereas if the general member of  $|R|$  is supported on a curve of genus  $g \geq 2$ , then  $X$  is of general type.

Finally, if  $R$  does not move in a linear system and it is supported only on rational curves, then the minimal model of  $X$  must have trivial canonical bundle and so we get the case (1).

Conversely, if  $|M| = \emptyset$ ,  $\kappa(X)$  cannot be 1 or 2, and hence  $\kappa(X) = 0$ . If  $|M| \neq \emptyset$  and the general member of  $|M|$  is supported on curves of genus  $g \geq 2$ ,  $\kappa(X)$  cannot be 0 or 1, and hence it is 2.

If  $|M| \neq \emptyset$  and the general member of  $|M|$  is supported on elliptic curves, by adjunction,  $M \cdot M = 0$  and so  $X$  would have a genus 1 fibration, which is impossible for surfaces of general type. Since  $|M| \neq \emptyset$ ,  $\kappa(X) \neq 0$ .  $\square$

Similar results work in case  $X$  is normal, but not necessarily Gorenstein. The main problem here is that the canonical divisor is not Cartier, so we have to consider the canonical resolution of  $X$ .

**PROPOSITION 2.9.** *The results of Proposition 2.8 hold even if  $X$  is normal (not necessarily Gorenstein).*

*Proof.* If  $X$  is normal, we consider the canonical resolution  $\tilde{X}$ , as in Proposition 2.5, since  $\kappa(X) = \kappa(\tilde{X})$ . The map  $\sigma : \tilde{Y} \rightarrow Y$  is a blowup and introduces an exceptional divisor  $E$ , which does not move in a linear system. The canonical bundle of  $\tilde{Y}$  is  $K_{\tilde{Y}} = K_Y + E = E$ . We apply the Hurwitz ramification formula to the triple cover  $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ :

$$K_{\tilde{X}} = \tilde{f}^*E + R_{\tilde{f}},$$

where  $R_{\tilde{f}}$  is the ramification divisor of  $\tilde{f}$ .

Let analyze both the summands:  $\tilde{f}^*E$  has negative self-intersection and is the exceptional locus of  $\tilde{\sigma}$  for the commutativity of the diagram in particular  $\tilde{\sigma}^*(B)\tilde{f}^*E = 0$  for all  $B \in \text{Pic}(X)$ . Hence, its movable part is trivial; otherwise, there would be at least one curve in  $|\tilde{f}^*E|$  which is not contracted by  $\tilde{\sigma}$ , say  $\tilde{D}$ , so  $\tilde{D}\tilde{f}^*E > 0$ . Denoted by  $D$  the image of this curve,  $\tilde{\sigma}^*(D) \sim \tilde{D} + k\tilde{f}^*E \sim (k+1)\tilde{f}^*E$ , for a nonnegative  $k$ . However, this contradicts  $\sigma^*(D)\tilde{f}^*E = 0$ .

We observe that  $R_{\tilde{f}}$  are curves contained in the linear system of

$$\tilde{\sigma}^*R = \tilde{\sigma}^*(M + F) = \tilde{M} + \tilde{F} + k\tilde{f}^*E,$$

where  $\tilde{M}$  and  $\tilde{F}$  are the strict transforms of  $M$  and  $F$ , respectively, using the notation of Proposition 2.8. Since  $\tilde{f}^*E$  has no movable part, the movable part of  $R_{\tilde{f}}$  is contained in  $\tilde{M}$ . Since  $\sigma$  is a blowup,  $|M|$  is isomorphic to  $|\tilde{M}|$ .

To conclude, it suffices to apply Proposition 2.8 to the cover  $\tilde{f}$ . □

**2.10**

The situation becomes a little bit easier if we consider only the Galois case. Indeed, by [T, Th. 2.1], a triple cover is Galois if and only if it is totally ramified over its branch locus. Moreover, in this instance, the branch locus is exactly  $D' = D/2 = B + C$  [T, Th. 1.3(2)]. In this case,  $X$  is smooth if and only if  $D$  is smooth [T, Th. 3.2]. A Galois triple cover  $f : X \rightarrow Y$  (with  $Y$  smooth) is first of all a cyclic  $\mathbb{Z}/3\mathbb{Z}$ -cover, and hence it can be treated as a cyclic cover. Moreover, it is determined by two curves  $B$  and  $C$  in  $Y$  and by two divisors  $L$  and  $M$  on  $Y$  such that  $B \in |2L - M|$  and  $C \in |2M - L|$ . As already remarked, the branch locus of  $f$  is  $B + C$  and  $3L \equiv 2B + C, 3M \equiv B + 2C$ . Thus, the class of the branch locus is  $B + C = L + M$ . The surface  $X$  is normal if and only if  $B + C$  is reduced. Otherwise, it is possible to consider the normalization, which is associated with another triple cover as explained in [Mi1, Prop. 7.5]. The singularities of  $X$  lie over the singularities of  $D' = B + C$  (see Proposition 2.4).

If  $B + C$  is smooth, we have

$$\chi(\mathcal{O}_X) = 3\chi(\mathcal{O}_Y) + \frac{1}{2}(L^2 + K_Y L) + \frac{1}{2}(M^2 + K_Y M), \tag{2.7}$$

$$K_X^2 = 3K_Y^2 + 4(L^2 + K_Y L) + 4(M^2 + K_Y M) - 4LM, \tag{2.8}$$



$$q(X) = q(Y) + h^1(Y, \mathcal{O}_Y(K_Y + L)) + h^1(Y, \mathcal{O}_Y(K_Y + M)), \tag{2.9}$$

$$p_g(X) = p_g(Y) + h^0(Y, \mathcal{O}_Y(K_Y + L)) + h^0(Y, \mathcal{O}_Y(K_Y + M)). \tag{2.10}$$

Of course, one can apply Theorem 2.1 to the Galois case. For this, let  $\xi$  be a primitive cube root of unity, generating  $\mathbb{Z}/3\mathbb{Z}$ , and we obtain the following proposition.

PROPOSITION 2.11 [Mi1, Prop. 7.1], [T, Th. 1.3]. *If  $f: X \rightarrow Y$  is a Galois triple cover, then:*

- (i) *The sheaf  $f_*\mathcal{O}_X$  splits into eigenspaces as  $\mathcal{O}_Y \oplus \mathcal{L}^{-1} \oplus \mathcal{M}^{-1}$  where  $\mathcal{O}_Y, \mathcal{L}^{-1}$ , and  $\mathcal{M}^{-1}$  are the eigenspaces for 1,  $\xi$ , and  $\xi^2$ , respectively.*
- (ii) *The Tschirnhausen bundle  $\mathcal{E}$  for  $f$  is the sum of eigenspaces  $\mathcal{L}^{-1} \oplus \mathcal{M}^{-1}$ .*
- (iii) *The branch locus of  $f$  is the divisor  $D'$  such that  $\mathcal{O}(D') = \mathcal{L} \otimes \mathcal{M}$ .*

*Proof.* (i) and (ii) are contained in [Mi1, Prop. 7.1]. Recall that, by [T, Th. 2.1], a triple cover is Galois if and only if it is totally ramified over its branch locus. Moreover, in this case, the branch locus is exactly  $D' = D/2 = B + C$  [T, Th. 1.3(2)]. □

**2.12**

Being Galois triple cover cyclic, one can compare the previous result with the standard theory of cyclic covers (see, e.g., [BHPV, Chap. I.17]). Both the line bundles  $\mathcal{M}$  of Proposition 2.11 and  $\mathcal{L}^2$  correspond to the same eigenspace of  $f_*\mathcal{O}_X$ , the one relative to the eigenvalue  $\xi^2$ . Nevertheless, they can differ in the Picard group by a 3-torsion element and an integer multiple of divisors supported on the codimension 1 subvarieties in the branch locus (see also [T, §1.4]). The similar situation must occur for  $\mathcal{M}^2$  and  $\mathcal{L}$ .

Vice versa, given a Tschirnhausen bundle  $\mathcal{E}$  and a section  $\eta \in H^0(Y, S^3\mathcal{E}^\vee \otimes \wedge^2 \mathcal{E})$ , it determines a triple cover, and to see if it is Galois, one has to check that  $\mathcal{O}_Y \oplus \mathcal{E}$  is a representation of the group  $\mathbb{Z}/3\mathbb{Z}$ . In particular,  $\mathcal{E}$  must split according to the two characters  $\xi$  and  $\xi^2$ . Therefore,  $\mathcal{E} = \mathcal{L}^{-1} \oplus \mathcal{M}^{-1}$  where  $\mathcal{L}^2$  differs from  $\mathcal{M}$  by 3-torsion element and integer multiple of divisors supported on the codimension 1 subvarieties in the branch locus and  $\mathcal{M}^2$  differs from  $\mathcal{L}$  by 3-torsion element and integer multiple of divisors supported on the codimension 1 subvarieties in the branch locus. The choice of  $\eta$  determines uniquely the cover.

**2.13**

Finally, we call a *Galois triple cover data* over  $Y$  a pair of line bundles  $\mathcal{L}$  and  $\mathcal{M}$  on  $Y$  and two sections

$$b \in H^0(\mathcal{L}^2 \otimes \mathcal{M}^{-1}), \quad c \in H^0(\mathcal{L}^{-1} \otimes \mathcal{M}^2).$$

To give a Galois triple cover data, we use alternatively the quadruple  $(b, c, \mathcal{L}, \mathcal{M})$  or  $(B, C, \mathcal{L}, \mathcal{M})$  or even  $(B, C, L, M)$  with their clear meaning, that is, the different cases of the letter give a different incarnation of the objects treated, once being a section, once a divisor, and once a sheaf.

**2.14**

We assume that  $X$  is normal and denote by  $X'$  the *minimal resolution* of the singularities of  $X$ . We observe that in general  $X'$  coincides neither with the minimal model  $X^\circ$  nor with the canonical resolution  $\tilde{X}$ . One cannot apply neither the formulae of Proposition 2.6

nor (2.7)–(2.10) if  $X$  is singular and in particular for the last ones if the branch locus is singular. We would like to define a class of singularities of  $f : X \rightarrow Y$  such that the formulae of Proposition 2.6 give the invariants of  $X'$  (instead of the one of  $X$ ).

We follow [PP, Def. 1.5].

**DEFINITION 2.15.** Let  $f : X \rightarrow Y$  be a triple cover of a smooth algebraic surface  $Y$ , with Tschirnhausen bundle  $\mathcal{E}$ . We say that  $X$  has only *negligible* (or *nonessential*) singularities if the invariants of the minimal resolution  $X'$  are given by the formulae in Proposition 2.6. We also call *negligible* singularities the corresponding singularities of the branch locus.

**PROPOSITION 2.16** [PP, Exams. 1.6 and 1.8]. *Let  $f : X \rightarrow Y$  be a triple cover of a smooth algebraic surface  $Y$ , with Tschirnhausen bundle  $\mathcal{E}$ . If the singularities of  $X$  are only of type  $\frac{1}{3}(1,1)$  and  $\frac{1}{3}(1,2)$ , then  $X$  has only negligible singularities.*

### §3. Triple covers of K3 surfaces: The Galois Case

From this section onward, we always consider as the base of a triple cover a K3 surface  $S$ . Unless otherwise stated,  $f : X \rightarrow S$  is a triple cover such that  $X$  is a normal connected surface.

#### 3.1 Galois triple cover of K3 surface

Let  $S$  be a K3 surface, and let  $f : X \rightarrow S$  be a Galois triple cover of  $S$  with branch locus  $\coprod_{i=1}^n D_i$  where  $D_i$  are (possibly singular and reducible) curves and thus the  $D_i$ 's are the  $n$  connected components of the branch locus. Requiring that  $X$  is normal implies that the  $D_i$ 's are reduced. Up to reordering the components, we can always assume that  $D_1^2 \geq D_i^2$  for every  $i = 1, \dots, n$ . Since  $B + C$  is the branch locus (with the same notation of Paragraph 2.10), there exist curves  $B_i$  and  $C_i$  (not necessarily connected) such that  $D_i = B_i + C_i$  and  $B = \sum_i B_i, C = \sum_i C_i$ .

**LEMMA 3.1.** *Let  $(B, C, L, M)$  be the Galois triple cover data of a Galois triple cover  $X \rightarrow S$ . Let  $B = \sum_{i=1}^n B_i, C = \sum_{i=1}^n C_i$ , and  $D_i = B_i + C_i$ . Then:*

- $B_i B_j = C_i C_j = B_i C_j = 0$ , if  $i \neq j$ ; and
- $B_i^2 \equiv_3 B_i C_i \equiv_3 C_i^2 \equiv_3 D_i^2$ .

*Proof.* Since  $B_i$  (resp.,  $C_i$ ) are contained in a connected component of the branch locus and  $B_j$  (resp.,  $C_j$ ) in another one, we have  $B_i B_j = C_i C_j = B_i C_j = 0$ , if  $i \neq j$ . Moreover, the last part of the statement follows directly by the conditions (not all independent):  $B_i L = B_i(2B + C)/3 = (2B_i^2 + B_i C_i)/3 \in \mathbb{Z}$ . Analogously,  $C_i L = C_i(2B + C)/3 \in \mathbb{Z}, L^2 \in 2\mathbb{Z}, B_i M = B_i(B + 2C)/3 \in \mathbb{Z}, C_i M = C_i(B + 2C)/3 \in \mathbb{Z}$ , and  $M^2 \in 2\mathbb{Z}$ . □

**PROPOSITION 3.2.** *If  $f : X \rightarrow S$  is a smooth Galois triple cover, then the branch locus is smooth. In particular:*

- all the  $D_i$ 's are irreducible smooth curves with positive genus;
- if moreover  $D_1^2 > 0$ , then  $n = 1$  (i.e., the branch locus is  $D_1$ ) and there exists a divisor  $H \in \text{Pic}(S)$  such that  $D = 3H$ ; and
- if at least one  $D_i$  has genus 1, that is,  $D_i^2 = 0$ , then all the  $D_i$ 's are smooth curves of genus 1. In particular,  $\varphi_{|D_1|} : S \rightarrow \mathbb{P}^1$  is an elliptic fibration, all the  $D_i$ 's are smooth fibers and  $f : X \rightarrow S$  is obtained by a base change of order 3 branched on the  $n$  smooth fibers  $D_i$ 's.

*Proof.* The triple cover  $f : X \rightarrow S$  is smooth if and only if the branch locus is smooth (see Paragraph 2.10), and thus each  $D_i$  is a smooth irreducible curve. In particular, either  $D_i = B_i$  or  $D_i = C_i$ . In both the cases, one obtains  $D_i^2 \equiv_3 0$ , by Lemma 3.1. If  $D_i$  is a smooth irreducible rational curve,  $D_i^2 = -2 \not\equiv_3 0$ , which is not admissible. So, for every  $D_i$ , we have  $g(D_i) \geq 0$ . If  $D_1^2 > 0$ , then, by the Hodge index theorem,  $D_i^2 < 0$  for each  $i > 1$ , which implies that  $D_i$  are rational curves for each  $i > 1$ , contradicting the first assertion. Therefore, if  $D_1^2 > 0$ , there are no other components in the branch locus. This implies that  $B = D_1$  and  $C = 0$  (or vice versa). So  $L = B/3 \in Pic(S)$ , that is, there exists a divisor  $H$  such that  $3H \equiv D$ .

If  $D_1^2 = 0$  and it is a smooth irreducible curve, then it is a genus 1 curve on  $S$  and  $\varphi_{|D_1|} : S \rightarrow \mathbb{P}^1$  is a genus 1 fibration. In particular, any  $D_j$  orthogonal to  $D_i$  is contained in a fiber of  $\varphi_{|D_1|}$ , and thus it can be either a rational curve (component of a reducible fiber) or a genus 1 curve. Since there are no rational curves contained in the branch locus, we conclude that all the  $D_i$ 's are smooth fibers of the same fibration and that the triple cover  $X \rightarrow S$  is branched over smooth fibers of the fibration  $S \rightarrow \mathbb{P}^1$ . This induces a genus 1 fibration  $X \rightarrow C$ , where  $C$  is a smooth curve, such that there is a 3 : 1 map  $g : C \rightarrow \mathbb{P}^1$  and  $X \simeq S \times_g \mathbb{P}^1$ . □

Let us now consider the case of a singular branch locus: in order to construct  $X$ , we first consider its canonical desingularization  $\tilde{X}$ . So we blow up  $S$  in such a way that the strict transform of the branch locus becomes smooth and then we take the triple cover (see Proposition 2.5). The surface  $X$  is a contraction of  $\tilde{X}$ . This gives information both on the singularities of  $X$  and on the construction of a smooth model of it. So, now, we perform a local analysis near the singularities of the branch locus. To simplify the treatment, we work locally around a singular point  $P$  and we assume it is the unique singularity of the branch locus.

### 3.3 Singularities of the branch locus of type 1.

Let  $f : X \rightarrow S$  be a triple cover. Let  $W$  and  $V$  be two curves on  $S$ , meeting transversally in the point  $P$ . Let  $W$  and  $V$  be contained in the branch locus, and assume that  $P$  is an ordinary double point of the branch. Let us assume that locally the equation of the branch locus of the triple cover near to  $P = (0,0)$  is  $xy$ . We blow up  $P$  obtaining  $\beta_1 : S_1 \rightarrow S$  which introduces an exceptional divisor  $E_P$ . We denote by  $W_1$  and  $V_1$  the strict transforms of  $W$  and  $V$  with respect to  $\beta_1$ . The divisor  $E_P$  appears with multiplicity 2, so it is still contained in the branch locus of the triple cover and it intersects  $W_1$  and  $V_1$  in two points  $R$  and  $Q$ . We further blow up  $R$  and  $Q$ , obtaining the map  $\beta_2 : S_2 \rightarrow S_1$ , the exceptional divisors  $E_R$  and  $E_Q$ , and the strict transforms  $W_2, V_2$ , and  $\tilde{E}_P$  of  $W_1, V_1$ , and  $E_P$ , respectively. Denoted by  $w = W^2$  and  $v = V^2$ , the intersection properties on  $S_2$  are the following:  $E_R^2 = E_Q^2 = -1$ ,  $W_2^2 = w - 2$ ,  $V_2^2 = v - 2$ ,  $\tilde{E}_P^2 = -3$ , and  $E_R W_2 = E_Q V_2 = E_R \tilde{E}_P = E_Q \tilde{E}_P = 1$ , and the other intersections are trivial. The triple cover  $f_2 : X_2 \rightarrow S_2$ , induced by the one of  $S$ , is branched on  $W_2, V_2$ , and  $\tilde{E}_P$ , so the branch locus is smooth and  $X_2$  is the canonical resolution of  $f : X \rightarrow S$ . The self-intersections of the inverse image of some curves on  $S_2$  are the following:

$$\begin{aligned} (f_2^{-1}(E_R))^2 &= (f_2^{-1}(E_Q))^2 = -3, & (f_2^{-1}(W_2))^2 &= (W^2 - 2)/3, \\ (f_2^{-1}(V_2))^2 &= (V^2 - 2)/3, & (f_2^{-1}(\tilde{E}_P))^2 &= -1. \end{aligned}$$

To get  $X$ , we contract  $f_2^{-1}(E_R)$ ,  $f_2^{-1}(E_Q)$ , and  $f_2^{-1}(\widetilde{E}_P)$ . Since  $(f_2^{-1}(\widetilde{E}_P))^2 = -1$ , the contraction  $\gamma_2 : X_2 \rightarrow X_1$  gives the minimal resolution  $X' = X_1$  of  $X$ . Moreover,  $X_2$  coincides with the canonical resolution  $\widetilde{X}$ . Then one contracts the curves  $\gamma_2(f_2^{-1}(E_R))$  and  $\gamma_2(f_2^{-1}(E_Q))$  which are  $-2$  curves meeting in a point (i.e., a Hirzebruch–Jung string of type  $\frac{1}{3}(1, 2)$ ). The contraction  $\gamma_1 : X_1 \rightarrow X$  of these two curves produces the surface  $X$  (triple cover of  $S$ ). The image of these two curves under  $\gamma_1$  is the point  $f^{-1}(P)$ , and thus  $f^{-1}(P)$  is a singular point on  $X$  of type  $A_2$ , that is, of type  $\frac{1}{3}(1, 2)$ . We have the following diagram (see also Figure 1):

$$\begin{array}{ccccc} X & \xleftarrow{\gamma_1} & X_1 & \xleftarrow{\gamma_2} & X_2 \\ f \downarrow & & & & f_2 \downarrow \\ S & \xleftarrow{\beta_1} & S_1 & \xleftarrow{\beta_2} & S_2. \end{array}$$

**3.4 Singularities of the branch locus of type 2.**

Let  $f : X \rightarrow S$  be a triple cover. Let  $W$  and  $V$  be two curves on  $S$ , meeting transversally in the point  $P$ . Let  $W$  and  $V$  be contained in the branch locus, and assume that  $P$  is an ordinary double point of the branch. Let us assume that locally the equation of the branch locus of the triple cover near to  $P = (0, 0)$  is  $xy^2$ . We blow up  $P$  obtaining  $\beta_1 : S_1 \rightarrow S$  which introduces an exceptional divisor  $E_P$ . We denote by  $W_1$  and  $V_1$  the strict transforms of  $W$  and  $V$  with respect to  $\beta_1$ . The divisor  $E_P$  appears with multiplicity 3, so it is not contained in the branch locus of the triple cover. The triple cover  $f_1 : X_1 \rightarrow S_1$ , induced by the one of  $S$ , is branched on  $W_1 \cup V_1$  and since the branch locus is smooth,  $X_1$  is the canonical resolution  $\widetilde{X}$  of  $X$ . The intersection properties on  $S_1$  are the following:  $W_1^2 = W^2 - 1$ ,  $V_1^2 = V^2 - 1$ ,  $E_P^2 = -1$ , and  $W_1 E_P = V_1 E_P = 1$ , and the other intersections are trivial. The self-intersections of the inverse image of some curves on  $S_1$  are the following:

$$(f_1^{-1}(E_P))^2 = -3, (f_2^{-1}(W_1))^2 = (W^2 - 1)/3, (f_2^{-1}(V_1))^2 = (V^2 - 1)/3.$$

The contraction  $\gamma_1 : X_1 \rightarrow X$  of  $E_P$  produces the surface  $X$  (triple cover of  $S$ ). The image of this curve under  $\gamma_1$  is the point  $f^{-1}(P)$ , and thus  $f^{-1}(P)$  is a singular point on  $X$  of type  $\frac{1}{3}(1, 1)$ . In particular,  $X_1$  is also the minimal resolution  $X'$  of  $X$ , which coincides in this case with  $\widetilde{X}$ . We have the following diagram:

$$\begin{array}{ccc} X & \xleftarrow{\gamma_1} & X_1 \\ f \downarrow & & f_1 \downarrow \\ S & \xleftarrow{\beta_1} & S_1. \end{array}$$

We can summarize the above discussion with the following proposition.

PROPOSITION 3.5. *Let  $f : X \rightarrow S$  be a Galois triple cover. Let  $P$  be a singular point of the branch locus.*

- *If the local equation of the branch locus near  $P$  is  $xy$ , then  $f^{-1}(P)$  is a singularity of  $X$  type  $\frac{1}{3}(1, 2)$ .*
- *If the local equation of the branch locus near  $P$  is  $xy^2$ , then  $f^{-1}(P)$  is a singularity of  $X$  type  $\frac{1}{3}(1, 1)$ .*

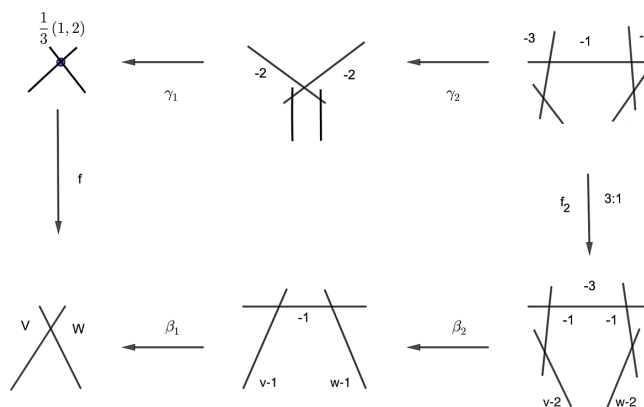


Figure 1.

In both the previous cases,  $f^{-1}(P)$  is a negligible singularity by Proposition 2.16.

REMARK 3.6. Let  $f : X \rightarrow S$  be a triple cover with Galois cover data  $(B, C, L, M)$ .

Let  $V$  and  $W$  be two irreducible components of the branch locus, meeting in a point  $P$ . If  $P$  is a singularity of type 1, then  $V$  and  $W$  are both components either of  $B$  or of  $C$ . If  $P$  is a point of type 2, then, without loss of generality,  $V$  is a component of  $B$  and  $W$  is a component of  $C$ .

We observe that the ordinary nodes of a curve (in the branch locus) are singularities of type 1.

By considering the contraction  $\gamma_1$  described in Paragraphs 3.3 and 3.4, one obtains the following.

COROLLARY 3.7. Let  $X \rightarrow S$  be a Galois triple cover such that the branch locus contains  $c$  singularities of type 1 and  $b$  singularities of type 2 and no other singular points. The Picard number of the minimal resolution  $X'$  of  $X$  is  $\rho(X') = \rho(X) + 2c + b$ , and the one of the canonical resolution  $\tilde{X}$  is  $\rho(\tilde{X}) = \rho(X) + 3c + b$  (see also Figure 1).

In the following lemma and examples, we show that singularities, both of type 1 and of type 2, appear as the branch locus of triple covers of K3 surfaces.

LEMMA 3.8. Let  $N_1$  and  $N_2$  be two rational curves contained in the branch locus of a Galois triple cover  $X \rightarrow S$  such that any other curves contained in the branch locus are disjoint from them. Let us assume that  $N_1 N_2 = k$ . Then:

- (i)  $k \not\equiv_3 0$ ;
- (ii) if  $k \equiv_3 1$ , the points of intersection of  $N_1$  and  $N_2$  are of type 2; and
- (iii) if  $k \equiv_3 2$ , the points of intersection of  $N_1$  and  $N_2$  are of type 1.

*Proof.* Since  $N_1$  and  $N_2$  have trivial intersection with any other components of the branch locus, the condition  $LN_i \in \mathbb{Z}$  restricts to the condition  $N_i(\alpha N_1 + \beta N_2)/3 \in \mathbb{Z}$ , where  $\alpha \in \{1, 2\}$  and  $\beta \in \{1, 2\}$  and  $(\alpha N_1 + \beta N_2)/3$  is the summand of  $L$  in which  $N_1$  and  $N_2$  appear with a nontrivial coefficient. Up to replace possibly  $L$  with  $M$ , we can assume that  $\alpha = 1$ . So the condition is now  $N_i(N_1 + \beta N_2)/3 \in \mathbb{Z}$ , which implies that  $-2 + \beta k \equiv_3 0$  and  $k - 2\beta \equiv_3 0$ . These conditions imply that  $k \not\equiv_3 0$ ,  $k \equiv_3 1$  if and only if  $\beta = 2$ , and  $k \equiv_3 2$  if and only if  $\beta = 1$ . □

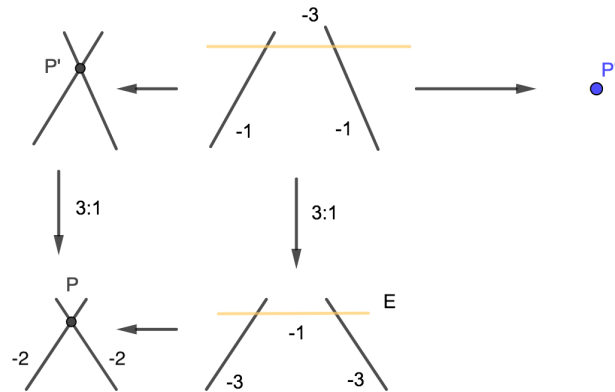


Figure 2.

EXAMPLE 3.9. Let us consider an elliptic fibration  $\mathcal{E} : S \rightarrow \mathbb{P}^1$  with two fibers of type  $I_2$  over the points  $p_1$  and  $p_2$ . The base change of order 3  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  branched over the points  $p_1$  and  $p_2$  induces a triple cover of  $S$  whose branch locus contains four singularities of type 1. Indeed, the branch curves are rational curves meeting in two points, and then the singularities are of type 1 by Lemma 3.8.

The following example contains singularities of type 2 and, moreover, shows that the minimal resolution  $X'$  of  $X$  constructed above is not necessarily a minimal model.

EXAMPLE 3.10. Let  $f : S \rightarrow \mathbb{P}^1$  be an elliptic fibration with six fibers of type  $I_3$  and a 3-torsion section. Generically,  $\rho(S) = 14$  (see [GS2, §4.1]). Let  $\Theta_i^{(j)}$ ,  $i = 0, 1, 2$ , and  $j = 1, \dots, 6$ , be the irreducible components of the reducible fibers. There exists a  $3 : 1$  cover  $X \rightarrow S$  branched over  $\Theta_1^{(j)}$ ,  $\Theta_2^{(j)}$  for  $j = 1, \dots, 6$ . The minimal model  $X^\circ$  is a K3 surface (see [G1, Prop. 4.1]). In particular, this triple cover is branched over six configurations of type  $A_2$  of rational curves and the branch data are  $B = \sum_{j=1}^6 \Theta_1^{(j)}$ ,  $C = \sum_{j=1}^6 \Theta_2^{(j)}$ ,  $L = (2B + C)/3$ , and  $M = (B + 2C)/3$ .

The surface  $X^\circ$  has Picard number equal to 14 (see [GS2, Prop. 4.1]), but the minimal resolution of  $X'$  has Picard number  $\rho(X') = \rho(S) + 6$ . In particular, this implies that  $X^\circ \neq X'$ , that is, the minimal resolution of  $X$  is not minimal.

The minimal resolution of  $X$  is not a minimal model if there is configuration of type  $A_2$  in the branch locus of  $X \rightarrow S$ ; indeed, in this case, one has to contract two  $(-1)$ -curves for each of these configurations, as shown in Figure 2.

REMARK 3.11. Let  $f : X \rightarrow S$  be a triple cover,  $X'$  the minimal resolution of  $X$ , and  $X^\circ$  the smooth minimal model of  $X$ . By the above construction, one obtains that if a singularity of type 2 lies on exactly one rational curve which contains no other singularities of the branch locus, then its inverse image in  $X'$  is contractible and so  $K_{X^\circ}^2 \geq K_{X'}^2 + 1$ . If the singularity of type 2 lies on the intersection of two rational curves which do not contain other singularities of the branch locus, then, in  $X'$ , there are three contractible curves (this is the case of Figure 2) and so  $K_{X^\circ}^2 \geq K_{X'}^2 + 3$ . We refer to the last situation as  $A_2$ -configuration in the branch locus.

In Theorem 4.12, we analyze other type of singularities which can appear in the branch locus of a triple cover.

PROPOSITION 3.12. *Let  $X$  be a normal variety which is a Galois triple cover of  $S$ , whose branch locus has  $n$  connected components, that is, the  $n$  connected reducible curves  $D_1, \dots, D_n$ . Then:*

- (1) *Each curve  $D_i$  is reduced.*
- (2) *At most one  $D_i$  is a curve with positive self-intersection.*
- (3) *If the singularities of  $X$  are negligible, denoted by  $(B, C, L, M)$  the Galois triple cover data, the invariants of the minimal resolution  $X'$ , are*

$$\begin{aligned} \chi(\mathcal{O}_{X'}) &= 6 + \frac{1}{2}(L^2 + M^2), \\ K_{X'}^2 &= 2L^2 + 2M^2 + LM, \\ e(X') &= 72 + 4(L^2 + M^2) - LM, \quad h^{1,0}(X') = h^1(L) + h^1(M). \end{aligned}$$

*Proof.* Condition (1) is needed to obtain a normal cover  $X \rightarrow S$  (see Paragraph 2.10). Condition (2) follows by the Hodge index theorem.

Since the singularities are negligible, the numerical invariants for  $X'$  can be calculated using the Tschirnhausen bundle  $\mathcal{E} = \mathcal{L}^{-1} \oplus \mathcal{M}^{-1} = \mathcal{O}_S(-L) \oplus \mathcal{O}_S(-M)$ . It has the following Chern classes:

$$c_1(\mathcal{E}) = -L - M, \quad c_2(\mathcal{E}) = L \cdot M.$$

To have a connected covering, we have

$$0 = h^0(\mathcal{E}) = h^0(-L) + h^0(-M) \Rightarrow h^0(-L) = h^0(-M) = 0.$$

In addition, by the Hirzebruch–Riemann–Roch theorem, we have

$$\begin{aligned} \chi(\mathcal{E}) &= h^2(\mathcal{E}) - h^1(\mathcal{E}) = h^0(L) + h^0(M) - h^1(\mathcal{E}) = \\ &= \int_S \text{Td}(S) \cdot \text{ch}(\mathcal{E}) = \int_S (1, 0, 2) \cdot \left( 2, -L - M, \frac{L^2 + M^2}{2} \right) = \frac{L^2 + M^2}{2} + 4. \end{aligned} \tag{3.1}$$

So, by Proposition 2.6(ii), we have  $\chi(\mathcal{O}_{X'}) = 6 + \frac{1}{2}(L^2 + M^2)$ , by Proposition 2.6(iii), we have  $K_{X'}^2 = 2L^2 + 2M^2 + LM$ , and finally, by Proposition 2.6(iv), we have  $e(X') = 72 + 4(L^2 + M^2) - LM$ .

By (3.1) and the genus formula,

$$\begin{aligned} h^1(\mathcal{E}) &= h^0(L) + h^0(M) - \chi(\mathcal{E}) = (\chi(L) + h^1(L)) + (\chi(M) + h^1(M)) - \chi(\mathcal{E}) = \\ &= \left( \frac{L^2}{2} + 2 + h^1(L) \right) + \left( \frac{M^2}{2} + 2 + h^1(M) \right) - \frac{L^2 + M^2}{2} - 4 = h^1(L) + h^1(M), \end{aligned} \tag{3.2}$$

which allows to compute  $h^{1,0}(X')$ , by Proposition 2.6(i). □

REMARK 3.13. If the branch locus of the cover  $f : X \rightarrow S$  is smooth, then  $BC = 0$  and so the formulae in Proposition 3.12 and (2.8), which compute the self-intersection of the canonical divisor of  $X$ , coincide and give  $K_X^2 = 12(B^2 + C^2)$ .

LEMMA 3.14. *Let us consider a connected reducible reduced curve,  $C$ , on a K3 surface  $S$ , whose irreducible components are rational curves (i.e., negative curves). If  $C^2 \leq 0$ , then  $C^2 = -2$ .*

*In particular, a reducible reduced negative curve  $D$  such that  $D^2 < -2$  has more than one connected component.*

*Proof.* Let us call  $C_i$  the irreducible components of  $C$ . Then  $C = \sum_{i=1}^n C_i$  and

$$(C)^2 = -2n + 2 \sum_{i=1}^n \sum_{j=i+1}^n C_i C_j.$$

The curve  $C$  is connected, so each curve  $C_i$  intersects at least one other curve, and one has  $\sum_{i=1}^n \sum_{j=i+1}^n C_i C_j \geq n - 1$ . Hence,  $C^2 \geq -2n + 2(n - 1) = -2$ . If  $C^2 < 0$ , then  $C^2 = -2$ .  $\square$

LEMMA 3.15. *Let  $S$  be a K3 surface, and let  $\Lambda$  be a sublattice of  $NS(S)$ . If  $\Lambda$  is generated by irreducible curves and is degenerate, then all the curves represented by classes in  $\Lambda$  are either fibers or components of fibers of a genus 1 fibration on  $S$ .*

*Proof.* Since  $\Lambda$  is degenerate, there exists  $v \in \Lambda$  such that  $v^2 = 0$  and  $vw = 0$  for every  $w \in \Lambda$ . By the Riemann–Roch theorem, either  $v$  or  $-v$  is effective, so we assume that  $v$  is effective and hence  $\varphi_{|v|}$  is a fibration whose smooth fibers are genus 1 curve (cf., e.g., [K, Proof of Lemma 2.1]). Since any other class  $w$  in  $\Lambda$  has a trivial intersection with  $v$ , if a curve is represented by  $w$ , it is contracted by  $\varphi_{|v|}$  and hence it is contained in a fiber of the fibration  $\varphi_{|v|}$ .  $\square$

Let  $\Lambda_{D_i}$  be the lattice generated by the irreducible components of an effective divisor  $D_i$ .

THEOREM 3.16. *Let  $f: X \rightarrow S$  be a normal Galois triple cover of a K3 surface, whose branch locus has  $n \geq 1$  connected components, that is, the  $n$  connected possibly reducible curves  $D_1, \dots, D_n$ .*

- (1) *If  $\kappa(X) = 0$ , then all the lattices  $\Lambda_{D_i}$  are  $A_2(-1)$  (in particular, negative definite).*
- (2) *If  $\kappa(X) = 2$ , then there exists a lattice  $\Lambda_{D_i}$  such that  $\text{sgn}(\Lambda_{D_i}) = (1, \text{rank}(\Lambda_{D_i}) - 1)$  and all the other lattices  $\Lambda_{D_j}$  are negative definite.*
- (3) *If  $\kappa(X) = 1$ , then there are no lattices  $\Lambda_{D_i}$  which are indefinite and there exists at least on  $\Lambda_{D_i}$  which is degenerate.*

*Proof.* (1) If  $\kappa(X) = 0$ , each component of  $D_i$  is a negative curve, and thus by adjunction a  $(-2)$ -curve, since the intersection with the canonical bundle is trivial for every curve on  $X$  (if  $X$  is a K3 surface; see also Lemma 3.14). Therefore, the lattice  $\Lambda_{D_i}$  is a negative definite root lattice, and thus  $\Lambda_{D_i}(-1)$  is an A–D–E lattice [BHPV, Lem. 2.12, Chap. I]. By [BHPV, Chap. I, §17], the existence of a triple cover branched in  $D_i$  implies that a linear combination of the components of the  $D_i$ ’s with noninteger coefficients in  $\frac{1}{3}\mathbb{Z}$  is contained in the Picard group of  $S$ . So the discriminant group of  $\Lambda_{D_i}$  contains  $\mathbb{Z}/3\mathbb{Z}$ , which implies that either  $\Lambda_{D_i} \simeq A_{3k-1}(-1)$  with  $k > 1$  or  $\Lambda_{D_i} \simeq E_6(-1)$ . In both the cases,  $\Lambda_{D_i}^\vee/\Lambda_{D_i}$  is a cyclic group, generated by a class  $l$  (supported on the components of  $D_i$ ). The generators of the subgroups  $\mathbb{Z}/3\mathbb{Z}$  in  $\Lambda_{D_i}^\vee/\Lambda_{D_i}$  are multiple of  $l$ , and they are always supported on a disjoint  $A_2$ -configuration of curves. So each  $D_i$  is the sum  $B_i + C_i$ , where  $B_i$  and  $C_i$  are two rational curves meeting in a point, that is,  $\langle B_i, C_i \rangle \simeq A_2(-1)$ .

(2) If  $\kappa(X) = 2$ , then there exists a curve  $C \in |R|$ , where  $R$  is the ramification of  $f$  such that  $g(C) > 1$  (see Proposition 2.9) and  $C^2 \geq 0$ . By the projection formula,  $(f(C))^2 \geq 0$  and so  $g(f(C)) \geq 1$ . There exists a (possibly reducible) branch curve  $\Gamma$  linearly equivalent to  $f(C)$  and contained in  $D_i$ . If  $g(f(C)) > 1$ , then  $\Gamma^2 > 0$  and hence  $\Lambda_{D_i}$  satisfies the hypothesis, by the Hodge index theorem.



If  $g(f(C)) = 1$ ,  $f(C)$  defines a genus 1 fibration  $\mathcal{E}$  on  $S$ . Moreover,  $f(C)$  intersects the branch locus, that is,  $f(C) \cap (\cup_i D_i) \neq \emptyset$ ; otherwise,  $g(C) = 1$ . This implies that  $\Gamma$  in the union of a fiber of  $\mathcal{E}$  and some horizontal curves. In particular,  $\Gamma$ , and hence  $D_i$ , contains the class  $F$  of the fiber of the fibration  $\mathcal{E}$  and at least a horizontal curve  $\Delta$ . The class  $\alpha F + \delta \Delta$  is contained in  $\Lambda_{D_i}$ , and it has a positive self-intersection for  $\alpha$  sufficiently big. By the Hodge index theorem, the signatures of all the other lattices  $\Lambda_{D_i}$  are the required ones.

(3) If  $\kappa(X) = 1$ , then there exists a curve  $C \in |R|$ , where  $R$  is the ramification of  $f$  such that  $g(C) = 1$  and  $C$  defines a genus 1 fibration on  $X$ . Hence,  $f(C)$  defines a genus 1 fibration on  $S$ . All the irreducible components of  $|f(C)|$  are genus 1 curve and  $f(C)^2 = 0$ . As above, there exists a (possibly reducible) branch curve  $\Gamma$  linearly equivalent to  $f(C)$  and contained in  $D_i$ . Then  $\Gamma \in |f(C)|$  and it is a fiber (irreducible or not), so it coincides with  $D_i$  for a certain  $i$  and  $\Lambda_{D_i}$  is degenerate. All the other  $D_j$ 's are contained in (or coincides with) fibers, so each  $\Lambda_{D_j}$  cannot contain a class with positive self-intersection.  $\square$

**COROLLARY 3.17.** *Let  $f: X \rightarrow S$  be a normal Galois triple cover of a K3 surface, whose branch locus has  $n \geq 1$  connected components, that is, the  $n$  connected reducible and reduced curves  $D_1, \dots, D_n$ .*

- (1) *If all the  $\Lambda_{D_i}$  are negative definite, then  $\kappa(X) = 0$ ,  $n = 6$  or  $n = 9$ , and the minimal model of  $X$  is either a K3 or an abelian surface.*
- (2) *If  $\text{sgn}(\Lambda_{D_1}) = (1, \text{rank}(\Lambda_{D_1}) - 1)$ , then  $\kappa(X) = 2$ ,  $\Lambda_{D_j} \simeq A_2(-1)$  for all  $j \neq 1$ , and  $n \leq 10$ .*
- (3) *If  $\Lambda_{D_1}$  is degenerate, then  $\Lambda_{D_j}$  for  $j \neq 1$  is either degenerate or negative definite and  $\kappa(X) = 1$ .*

*Proof.* The assumption that the curve  $D_i$ 's are reduced guarantees that  $X$  is normal. (1) If all the  $\Lambda_{D_i}$  are negative definite, we already showed in Theorem 3.16 that  $\Lambda_{D_i} \simeq A_2(-1)$ . The sets of  $A_2$ -configurations which are 3-divisible (and thus are in the branch locus of a triple cover) necessarily contain either six or nine  $A_2$ -configurations [Ba, Lem. 1]. In the first case, the minimal model of the triple cover is a K3 surface, and in the latter case, it is an abelian surface and in particular  $\kappa(X) = 0$ .

(2) If  $\text{sgn}(\Lambda_{D_1}) = (1, \text{rank}(\Lambda_{D_1}) - 1)$ , by the Hodge index theorem, the lattices  $\Lambda_{D_j}$  with  $j \neq 1$  are negative definite. As in proof of (1) of Theorem 3.16, this implies that  $\Lambda_{D_j}$  coincides with  $A_2(-1)$ . The maximal number of disjoint  $A_2$ -configurations of rational curves on a K3 surface is 9, and hence  $n \leq 10$ . To show that  $\kappa(X) = 2$ , we observe that if  $\kappa(X) = 0$ , then by Theorem 3.16, all  $\Lambda_{D_i}$  should be negative definite, which contradicts the hypothesis on  $\Lambda_{D_1}$ . If  $\kappa(X) = 1$ , then by Theorem 3.16, at least one  $\Lambda_{D_i}$  should be degenerate, which is again impossible.

(3) If  $\Lambda_{D_1}$  is degenerate, then  $D_1$  consists of a fiber of an elliptic fibration. So  $D_j, j \neq 1$ , are contained in fibers and then  $\Lambda_{D_j}$  cannot contain a positive class. In particular,  $\kappa(X)$  cannot be 2, because there is no indefinite lattice  $\Lambda_{D_i}$  and cannot be 0, because not all the lattices  $\Lambda_{D_i}$  are negative definite.  $\square$

We now consider the case in which one connected component  $D_i$  of the branch locus is irreducible. The following corollary shows that if its self-intersection is nonnegative, we can easily determine the Kodaira dimension of  $X$ .

**COROLLARY 3.18.** *Let  $D_i$  be an irreducible reduced curve in the branch locus of a Galois triple cover  $f: X \rightarrow S$ . If  $D_i^2 > 0$ , then  $\kappa(X) = 2$ . If  $D_i^2 = 0$ , then  $\kappa(X) = 1$ .*

If  $D_i^2 \geq 0$ , then the minimal model  $X^\circ$  of  $X$  is obtained by contracting on  $X'$  three curves for each  $A_2$ -configuration in the branch locus.

*Proof.* The first part of the statement follows directly by Corollary 3.17.

Let  $D_1$  be a smooth irreducible curve, then the singularities of the branch locus are of type 2 and they are the singular points of each  $A_2$ -configuration in the branch locus.

In this case,  $X^\circ$  is obtained by contracting on  $X'$  three curves for each  $A_2$ -configuration in the branch locus. Indeed, by Remark 3.11, one has to contract at least three curves for each  $A_2$ -configuration, and we call the surface obtained by these contractions  $X_m$ . It remains to prove that  $X_m$  and  $X^\circ$  coincide, and so that there are no other possible contractions to a smooth surface.

On  $X_m$ , there is an automorphism  $\sigma_m$  of order 3, induced by the Galois  $\mathbb{Z}/3\mathbb{Z}$ -cover automorphism  $\sigma$  on  $X$ . The surface  $S_m := X_m/\sigma_m$  is the singular surface obtained by  $S$  contracting all the  $A_2$ -configurations in the branch locus. If there were a  $(-1)$ -curve  $E$  on  $X_m$ , then there are two cases: either  $E$  is disjoint from the ramification locus of  $f_m : X_m \rightarrow S_m$ , or  $E$  meets it (not being contained).

If  $E$  is disjoint from the ramification locus, then there exists a  $(-1)$ -curve, still denoted by  $E$ , on  $X$  which is mapped to  $E$  (because we blow up and down points away from  $E$ ). Hence,  $\sigma(E) \cap E$  is empty, and therefore  $\sigma(E)$  and  $\sigma^2(E)$  are other two  $(-1)$ -curves on  $X$ . So  $f(E) = f(\sigma(E)) = f(\sigma^2(E)) \subset S$  is a  $(-1)$ -curve. This is absurd because  $S$  is a K3 surface.

Hence,  $E$  meets the ramification locus  $R_m$  and  $\sigma_m(E) = E$ . Otherwise, there should be different  $(-1)$ -curves ( $E$  and  $\sigma_m(E)$ ) meeting in the point  $E \cap R_m$  and this is impossible on a surface with nonnegative Kodaira dimension. Moreover,  $\sigma$  is an automorphism of order 3 of the rational curve  $E$ , and then  $E \cap R_m$  consists of two points.

Let  $\beta : X_m \rightarrow X^\circ$  be the contraction of  $E$ , and let  $\sigma^\circ$  be the automorphism induced by  $\sigma_m$  on  $X^\circ$ . This induces the contraction  $S_m \rightarrow S^\circ$  of the curve  $f_m(E)$ . Since  $E \cap R_m$  consists of two points, the contraction of  $f_m(E)$  identifies two singular points on  $S_m$  and introduces a singularity on the image of  $D_1$ . By construction, the smooth  $X^\circ$  is a triple cover of  $S^\circ$  and thus the singularities of  $S^\circ$  cannot be worse than  $(\mathbb{C}^2, 0)/\mathbb{Z}/3\mathbb{Z}$ , so two singularities of  $S_m$  cannot be identified. Moreover, there cannot be a singularity on the branch curve image of  $D_1$ , and  $S^\circ$  has to coincide with  $S_m$ . We conclude that  $X_m$  coincides with  $X^\circ$ .  $\square$

#### §4. Examples of Galois covers of K3 surfaces

By Theorem 3.16 and Corollary 3.17,  $k(X) = 0$  if and only if for all the components  $D_i$  of the branch locus,  $\Lambda_{D_i}$  is negative. In this case, the minimal model of  $X$  is either a K3 surface or an abelian surface, and these cases are well known (see, e.g., [Ba]).

We now provide examples for the other cases. First, we consider surfaces of general type: in Propositions 4.1 and 4.6, we compute the invariants of  $X^\circ$  if it is of general type and we make specific assumption on the component  $D_1$  of the branch locus such that  $\Lambda_{D_1}$  is indefinite; in Corollary 4.3, we provide an example of surface  $X^\circ$  with  $p_g = 2$ ; and in Theorems 4.16 and 4.17, examples of  $X^\circ$  with  $q \neq 0$ . Then we consider the case  $\kappa(X^\circ) = 1$ , and we classify the invariants of  $X^\circ$  in Proposition 4.11.

##### 4.1 The covering surface $X$ is of general type

Case (2) of Theorem 3.16 is the most general one, but under some assumptions, we are able to give a more detailed description of these covers.

We first assume to be in the hypothesis of Corollary 3.18.

PROPOSITION 4.1. *Let  $D_1$  be a connected component of the branch locus of a Galois triple cover  $f : X \rightarrow S$ , which is also irreducible, reduced, and of positive genus. Then  $D_1^2 = 6d$ , for an integer  $d > 0$ ;  $d \equiv_3 n - 1$ , and denoted by  $k$  the integer such that  $d = n - 1 + 3k$ , one has  $k \geq -2$ .*

If  $D_1$  is moreover smooth, then

$$\chi(X^\circ) = 5 + n + 5k, \quad K_{X^\circ}^2 = 8n - 8 + 24k, \quad e(X^\circ) = 68 + 4n + 36k. \tag{4.1}$$

*Proof.* By Corollary 3.17, the components  $D_j$  with  $j > 1$  consist of two rational curves meeting in a point. We denote by  $A_1^j$  and  $A_2^j$  the two components of  $D_j$ . Since  $D_1$  is irreducible, it is a component of  $B$  (or equivalently of  $C$ ). Then the data of the triple cover are

$$B = D_1 + \sum_{j=1}^{n-1} A_1^j, \quad C = \sum_{j=1}^{n-1} A_2^j,$$

$$L = \frac{D_1 + \sum_{j=1}^{n-1} (A_1^j + 2A_2^j)}{3}, \quad M = \frac{2D_1 + \sum_{j=1}^{n-1} (2A_1^j + A_2^j)}{3}.$$

By  $LD_1 \in \mathbb{Z}$  and  $L^2 \in 2\mathbb{Z}$ , it follows that  $D_1^2 \equiv_3 0$  and  $d \equiv_3 n - 1$ . Since  $n \leq 10$  and  $d > 0$ ,  $9 + 3k > 0$ , so  $k \geq -2$ . The formula (4.1) follows by Proposition 3.12 since  $L^2 = 2k$ ,  $M^2 = 2n + 8k - 2$ , and  $LM = n - 1 + 4k$ , and  $X^\circ$  is obtained by  $X$  contracting  $3(n - 1)$  curves. We conclude by Corollary 3.18.  $\square$

4.2

In the situation of the previous corollary, if  $L^2 \geq -2$ , there exists a member of the linear system  $|D_1|$  which splits into the union of a curve  $G$  and the curves  $A_i^j$ , where  $G \simeq \frac{D_1 - \sum_{j=1}^{n-1} (A_1^j + 2A_2^j)}{3}$  and hence  $D_1 \simeq 3G + \sum_{j=1}^{n-1} (A_1^j + 2A_2^j)$ . We observe that  $GA_2^{(j)} = 1$  and  $GA_1^{(j)} = 0$ . Since  $G^2 = 2k$ , if  $G$  is an irreducible and smooth curve, then  $g(G) = k + 1$ . In particular,  $G$  is rational if  $k = -1$ . A limit case is the one with  $k = -1$  and  $n = 3$ . It is not an example of case (1) of the Theorem 3.16, because  $D_1^2 = 0$ , but it is still instructive, since the interpretations of the curves  $G, A_i^{(j)}$  in this situation are well known:  $D_1$  is a fiber of type  $IV^*$  of an elliptic fibration and the curves  $G$  and  $A_i^{(j)}$  are its components.

Similarly, there is a member of  $|2D_1|$  which splits in the union of a curve  $F$  and the union of the curves  $A_i^j$ , where  $F \simeq \frac{2D_1 - \sum_{j=1}^{n-1} (2A_1^j + A_2^j)}{3}$ , and hence  $2D_1 \simeq 3F + \sum_{j=1}^{n-1} (2A_1^j + A_2^j)$ . Since  $F^2 = 2n + 8k$ , if  $F$  is irreducible and smooth curve, then  $g(F) = n + 4k + 1$ . In particular,  $F$  is rational if  $n + 4k = -1$  and elliptic if  $n = -4k$ .

COROLLARY 4.3. *There exists a smooth Galois triple cover  $X$  of a K3 surface  $S$  whose branch locus consists of a smooth curve of genus 4 and seven  $A_2$ -configurations of rational curves such that, denoted by  $X^\circ$  the minimal model of  $X$ , it holds*

$$\chi(X^\circ) = 3, \quad q(X^\circ) = 0, \quad p_g(X^\circ) = 2, \quad K_{X^\circ}^2 = 8.$$

*Proof.* Let  $S$  be a K3 surface with an elliptic fibration such that the reducible fibers are  $I_2 + 6I_3$  and the Mordell–Weil group is  $\mathbb{Z}/3\mathbb{Z}$ . The existence of a K3 surface with such an

elliptic fibration is guaranteed by [Sh, Table 1, case 835]. We denote by  $F$  the class of the fiber of this fibration, by  $\mathcal{O}$  the class of the zero section, by  $A_2^{(1)}$  the class of the irreducible component of the fiber  $I_2$  which meet the section  $\mathcal{O}$ , and by  $A_h^{(j)}$ ,  $h = 1, 2$ ,  $j = 2, \dots, 7$ , the classes of the two irreducible components not meeting the zero section of the  $j$ th reducible fiber, which is a fiber of type  $I_3$ . We observe that the class of the 3-torsion section  $P$ , which generates the Mordell–Weil group, can be written in terms of the previous curves as

$$P = 2F + \mathcal{O} - \frac{1}{3} \left( \sum_{j=2}^7 A_1^{(j)} + 2A_2^{(j)} \right).$$

Moreover, we observe that there are seven disjoint  $A_2$ -configurations on this surface, which are given by  $\mathcal{O}$ ,  $A_2^{(1)}$ ,  $A_h^{(j)}$ ,  $h = 1, 2$ ,  $j = 2, \dots, 7$ . Let us consider the divisor  $3F + A_2^{(1)} + 2\mathcal{O} = D_1$ , and we notice that  $D_1^2 = 6$ ,  $D_1F = 2$ ,  $D_1A_2^{(1)} = 0$ , and  $D_1\mathcal{O} = 0$ . One can check that  $D_1$  is a big and nef divisor, and hence in its linear system, there is a smooth irreducible curve of genus 4, still denoted by  $D_1$ . We claim that there exists a triple cover of  $S$  branched over  $D_1$  and the seven  $A_2$ -configurations  $\mathcal{O}$ ,  $A_2^{(1)}$ ,  $A_h^{(j)}$ ,  $h = 1, 2$ ,  $j = 2, \dots, 7$ . Indeed, the divisor

$$L := \left( D_1 + \mathcal{O} + \sum_{i=2}^7 A_1^{(i)} + 2 \sum_{j=1}^7 A_2^{(j)} \right) / 3 = 3F + A_2^{(1)} + 2\mathcal{O} - P$$

is contained in  $NS(S)$  and so

$$B := D_1 + \mathcal{O} + \sum_{i=2}^7 A_1^{(i)}, \quad C := \sum_{j=1}^7 A_2^{(j)}, \quad L = (B + 2C)/3, \quad M = (2B + C)/3$$

form triple cover data on  $S$ . So there exists a triple cover  $X \rightarrow S$  which satisfies the condition of Proposition 4.1 with  $k = -2$ ,  $n = 8$ , and  $d = 1$ , and then we deduced the properties of  $X^\circ$ .  $\square$

REMARK 4.4. If  $S$  is a K3 surface and  $p_g(X) = 2$ , the cover  $f : X \rightarrow S$  induces a splitting of the Hodge structure on  $T_X$  in a direct sum of two Hodge structures of K3 type (i.e., the Hodge structure of weight 2 of type  $(1, \star, 1)$ ); indeed,

$$T_X = f^*(T_S) \oplus (f^*(T_S))^\perp.$$

The Hodge structure of  $T_X$  is of type  $(2, \star, 2)$ , and the one of  $f^*(T_S)$  is of type  $(1, \star', 1)$  since it is induced by the Hodge structure of the K3 surface  $S$ . Hence, both  $f^*(T_S)$  and  $(f^*(T_S))^\perp$  carry a K3-type Hodge structure.

The surfaces with  $p_g = 2$  such that the transcendental Hodge structure splits in the direct sum of two K3-type Hodge structures are studied in several context (see, e.g., [G3], [L1], [L2], [Mo], [PZ]), and in general, it is interesting to look for K3 surfaces geometrically associated with the K3-type Hodge structure. In the case of covers of K3 surfaces  $S$  (and in particular in the setting of Corollary 4.3), at least one of the two K3-type Hodge structures is of course geometrically associated with the K3 surface  $S$ ; indeed, it is the pull back of the Hodge structure of  $S$ .

We give examples of case (2) of the Theorem 3.16 such that  $D_1$  is reduced and reducible. In particular, we consider the case  $D_1^2 = 0$ , but  $\Lambda_{D_1}$  contains a class with a positive

self-intersection. In this case, the support of  $D_1$  consists of a certain number of fibers  $F_i$  and some rational horizontal curves  $P_j$  such that

$$(D_1)^2 = \left( r \sum_i F_i + \sum_j P_j \right)^2 = 0.$$

We denote by  $F$  the class of the fiber, and therefore  $F_i P_j = F P_j$  for all  $i$ . Since  $(D_1)^2 = \sum_{k=1}^s D_1 P_k + r F P_k$ ,  $D_1^2 = 0$  and  $F P_k > 0$ , it follows that

$$\exists j \text{ such that } D_1 P_j < 0.$$

In particular,  $D_1$  is not nef.

The following lemma shows that each curve  $P_j$  such that  $D_1 P_j < 0$  is a section orthogonal to all the other horizontal curves in  $D_1$ .

LEMMA 4.5. *Let  $F$  be the class of the fiber of an elliptic fibration and  $P_j$ 's irreducible horizontal curves. Let  $D_1 = rF + \sum_{j=1}^k P_j$  be such that  $D_1^2 = 0$  and  $D_1$  is reduced.*

*There exists  $j$  such that  $D_1 P_j < 0$  if and only if  $F P_j = 1$ ,  $P_j P_i = 0$  for every  $j \neq i$  and  $r = 1$*

*Proof.* We already observed that  $D_1$  intersects negatively at least one of its components, say  $P_j$ . Hence,  $P_j$  is a fixed component of a non-nef divisor and  $P_j^2 = -2$ . So

$$D_1 P_j = r F P_j + \sum_{i \neq j} P_i P_j - 2 < 0.$$

So  $r F P_j + \sum_{i \neq j} P_i P_j \leq 1$ . We observe that  $P_j$  is horizontal, so  $F P_j > 0$ , and  $P_j P_i \geq 0$  because  $P_i$  are irreducible effective curves. Hence, the only possibility is  $r = 1$ ,  $F P_j = 1$ , and  $P_i P_j = 0$ . □

We recall that by (2) of Corollary 3.17, if  $D_1$  is as above, in the branch locus, there are  $n - 1$  components  $D_h$ ,  $n \geq 1$  which are  $A_2$ -configurations of curves. We denote by  $A_1^{(h)}$ ,  $A_2^{(h)}$  the components of such configurations.

PROPOSITION 4.6. *Let  $S$  be a K3 surface admitting an elliptic fibration with  $k$  disjoint sections  $P_j$  and whose class of the fiber is  $F$ , and let  $D_1 = F + \sum_j P_j$ . There exists a triple cover  $X \rightarrow S$  branched on  $D_1$  and other  $n - 1$  irreducible components  $D_h$  if and only if*

$$D_1 = F + \sum_{j=1}^k P_j, \quad k \equiv_3 0, \quad n \equiv_3 1 + \frac{k}{3}, \tag{4.2}$$

and the data of the triple cover are determined by

$$B = F + \sum_{h=1}^{n-1} A_1^{(h)}, \quad C = \sum_{i=1}^k P_i + \sum_{h=1}^{n-1} A_2^{(h)}.$$

The surface  $X^\circ$  is of general type, and its numerical invariants are

$$\chi(\mathcal{O}_{X^\circ}) = \frac{60 - 6n - k}{9}, \quad K_{X^\circ}^2 = \frac{2k}{3}.$$

*Proof.* Since  $L = (B + 2C)/3$  and  $LP_i \in \mathbb{Z}$ , it follows that if  $F$  is a component of  $B$ , the  $P_i$ 's must be components of  $C$ . The divisor

$$L = \frac{F + 2\sum_{j=1}^k P_j + \sum_{h=1}^{n-1} (A_1^{(h)} + 2A_2^{(h)})}{3}$$

has to be contained in  $NS(S)$ ,  $LF \in \mathbb{Z}$ ,  $LP_i \in \mathbb{Z}$ , for every  $i = 1, \dots, h$ , and  $L^2 \in 2\mathbb{Z}$ . These conditions imply

$$\frac{2k}{3} \in \mathbb{Z}, \quad -1 \in \mathbb{Z}, \quad \text{and} \quad \frac{-4k - 6(n-1)}{9} = 2 \frac{-2k - 3n + 3}{9} \in 2\mathbb{Z}.$$

Recall that

$$M = \frac{2F + \sum_{j=1}^k P_j + \sum_{h=1}^{n-1} (2A_1^{(h)} + A_2^{(h)})}{3},$$

so

$$L^2 = \frac{6 - 4k - 6n}{9}, \quad M^2 = \frac{2k + 6 - 6n}{9}, \quad \text{and} \quad LM = \frac{k + 3 - 3n}{9},$$

which gives  $\chi(X)$  and  $K_X^2$ . Since the singularities of  $X$  are negligible, these are the invariants of the minimal resolution of the cover. To obtain a minimal model, one has to contract the  $(-1)$ -curves. Each  $A_2$ -configuration produces three  $(-1)$ -curves in the minimal resolution of triple cover, and each curve  $P_j$  produces one  $(-1)$ -curve, by Remark 3.11. So one contracts at least  $3(n-1) + k$  curves to obtain the minimal model from the minimal resolution and so  $K_{X^\circ} \geq K_X^2 + 3(n-1) + k = 2k/3$ .

As in the proof of Corollary 3.18, if there were other  $(-1)$ -curves, they have to intersect the ramification locus, and we already excluded the cases coming from rational curves intersecting the components  $D_j, j \geq 2$ . We consider the canonical resolution of the triple cover  $\tilde{f}: \tilde{X} \rightarrow \tilde{S}$  as in the diagram (2.3). Thanks to the particular configuration of curves in  $D_1$ , one checks that the only  $(-1)$ -curves on  $\tilde{X}$  mapped to  $\sigma^{-1}(D_1)$  are the strict transforms of the triple cover of the curves  $P_i$ . After their contraction, there are no other  $(-1)$ -curves in the strict transform of  $\sigma^{-1}(D_1)$ .

Consider a rational curve  $C \subset S$  with  $CD_1 > 0$  and observe that  $C$  is not mapped to  $\sigma^{-1}(D_1)$ . We denote by  $\tilde{C}$  the strict transform of  $C$ . If  $\tilde{f}^{-1}(\tilde{C})$  is an irreducible rational curve, it intersects the ramification locus in at most two points. Moreover,  $(\tilde{f}^{-1}(\tilde{C}))^2 = 3\tilde{C}^2 \leq -6$ . Since we can contract at most two curves meeting  $\tilde{f}^{-1}(\tilde{C})$ , we cannot obtain a  $(-1)$ -curve. If  $\tilde{f}^{-1}(\tilde{C})$  splits in three curves and they meet, then they cannot be  $(-1)$ -curves (because  $X$  is of general type). If  $\tilde{f}^{-1}(\tilde{C})$  splits in three curves, these cannot meet,  $\tilde{f}^{-1}(\tilde{C})$  does not meet the ramification locus, and  $(\tilde{f}^{-1}(\tilde{C}))^2 = (\tilde{C}^2)$ . In particular, they are not  $(-1)$ -curves. The  $(-1)$ -curves on  $\tilde{X}$  are contained in the ramification locus, and thus  $\tilde{f}^{-1}(\tilde{C})$  does not meet these curves. □

In the previous proposition, we do not discuss the existence of the K3 surfaces considered, so a priori it is possible that the hypothesis of the proposition are empty. This is not the case, by the following corollary.

**COROLLARY 4.7.** *Let  $k$  and  $n$  be two positive integers such that  $k + 2n - 1 \leq 11$ . Then there exists a K3 surface  $S$  with an elliptic fibration with  $k$  disjoint sections  $P_j$  and  $n - 1$  fibers of type  $I_3$  such that the sections  $P_j$  all meet the same component of each  $I_3$ -fiber.*

In particular, there exists a triple cover  $X \rightarrow S$  as in Proposition 4.6 if  $(k, n) = (3, 2), (6, 3), (9, 1)$  and in these cases  $\chi(X) = 5, 4, 5$  and  $K_{X^\circ}^2 = 2, 4, 6$ , respectively.

*Proof.* Let  $\Lambda$  be a lattice generated by the following classes:

$$F, P_j, (j = 1, \dots, k), A_1^{(h)}, A_2^{(h)}, \frac{F + 2 \sum_{i=1}^j P_j + \sum_{h=1}^{n-1} (A_1^{(h)} + 2A_2^{(h)})}{3} = L,$$

where the intersections which are not zero are

$$FP_j = A_1^{(h)} A_2^{(h)} = 1, P_j^2 = (A_i^{(h)})^2 = -2.$$

The lattice  $\Lambda$  is even and hyperbolic. If the rank of the lattice  $\Lambda$  is less than 12, then it admits a primitive embedding  $\Lambda_{K3}$  and there exists a projective K3 surface  $S$  such that  $NS(S) = \Lambda$  (by the surjectivity of the period map). If  $(k, n)$  is such that  $\text{rank}(\Lambda) \leq 12$  and satisfies the condition (4.2), then  $(k, n) = (3, 2), (6, 3), (9, 1)$ . The classes  $F$  and  $F + P_j$  span a copy of  $U$  inside  $\Lambda$ , then there exists a negative definite lattice  $K$  such that  $U \oplus K \simeq \Lambda$ . In [K, Proof of Lem. 2.1], it is proved that if  $NS(S) \simeq \Lambda$  is as described, there exists an elliptic fibration. The  $2(n-1)(-2)$ -curves  $A_i^{(h)}$ , which are roots in  $K$ , are contained in singular fibers as in the statement. By the Shioda–Tate formula, the rank of the Mordell–Weil group of the elliptic fibration is the Picard number of the surface minus the rank of the trivial lattice. The latter is spanned by  $U$  and the roots contained in  $K$ . Hence, the Mordell–Weil group has rank  $k - 1$ , and hence there are  $k$  independent sections, which corresponds to the classes  $P_j$  (see [SS, Cor. 6.13]). □

### 4.8

We observe that the K3 surface associated with the values  $(9, 1)$  corresponds to a K3 surface with an elliptic fibration with nine disjoint sections and, generically, without reducible fibers. This K3 surface is obtained as base change of order 2 on a generic rational elliptic surface  $R$ . One can directly check this fact by considering the Néron–Severi group of  $S$ , which is generated by  $L$  and by the classes  $P_1, \dots, P_9$  and it is isometric to  $U \oplus E_8(-2)$  which is the Néron–Severi group of a K3 surface corresponding to the double cover of a generic rational elliptic surfaces [GS1]. In particular, the rational elliptic surface is a blowup of  $\mathbb{P}^2$  in the base locus of a generic pencil of cubics, and hence the K3 surface  $S$  can be described as double cover of  $\mathbb{P}^2$  branched on two specific cubics of the pencil (see, e.g., [GS1]). The triple cover  $X \rightarrow S$  defines a  $6 : 1$  Galois cover  $X \rightarrow \mathbb{P}^2$  whose Galois group is  $\mathfrak{S}_3$ . In particular, the rational surface  $R$  admits a non-Galois triple cover  $W$  and by construction  $X$  is a double cover of  $W$ .

In Lemma 4.5 and hence in Proposition 4.6, we assume that the fibers appearing in the component  $D_1$  of the branch locus are smooth. However, of course, this is not the only possibility. Indeed, one can also consider reducible fibers. This gives many ways to construct the data of the triple cover. For example, let  $F$  be a reducible fiber with two components  $G_0$  and  $G_1$ , then either both  $G_0$  and  $G_1$  are contained in  $B$  or one is contained in  $B$  and the other in  $C$ . These choices produce different covers, and the situation can be easily generalized with fibers with many components.

We now describe one case where the fiber is of type  $I_2$  (and then it has two components), one component is contained in  $B$  and the other in  $C$ . Moreover, in the branch locus, there

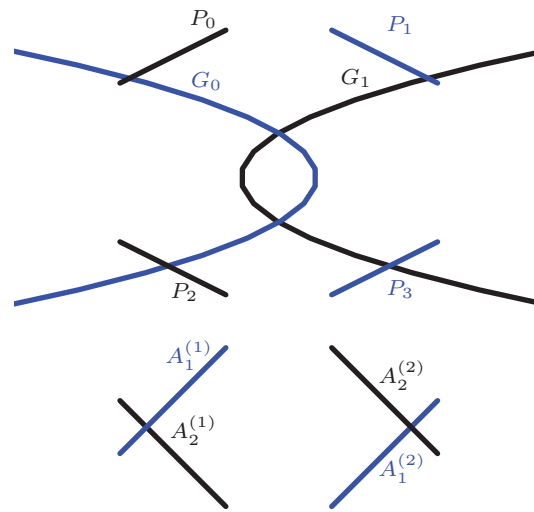


Figure 3.

are four sections and other two  $A_2$ -configurations of rational curves. This means that  $n = 3$  with the notation of case (2) of Theorem 3.16.

EXAMPLE 4.9. Let us consider a K3 surface  $S$  and the configuration of  $(-2)$ -curves as in Figure 2. The existence of a K3 surface with the required fibration is guaranteed by the surjectivity of the period map as in Corollary 4.7.

We then consider the triple cover such that

$$D_1 = G_0 + G_1 + P_0 + P_1 + P_2 + P_3, \quad D_2 = A_1^{(1)} + A_2^{(1)}, \quad D_3 = A_1^{(2)} + A_2^{(2)}.$$

The triple cover data are

$$B = G_0 + P_1 + P_3 + A_1^{(1)} + A_1^{(2)}, \quad C = G_1 + P_0 + P_2 + A_2^{(1)} + A_2^{(2)}, \quad L = \frac{B + 2C}{3}, \quad M = \frac{2B + C}{3}.$$

We observe that  $\Lambda_{D_1}$  is indefinite (e.g.,  $2(G_0 + G_1) + P_0$  has a positive self-intersection) and  $X$  is of general type by Theorem 3.16.

A straight forward calculation shows that  $B^2 = -10$ ,  $C^2 = -10$ , and  $BC = 8$ , which yields  $L^2 = -2$ ,  $M^2 = -2$ , and  $ML = 0$ . Since all the singularities in the branch locus are of type 2, and in particular negligible, by Proposition 3.12, we obtain

$$\chi(X') = 4, \quad K_{X'}^2 = -8.$$

Moreover, since  $L^2 = M^2 = -2$ , it follows that  $h^1(S, L) = h^1(S, M) = 0$ , and then  $q(X') = 0$ . Therefore,  $p_g(X') = 3$ . The surface  $X'$  is smooth but not minimal. Indeed, each configuration of type  $A_2$  in the branch locus corresponds to three  $(-1)$ -curves on the minimal resolution  $X'$  of the cover and each curve  $P_j$  corresponds to a  $(-1)$ -curve on  $X'$  (see Remark 3.11). So we have to contract at least  $3 \cdot 2 + 4 = 10$  curves on  $X'$ , and we denote by  $X_m$  the surfaces obtained contracting these 10 curves on  $X'$ . Then  $K_{X_m}^2 = -8 + 10 = 2$ . The surface  $X_m$  is minimal, to prove that directly it is not straightforward. Nevertheless, it is possible to see it using a different construction of  $X_m$ . In [BP], Penegini and Bini introduce a Calabi–Yau threefold  $Y_6$  with Hodge numbers  $(10, 10)$ . To some extent, this is special. In fact, its group of automorphisms contains a subgroup  $G$  isomorphic to  $\mathbb{Z}/6$ . Moreover, this Calabi–



Yau threefold can be described as a small resolution of a (3, 3) complete intersection  $Y$  in  $\mathbb{P}^5$  with 72 ordinary double points. Furthermore, the group  $G$  extends to a group of automorphisms of  $\mathbb{P}^5$ . Thus, there are six invariant hyperplane sections corresponding to irreducible characters of the abelian group  $\mathbb{Z}/6$ . Since the intersections of  $Y$  with these sections are invariant with respect to this group, it was natural to investigate them and their quotients. Out of the six invariant sections mentioned before, three of them are irreducible. These are singular surfaces of general type; on the minimal model  $\Sigma$  of one of them, let act  $\mathbb{Z}/2 \leq G$ , then it is not hard to prove that the minimal resolution of  $\Sigma/(\mathbb{Z}/2)$  is a minimal surface and is isomorphic to  $X_m$ . Hence,  $X_m$  is the minimal model of  $X$ , that is, it is  $X^\circ$ .

**4.9 Examples of case (3) of Theorem 3.16**

Case (3) of Theorem 3.16 implies that  $D_1$  is a fiber of an elliptic fibration. So  $D_j, j > 1$ , is necessarily contained in a fiber, and this naturally gives two cases: either, for all  $j = 1, \dots, n$ ,  $D_j$  is a full fiber or at least one of the  $D_j$  is supported on rational components of a fiber, but it does not coincide with the full fiber. Both cases are possible, and we now discuss them.

PROPOSITION 4.10. *Let  $X \rightarrow S$  be a triple cover as in case (3) of Theorem 3.16. This implies that  $\varphi_{|D_1|} : S \rightarrow \mathbb{P}^1$  is an elliptic fibration. Suppose that all the  $D_j$  are linearly equivalent to  $D_1$ , then  $X$  is obtained by a base change of order 3.*

*Proof.* By the proof of Corollary 3.17,  $D_1$  is the fiber of an elliptic fibration  $\varphi_{|D_1|} : S \rightarrow \mathbb{P}^1$ . By hypothesis, the  $D_j$ 's are fibers of the fibration  $\varphi_{|D_1|}$  too. The  $3 : 1$  cover  $X \rightarrow S$  induces a  $3 : 1$  cover  $C \rightarrow \mathbb{P}^1$  where  $C$  is a smooth curve. The branch locus of  $C \rightarrow \mathbb{P}^1$  is the image of the branch locus of  $X \rightarrow S$  and so it is  $\varphi_{|D_1|}(\bigcup_j D_j)$ . □

PROPOSITION 4.11. *Let  $S$  be a K3 surface endowed with an elliptic fibration  $\varphi_{|F|} : S \rightarrow \mathbb{P}^1$ . Let us consider  $p_1, \dots, p_n$  points in  $\mathbb{P}^1$  and a triple Galois cover  $g : W \rightarrow \mathbb{P}^1$  totally branched on  $p_1, \dots, p_n$ . The fiber product  $X := S \times_g \mathbb{P}^1$  is a triple cover of  $S$  branched on  $n$  fibers. If the fibers over the points  $p_i$  are reduced,  $X$  is also normal. Denoted by  $X^\circ$  the minimal model of  $X$ , if  $X^\circ$  is not a product, then*

$$h^{1,0}(X^\circ) = g(W) = n - 2 \text{ and } K_{X^\circ}^2 = 0.$$

*If  $X^\circ$  is a product, then  $h^{1,0}(X^\circ) = g(W) + 1 = n - 1$ .*

*If all the branch fibers are reduced and not of type IV, then*

$$e(X^\circ) = 72, \quad \chi(X^\circ) = 6, \text{ and } p_g(X^\circ) = 3 + n.$$

*Proof.* The cover automorphism of  $g : W \rightarrow \mathbb{P}^1$  acts as  $\zeta_3$  locally near the first  $b_1$  ramification points and as  $\zeta_3^2$  locally near the other  $b_2 = n - b_1$  ramification points. Notice that  $b_1 + 2b_2 \equiv_3 0$ . Let us consider the fiber product  $S \times_g \mathbb{P}^1$ . It is the triple cover  $X$  of  $S$  branched over the fibers  $F_{p_i} = \varphi_{|F|}^{-1}(p_i)$ . So we have a Galois triple cover  $X \rightarrow S$ . If all the fibers  $F_{p_i}$  are reduced, the cover  $X$  is normal, and we can apply the previous theory. The branching data are  $B = \sum_{i=1}^{b_1} F_{p_i}, C = \sum_{j=b_1+1}^n F_{p_j}, L \simeq (b_1 + 2b_2)F/3$ , and  $M \simeq (2b_1 + b_2)F/3$ .

Now, let us compute the triple cover invariants. The surface  $X$  is endowed with the elliptic fibration. Let us denote by  $X^\circ$  the minimal model of  $X$ , which is a smooth surface, admitting a relatively minimal elliptic fibration  $\mathcal{E} : X^\circ \rightarrow W$ . Assume that it admits at least one singular fiber (which is surely true if there exists a singular fiber of  $S$  which is not in

the branch locus). Then  $X^\circ$  is not a product and  $h^{1,0}(X^\circ) = g(W) = n - 2$ . Moreover, since  $X^\circ$  admits an elliptic fibration  $K_{X^\circ}^2 = 0$ . We recall that if a branch fiber is of type  $IV$ , the corresponding ramification fiber on  $X^\circ$  is smooth, and hence it is possible that after the base change the fibration  $X^\circ \rightarrow W$  has no singular fibers and it may be a product.

If the branch fibers are reduced and different from  $IV$ , the Euler characteristic of their strict transform is three times their Euler characteristic (see [Mil, VI.4.1]), then  $e(X^\circ) = 3e(S) = 72$ , so  $\chi(X^\circ) = e(X^\circ)/12 = 3\chi(S) = 6$ . □

We observe that even if there are nonreduced fibers  $F_{p_i}$  in the branch locus, the invariants of the minimal model of a normalization can be computed by the theory of the base change of elliptic fibrations (see [Mil, VI.4.1]).

**THEOREM 4.12.** *The singularities in the branch locus of a Galois triple cover are negligible (see Definition 2.15) if the local equation of the branch locus is in the following list:*

- $xy$  (type 1);
- $xy^2$  (type 2);
- $xy(x + y)$  (simple triple point);
- $x^2 - y^3$  (cusp); and
- $x(y - x^2)$ .

*Proof.* The singularities of types 1 and 2 were considered in [PP, Exams. 1.6 and 1.8] (see also Proposition 2.16). For the other cases, we use the results of Proposition 4.11, where we computed the invariants of  $X^\circ$  directly by considering the geometry of the elliptic fibration. We compare them with the ones obtained by applying point (3) of Proposition 3.12. If they coincide, this means that all the singularities which can appear in the branch locus are negligible. Since  $L^2 = M^2 = LM = 0$ , one obtains  $\chi(X') = 6$ ,  $e(X') = 72$ , and  $K_{X'}^2 = 0$ . Moreover,  $h^1(L) = h^1(\frac{b_1+2b_2}{3}F) = \frac{b_1+2b_2}{3} - 1 = \frac{b_1+2b_2-3}{3}$  and  $h^1(M) = h^1(\frac{2b_1+b_2}{3}F) = \frac{2b_1+b_2}{3} - 1 = \frac{2b_1+b_2-3}{3}$  so that  $h^1(L) + h^1(M) = b_1 + b_2 - 2 = n - 2$ . So all the singularities appearing in the singular reduced fibers are negligible.

For example, a branch fiber of type  $III$  in  $S$  consists of two tangent rational curves. We deduce that the singularities in the branch locus obtained by tangency of two components are negligible singularities. More precisely, if there is a branch fiber of type  $III$  on  $S$ , it induces a fiber of type  $IV^*$  (whose dual graph of curves is  $\tilde{E}_6$ ) of  $\mathcal{E} : X' \rightarrow W$ . We deduce that if the branch locus of a totally ramified triple cover contains two tangent curves, the triple cover has a singularities of type  $E_6$  (see Figure 4).

If a branch fiber on  $S$  is singular, then the corresponding fiber on  $X^\circ$  is obtained by a base change of order 3; [Mi2, Table VI.4.1] shows that effect of a base change on singular fibers of an elliptic fibration.

By [Mi2, Table VI.4.1], one immediately obtains that, if the branch contains a cusp, the cover has a singularity of type  $D_4$ , and if the branch has a simple triple point, the cover has an elliptic singularity. Notice that one recovers the singularities of type 1 by considering fibers of type  $I_m$  in the branch locus. □

Note that the list of Theorem 4.12 is not necessarily complete.

We now consider the other possibility of case (3) of Theorem 3.16, and hence we assume that  $D_1$  is a fiber, but at least one of the other components of the branch locus is strictly contained in a fiber.

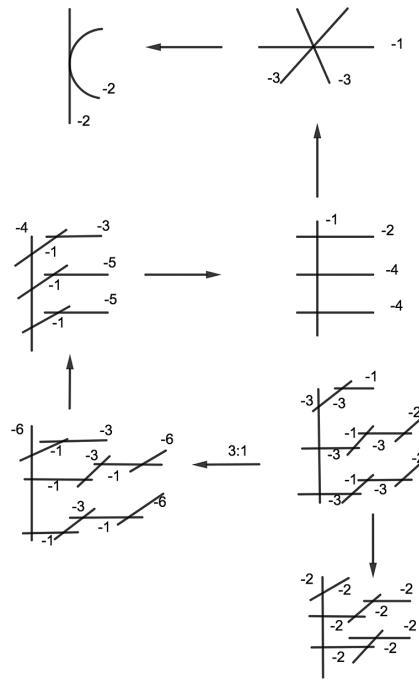


Figure 4.

In this case, the effect of the triple cover on the fibers strictly containing  $D_j$  is not the one of a base change of order 3. In the following proposition, we describe how the fibers change under a triple cover of this type.

**COROLLARY 4.13.** *Let  $D_1$  be a connected reducible (possibly nonreduced) component of the branch locus of a Galois triple cover  $f : X \rightarrow S$ . Let  $\varphi_{|D_1|} : S \rightarrow \mathbb{P}^1$  be the induced elliptic fibration (see the proof of Theorem 3.16). Let  $D_j \subset (F_S)_j$ , and we assume that  $D_j = (F_S)_j$  iff  $j \leq k$ . Given a fiber  $F_S$ , we denote by  $F_{X'}$  the corresponding fiber in the minimal model of the normalization of  $X$ . Then we have the following tables of types of singular fibers:*

$j \leq k$													
type $F_S$	type $F_{X'}$												
$I_m$	$I_{3m}$	<table border="1"> <thead> <tr> <th><math>j &gt; k</math></th> <th></th> </tr> <tr> <th>type <math>F_S</math></th> <th>type <math>F_{X'}</math></th> </tr> </thead> <tbody> <tr> <td><math>I_{3m}</math></td> <td><math>I_m</math></td> </tr> <tr> <td><math>IV</math></td> <td><math>IV</math></td> </tr> <tr> <td><math>IV^*</math></td> <td><math>I_0</math></td> </tr> </tbody> </table>	$j > k$		type $F_S$	type $F_{X'}$	$I_{3m}$	$I_m$	$IV$	$IV$	$IV^*$	$I_0$	
$j > k$													
type $F_S$	type $F_{X'}$												
$I_{3m}$	$I_m$												
$IV$	$IV$												
$IV^*$	$I_0$												
$I_m^*$	$I_{3m}^*$												
$II^*$	$I_0^*$												
$III^*$	$III$												
$II$	$I_0^*$												
$III$	$III^*$												
$IV$	$I_0$												

*Proof.* If  $j \leq k$ , then  $D_j = (F_S)_j$ , the fibers  $(F_S)_j$  are branch fibers, and the type of  $(F_X)_j$  is given in [Mil]. We already observed that if  $D_i$  is properly contained in a fiber  $F_S$ ,  $i \geq k$  and  $D_i$  is supported on an  $A_2$ -configuration.

Let us suppose that  $F_S$  properly contains  $r$  connected components  $D_i$ ,  $i \geq k$ . These are  $A_2$ -configurations, whose components are denoted by  $A_1^{(1)}, A_2^{(1)}, A_1^{(2)}, A_2^{(2)}, \dots, A_1^{(r)}, A_2^{(r)}$ .

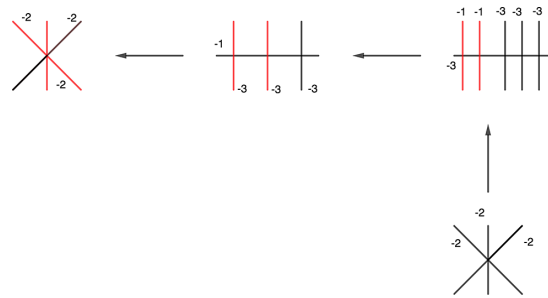


Figure 5.

Moreover,  $\sum_{h=1}^r (A_1^{(h)} + 2A_2^{(h)})/3$  necessarily has an integer intersection with all the components of  $F_S$ , and hence it is contained in the discriminant group of lattice associated with  $F_S$ . We conclude that  $F_S$  is necessarily one of the following:  $I_{3m}$ , with  $r = m$ ,  $IV$  with  $r = 1$ , and  $IV^*$  with  $r = 2$ .

There are  $mA_2$ -configurations contained in  $I_{3m}$ , and the birational inverse image of each of them in the minimal model  $X^\circ$  is a point. The remaining curves form an  $I_m$  fiber.

To obtain the minimal model in case  $IV^*$ , one first contracts the inverse image of the curves in the  $A_2$ -configurations each to a point. These three points lie on a  $(-1)$ -curve (which is the inverse image of the unique remaining curve in  $IV^*$ ). Contracting this curve, we obtain an  $I_0$  fiber.

The case  $IV$  is shown in Figure 5. □

#### 4.10 Irregular covers of K3 surfaces

Even if a K3 surface is a regular surface, it is of course possible to construct triple covers of K3 surfaces which are irregular surfaces. The easiest examples are provided by cases (1) and (3) of Theorem 3.16. Indeed, by case (1) of Theorem 3.16, the Galois triple cover of a K3 surface branched on nine  $A_2$ -configurations is an abelian surface. This case is effective, since it is known that there exist abelian surfaces admitting a symplectic automorphism of order 3 such that their quotient by this automorphism is birational to a K3 surface [Fu]. In Proposition 4.11, the irregularity of a surface  $X$  obtained as base change of order 3 on an elliptic fibration on a K3 surface  $S$  is computed. One checks that if  $n > 2$ , then the surface  $X$  is irregular. Once again, it is clear that there exist explicit examples of this situation; indeed, it suffices to construct a curve  $W$  (with the notation of Proposition 4.11) which is a Galois triple cover of  $\mathbb{P}^1$  branched on more than two points.

We observe that the previous constructions produce surfaces  $X$  whose Kodaira dimension is either 0 or 1. It is more complicated to construct examples of irregular covers of K3 surfaces which are surfaces of general type. This is essentially due to the difficulties in finding branch divisors on a K3 surface which are not supported on rational or elliptic curves, but such that the associated triple cover is irregular. Indeed, if  $X \rightarrow S$  is a triple cover of a K3 surface, then  $X$  is irregular if and only if at least one between  $L$  and  $M$  satisfies  $h^1(L) > 0$  or  $h^1(M) > 0$ , and there are not so many divisors with this property.

LEMMA 4.14. *Let  $D$  be a divisor on a K3 surface  $S$  such that  $-D$  is not effective. If  $h^1(D) \neq 0$ , then one of the following holds:*

- if  $D^2 < 0$ , then  $D^2 \leq -4$ , and if  $D^2 = -4$ , then  $D$  is effective;
- if  $D^2 = 0$  and  $D$  is nef, then  $D = nE$  where  $E$  is a genus 1 curve,  $n > 1$ , and  $h^1(S, D) = n - 1$ ; if  $D^2 = 0$  and  $D$  is not nef,  $D = M + F$  where  $M$  is its moving part and  $F$  is its fix part, and

$$F(F - 2D) > 0; \text{ and}$$

- if  $D^2 > 0$ , then  $D$  is not nef. In this case,  $D = M + F$  where  $M$  is its moving part and  $F$  is its fix part, and

$$F(F - 2D) > 0.$$

*Proof.* If  $D^2 < 0$ , then  $h^0(S, D) \leq 1$ . If  $D^2 = -2$ , then  $\chi(D) = 1$ , which would imply  $h^1(S, D) = 0$ .

If  $D^2 > 0$  and  $D$  is nef, then  $D$  is big and nef, and by the Kawamata–Viehweg vanishing theorem,  $h^1(S, D) = 0$ . Therefore, if  $D^2 > 0$  and  $h^1(D) \neq 0$ , then  $D$  is not nef. In particular, there is a fixed part  $F$  in  $|D|$  such that  $D = M + F$  with  $F^2 < 0$  and  $DF < 0$ . Moreover  $h^0(S, D) = h^0(S, D - F)$ . By the Riemann–Roch theorem,

$$\frac{1}{2}(D - F)^2 + 2 = h^0(S, D - F) = h^0(S, D) = \frac{1}{2}D^2 + 2 + h^1(S, D).$$

Finally, if  $D^2 = 0$  and  $D$  is not nef, the proof is the same as the case above. If  $D$  is nef, then  $D = nE$ , where  $E$  is a genus 1 curve. We recall the well-known fact  $h^1(S, nE) = n - 1$ . To recover this, note that  $L^2 = 0$  implies  $h^0(S, L) \geq 2$ , as by Serre duality  $h^2(S, L) = h^0(S, L^{-1}) = 0$  for the nontrivial nef line bundle  $L$  (intersect with an ample curve). It is enough to use the Riemann–Roch theorem for a K3 surface to compute  $\chi(\mathcal{O}_S(nE)) = 2$  and then use inductively the exact sequence in cohomology associated with the fundamental sequence of  $E$  tensorized with  $\mathcal{O}_S(nE)$ :

$$0 \rightarrow \mathcal{O}_S((n - 1)E) \rightarrow \mathcal{O}_S(nE) \rightarrow \mathcal{O}_E(nE|_E) \rightarrow 0. \quad \square$$

#### 4.15

In view of Lemma 4.14, we consider divisors on a K3 surface with very low self-intersection. In Theorems 4.16 and 4.17, we construct specific irregular surfaces of general type which are covers of K3 surfaces, and some notation are needed.

We denote by  $M_{(\mathbb{Z}/2\mathbb{Z})^4}$  a specific overlattice of  $\oplus_1^{15} A_1$ , constructed as follows: denoted by  $N_i$ , the 15 orthogonal classes with self-intersection  $-2$  which generate  $\oplus_1^{15} A_1$ , we add to the lattice  $\langle N_i \rangle$  the vectors

$$v_1 := \left( \sum_{i=1}^8 N_i \right) / 2, v_2 := \left( \sum_{i=1}^4 N_i + \sum_{j=9}^{12} N_j \right) / 2,$$

$$v_3 := (N_1 + N_2 + N_5 + N_6 + N_9 + N_{10} + N_{13} + N_{14}) / 2, v_4 := \left( \sum_{i=0}^7 N_{2i+1} \right) / 2.$$

The lattice  $M_{(\mathbb{Z}/2\mathbb{Z})^4} = \langle N_1, \dots, N_{15}, v_1, \dots, v_4 \rangle$  is an even negative definite lattice of discriminant  $(\mathbb{Z}/2\mathbb{Z})^7$ .

Similarly, we define  $M_{(\mathbb{Z}/2\mathbb{Z})^3}$  the overlattice of  $\oplus_1^{14} A_1$  obtained as above starting with 14 divisors  $N_i$  and adding the classes  $v_1, v_2, v_3$ . The lattice  $M_{(\mathbb{Z}/2\mathbb{Z})^3}$  is an even negative definite lattice of discriminant  $(\mathbb{Z}/2\mathbb{Z})^8$ .

Let us consider the rank 16 lattice  $R_{16} := \langle 6 \rangle \oplus M_{(\mathbb{Z}/2\mathbb{Z})^4}$ , and let us denote by  $H$  the generator of  $\langle 6 \rangle$ . By [GS3, Th. 8.3], there exists an even overlattice  $R'_{16}$  of index 2 of  $R_{16}$

obtained by adding to the lattice  $R_{16}$  the class  $(H + \sum_{i=1}^{15} N_i)/2$ . The discriminant group of  $R'_{16}$  is  $\mathbb{Z}/6\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^5$ . By [N, Th. 1.14.4 and Rem. 1.14.5],  $R'_{16}$  admits a primitive embedding in  $\Lambda_{K3}$ . So there is a K3 surface (indeed, a four-dimensional family of K3 surfaces) whose Néron–Severi group is isometric to  $R'_{16}$  lattice. Observe that this K3 surface has a model (given by the map  $\varphi_{|H|}$ ) as the complete intersection of a quadric and a cubic in  $\mathbb{P}^4$  with 15 nodes.

**THEOREM 4.16.** *Let  $S_{16}$  be a K3 surface such that  $NS(S_{16}) \simeq R'_{16}$ . On the surface  $S_{16}$ , there are a smooth curve  $G$  of genus 4 and 15 disjoint rational curves  $N_i$ ,  $i = 1, \dots, 15$ , such that  $GN_i = 1$ ,  $i = 1, \dots, 15$ . There exists a Galois triple cover  $\pi : X \rightarrow S_{16}$  branched on  $G \cup \cup_i N_i$ . The invariants of minimal model  $X^\circ$  of  $X$  are:*

$$p_g(X^\circ) = 6, \quad q(X^\circ) = 1, \quad c_1(X^\circ)^2 = 18.$$

*Proof.* By [GS3, Prop. 8.5], the divisors  $H$  and  $(H - \sum_{i=1}^{15} N_i)/2 \in NS(S_{16})$  are pseudoample. So  $(3H - \sum_{i=1}^{15} N_i)/2 \in NS(S_{16})$  is a pseudoample divisor whose self-intersection is 6. Hence, there is a smooth curve of genus 4, denoted by  $G$ , in  $\left| (3H - \sum_{i=1}^{15} N_i)/2 \right|$ . Moreover, one can assume that the divisors  $N_i$  are supported on irreducible rational curves (see [G2, Props. 2.3 and 5.1]). So, there are a smooth genus 4 curve  $G$  and 15 disjoint rational curves  $N_i$  on  $S$  such that  $GN_i = 1$ , for  $i = 1, \dots, 15$ . Set

$$B := G, \quad C := \sum_{i=1}^{15} N_i, \quad L := \frac{2B + C}{3} = H, \quad M := \frac{B + 2C}{3} = \left( H + \sum_{i=1}^{15} N_i \right) / 2.$$

The divisors  $B, C, L$ , and  $M$  satisfy the conditions in 2.10 and thus determine a triple cover of  $S_{16}$ . The branch locus is not smooth, and we consider the minimal resolution  $X'$  of  $X$ . The singularities of the branch locus are negligible, since they are transversal intersection of smooth curves, and thus the invariants of  $X'$  are obtained applying the formulae given in Proposition 3.12. Since  $L^2 = 6, M^2 = -6$ , and  $LM = 3$ , one obtains  $\chi(\mathcal{O}_{X'}) = 6, K_{X'}^2 = 3$ , and  $e(X') = 69$ . The surface  $X'$  is nonminimal, and the inverse images on  $X$  of the curves  $N_i$  are 15 disjoint exceptional curves  $E_i$ . We now prove that these are the unique  $(-1)$ -curves also even after their contraction. Indeed, suppose that there is a  $(-1)$ -curve on a contraction of  $X'$  which is mapped to a curve  $I$  on  $S_{16}$  with  $I \neq N_i$ . Then  $I$  is a rational curve, and in the triple cover, it has to split and intersect the exceptional curves  $E_i$ . Otherwise, the self-intersection of the inverse image of  $I$  is lower than  $-1$  (cf. the proof of Proposition 4.6). By direct inspection, one also sees that this case is not possible, since  $I$  has to split in three  $(-1)$ -curves which meet. So the minimal model  $X^\circ$  is obtained by  $X'$  contracting the curves  $E_i$ . So one obtains  $K_{X^\circ}^2 = 18$  and  $e(X^\circ) = 54$  and  $\chi(X) = \chi(X^\circ)$ . Moreover, one has

$$h^{1,0}(X^\circ) = h^{1,0}(X) = 0 + h^1(S, L) + h^1(S, M).$$

Since  $L = H$  is a pseudoample divisor,  $h^1(S, L) = 0$ . Since  $M^2 = -6$  and  $M$  is not effective, it follows  $h^1(S, M) = 1$ . One concludes that  $q(X^\circ) = q(X) = 1$  and  $p_g(X^\circ) = p_g(X) = \chi(X^\circ) = 6$ . □

Let us now consider the rank 15 lattice  $R_{15} := \langle 4 \rangle \oplus M_{(\mathbb{Z}/2\mathbb{Z})^3}$  with  $H$  the generators of  $\langle 4 \rangle$ . Let  $R'_{15}$  be the overlattice of  $R_{15}$  constructed by adding the class  $v = (H - \sum_{i=1}^{14} N_i)/2$ .

The lattice  $R'_{15}$  admits a primitive embedding in  $\Lambda_{K3}$ , and thus there exists a K3 surface  $S_{15}$  whose Néron–Severi group is isometric to  $R'_{15}$ .

**THEOREM 4.17.** *Let  $S_{15}$  be a K3 surface such that  $NS(S_{15}) \simeq R'_{15}$ . On the surface  $S_{15}$ , there are a smooth curve  $G$  of genus 2 and 14 disjoint rational curves  $N_i$ ,  $i = 1, \dots, 14$ , such that  $GN_i = 1$ ,  $i = 1, \dots, 14$ . There exists a Galois triple cover  $\pi : X \rightarrow S_{15}$  branched on  $G \cup \cup_i N_i$ . The invariants of the minimal model  $X^\circ$  of  $X$  are:*

$$p_g(X^\circ) = 4, \quad q(X^\circ) = 1, \quad c_1(X^\circ)^2 = 12.$$

*Proof.* The proof is analogous to the one of the previous proposition, but one has to choose  $G$  in the linear system  $\left| (3H - \sum_{i=1}^{14} N_i)/2 \right|$ . So one finds

$$B := G, \quad C := \sum_{i=1}^{14} N_i, \quad L := \frac{2B + C}{3} = H, \quad M := \frac{B + 2C}{3} = \left( H + \sum_{i=1}^{14} N_i \right) / 2,$$

and thus

$$L^2 = 4, \quad M^2 = -6, \quad LM = 2,$$

and one has to contract 14 curves to obtain  $X^\circ$  from the minimal resolution  $X'$  of  $X$ . So  $K_{X^\circ}^2 = K_{X'}^2 + 14 = -2 + 14 = 12$  and  $\chi(X^\circ) = \chi(X') = 5$ . As in the previous proof, one finds  $q(X^\circ) = 1$ . □

A different idea for finding irregular triple cover is to exploit the Albanese morphism and the Kummer construction, but the following remark shows that this approach is too naive.

**REMARK 4.18.** Due to the relation of an abelian surface  $A$  and its Kummer surface  $Km(A)$ , it is possible that a Galois triple cover  $Y \rightarrow A$  defines a triple cover  $X \rightarrow Km(A)$ . In particular, this happens if the involution  $\iota : a \rightarrow -a$  on  $A$  induces an involution on  $Y$ . In this case, one has the following diagram:

$$\begin{array}{ccc} Y & \xrightarrow{3:1} & A \\ | & & | \\ \downarrow 2:1 & & \downarrow 2:1 \\ X & \xrightarrow{3:1} & Km(A). \end{array}$$

Since  $q(A) = 2$ , it is easier to find cover  $Y$  such that  $q(Y) \neq 0$ . Nevertheless, let  $Y$  be a surface such that the Albanese variety  $Alb(Y)$  coincides with  $A$  (so, in particular, it has dimension 2 and the Albanese map is  $3 : 1$ ), then it holds

if  $\iota$  lifts to an involution  $\iota_Y$  of  $Y$ , the quotient  $X := Y/\iota_Y$  is a regular surface.

Indeed, the Albanese surface  $A$  is defined as  $H^0(Y, \Omega_1)^\vee / H_1(Y, \mathbb{Z})$ , and  $\iota$  acts on the space  $H^0(Y, \Omega_1)$  as  $-1$ . Thus,  $\iota_Y$  does not preserve the 1-holomorphic form of  $Y$ .

**§5. Triple cover of K3 surfaces: The split non-Galois case**

Now, we analyze the non-Galois triple covers  $f : X \rightarrow S$  of a K3 surface under the assumption that the Tschirnhausen bundle  $\mathcal{E}$  splits. We provide an example of such a triple cover for all possible Kodaira dimensions. Let  $\mathcal{E}$  be a direct sum of two line bundles  $\mathcal{L}^{-1}$  and  $\mathcal{M}^{-1}$  so that  $\mathcal{E}^\vee = \mathcal{L} \oplus \mathcal{M}$ . We have already observed in diagram (2.2) that the triple cover  $f$  is totally branched over  $D'$  and simply branched over  $D''$ . All the other covers in the diagram are Galois covers.

Since  $\mathcal{E}^\vee = \mathcal{L} \oplus \mathcal{M}$ , we have that

$$2L + 2M = 2D' + D'' \text{ is the branch locus of the triple cover } f : X \rightarrow S,$$

with  $D'' \neq 0$ , and there exist four effective divisors  $B', C', B''$ , and  $C''$  such that

$$L = \frac{2B' + C'}{3} + \frac{B''}{2}, \quad M = \frac{B' + 2C'}{3} + \frac{C''}{2}.$$

So  $L + M = B' + C' + (B'' + C'')/2$  is the branching data of an  $\mathfrak{S}_3$  cover of  $S$ , that is, the line bundle  $\mathcal{O}_S(L + M)$  returns the geometric line bundle  $\mathbb{L}$  of [CP, Th. 6.1]. This  $\mathfrak{S}_3$ -cover is the Galois closure of the non-Galois triple cover  $X \rightarrow S$ . Notice that in [CP, Th. 6.1], one assumes the branch locus to be smooth, but the results extend also to the nonsmooth case.

**5.1 A split non-Galois triple cover of a K3 surface with a K3 surface.**

We construct a split but not Galois triple cover  $f : X \rightarrow S$  such that  $S$  is a K3 surface and  $X$  is a singular surface whose minimal resolution is still a K3 surface. We refer to the diagram (2.2) for the notation.

Let  $Z$  be a K3 surface such that  $\mathfrak{S}_3 \subset \text{Aut}(Z)$  and  $\mathfrak{S}_3$  acts symplectically on  $Z$ . Then the quotient surface  $Z/\mathfrak{S}_3$  has three singularities of type  $A_2$  and eight singularities of type  $A_1$  (see [X, p. 78, case 6]).

The resolution of  $Z/\mathfrak{S}_3$  is a K3 surface  $S$ , which admits a Galois  $\mathfrak{S}_3$  cover, branched on the Jung–Hirzebruch strings which resolve the singularities. Let us denote by  $B_i \cup C_i$ ,  $i = 1, 2, 3$ , the  $i$ th  $A_2$  configuration and by  $N_j$ ,  $j = 1, \dots, 8$ , the  $j$ th  $A_1$  configuration. Then  $L = \left(\sum_{i=1}^3 2B_i + C_i\right)/3 + \sum_{j=1}^4 (N_j)/2$  and  $M = \left(\sum_{i=1}^3 B_i + 2C_i\right)/3 + \sum_{j=4}^8 (N_j)/2$ . The K3 surface  $S$  admits a  $2 : 1$  cover branched along  $\cup_{j=1}^8 N_j$ , which is the nonminimal surface  $W$ , whose minimal model is a K3 surface  $W^\circ$ . There are six  $A_2$ -configurations on  $W^\circ$ , inverse image of the three  $A_2$ -configurations in  $S$ . These six  $A_2$ -configurations form a 3-divisible set. The Galois triple cover of  $W^\circ$  branched on these six  $A_2$ -configurations is a nonminimal surface, whose minimal model is  $Z$ . The quotient of  $Z$  by an involution in  $\mathfrak{S}_3$  is the singular surface  $X$ , whose minimal resolution is another K3 surface,  $X^\circ$ . The surface  $X$  is by construction the non-Galois triple cover of  $S$  associated with  $\mathcal{E}^\vee = \mathcal{L}^{-1} \otimes \mathcal{M}^{-1}$ . The total ramification of  $S$  is on  $D' = \sum_{i=1}^3 (B_i + C_i)$ , and the simple one is on  $D'' = \sum_{i=1}^8 N_i$ .

**5.2 A split non-Galois triple cover of a K3 surface with a properly elliptic surface.**

Let us consider a K3 surface  $S$  endowed with an elliptic fibration  $\mathcal{E} : S \rightarrow \mathbb{P}^1$ . Let us consider  $g : C \rightarrow \mathbb{P}^1$  a split non-Galois triple cover of  $\mathbb{P}^1$ . This can be constructed by considering  $2k$  points  $P_i$ ,  $i = 1, \dots, k$ , and  $2h$  points  $Q_j$ ,  $j = 1, \dots, 2h$ .

Then one uses, as triple cover data,  $B' = \sum_{i=1}^k P_i$ ,  $C' = \sum_{i=k+1}^{2k} P_i$ ,  $B'' = \sum_{j=1}^{2r} Q_j$ , and  $C'' = \sum_{j=2r+1}^{2h} Q_j$  with  $r \leq h$ . So  $L = \frac{\sum_{i=1}^k 2P_i + P_{i+k}}{3} + \frac{\sum_{j=1}^{2r} Q_j}{2}$ ,  $M = \frac{\sum_{i=1}^k P_i + 2P_{i+k}}{3} + \frac{\sum_{j=2r+1}^{2h} Q_j}{2}$ , and there exists a split non-Galois triple cover of  $\mathbb{P}^1$  totally branched on  $\cup_{i=1}^{2k} P_i$  and simply branched on  $\cup_{j=1}^{2h} Q_j$ . The genus of the curve  $C$  is given by  $2g(C) - 2 = -6 + 2(2k) + 2h$ , so  $g(C) = 2k + h - 2 \geq 1$ .



Now, we consider the fiber product

$$\begin{array}{ccc}
 S \times_{\mathbb{P}^1} C & \longrightarrow & S \\
 \downarrow & & \downarrow \mathcal{E} \\
 C & \xrightarrow{g} & \mathbb{P}^1.
 \end{array}$$

If the fibers of  $\mathcal{E}$  over the points  $P_i$  and  $Q_j$  are smooth, the surface  $X := S \times_{\mathbb{P}^1} C$  is smooth and it is a triple non-Galois cover of  $S$  totally branched over  $\cup_i \mathcal{E}^{-1}(P_i)$  and simply branched over  $\cup_j \mathcal{E}^{-1}(Q_j)$ . The fibration  $\mathcal{E}$  induces a fibration  $\mathcal{E}_X : X \rightarrow C$  whose generic fiber is a smooth genus 1 curve and which has  $2h$  fibers with multiplicity 2.

It holds  $h^{1,0}(X) \geq g(C) \geq 1$ . The surface  $X$  is necessarily proper elliptic, that is,  $\kappa(X) = 1$  (cf. Proposition 2.9 and [Mi2, Lem. III.4.6]).

### 5.3 A split non-Galois triple cover of a K3 surface with a surface of general type.

Let  $S$  be a K3 surface which admits an even set of  $k$  disjoint rational curves  $N_i$ , so either  $k = 8$  or  $k = 16$ . There exists a pseudoample divisor  $H$  which is contained in  $\langle N_1, \dots, N_k \rangle^{\perp_{NS(S)}}$  with  $H^2 = 2h$  for a positive number  $h$ .

Then, putting  $C' = 3H, B' = 0, B'' = \sum_i N_i$ , and  $C'' = 0$ , one obtains the data of a split non-Galois triple cover:

$$L = \frac{C'}{3} + \frac{B''}{2} = H + \left( \sum_{i=1}^k N_i \right) / 2, \quad M = \frac{2C'}{3} = 2H.$$

We consider the rank 2 vector bundle  $\mathcal{E} = \mathcal{O}(-L) \oplus \mathcal{O}(-M)$ . Since  $S^3(\mathcal{E}^\vee) \otimes \wedge^2 \mathcal{E} = \mathcal{O}(2L - M) \oplus \mathcal{O}(2M - L) \oplus \mathcal{O}(L) \oplus \mathcal{O}(M)$  admits global sections,  $\mathcal{E}$  is the Tschirnhausen bundle of a triple cover (see Theorem 2.1).

This triple cover, denoted by  $f : X \rightarrow S$ , is totally ramified on  $C'$  (i.e., on a curve contained in the linear system  $|3H|$ ) and simply ramified on  $\cup_i N_i$ .

With the notation of (2.2), one has that  $W$  is a nonminimal surface and its minimal model is obtained by contracting  $k(-1)$ -curves. The minimal model of  $W$  is a K3 surface or an abelian surface according to the fact that  $k = 8$  or  $k = 16$ . In particular, denoted by  $E_i$  the  $(-1)$ -curves on  $W$ , one has

$$K_W = -\sum_{i=1}^k E_i, \quad K_W K_W = -k, \quad \chi(\mathcal{O}_W) = \frac{16-k}{4}, \quad h^{1,0}(W) = \frac{k-8}{4}, \quad e(W) = 48 - 2k.$$

The Galois triple cover  $\beta_2 : Z \rightarrow W$  is branched on a curve in the linear system  $|\beta_1^*(3H)|$ . Let us assume that the branch locus is a smooth curve in this linear system. Hence,  $\beta_2 : Z \rightarrow W$  is a smooth Galois triple cover, whose data are  $(B, C, L, M) = (0, \beta_1^* C', \beta_1^* H, 2\beta_1^* H)$  and whose invariant are

$$\begin{aligned}
 \chi(\mathcal{O}_Z) &= 3\chi(\mathcal{O}_W) + \frac{1}{2}(\beta_1^* H)^2 + \frac{1}{2}(2\beta_1^* H)^2 = && \frac{48-3k}{4} + 10h, \\
 K_Z^2 &= 3K_W^2 + 2(\beta_1^* H)^2 + 2(2\beta_1^* H)^2 + 2(\beta_1^* H)^2 = && -3k + 48h, \\
 h^i(Z, \mathcal{O}_Z) &= h^1(W, \mathcal{O}_W) + h^1(W, \beta_1^* H) + h^1(W, 2\beta_1^* H) = && \frac{k-8}{4}, \\
 e(Z) &= 3e(W) + 4((\beta_1^* H)^2 + (2\beta_1^* H)^2) - 2(\beta_1^* H)^2 = && 144 - 6k + 72h.
 \end{aligned}$$

We used that  $\beta_1^* D \beta_1^* D = 2D^2$  for every divisor  $D \in \text{Pic}(S)$  and that  $H$  is big and nef, so that  $\beta_1^* H$  is big and nef, and hence the vanishing theorems hold. Moreover, one obtains

$$h^{2,0}(Z) = \chi(Z) - 1 + h^{1,0}(Z) = 9 - \frac{k}{2} + 10h.$$

We want to apply the formulae [BHPV, Chap. 5, §22] to the double cover, branched on  $k$  rational curves  $Z \rightarrow X$ . The branch locus  $J$  is such that  $-2k = (K_X + J)J$  by adjunction. This implies that  $-k = K_X I + 2I^2$  where  $I$  is a divisor such that  $2I = J$ .

By construction,  $f^{-1}(N_i) = M_i \cup M'_i$ , where both  $M_i$  and  $M'_i$  are isomorphic to  $N_i$ , one of them has multiplicity 2 (because  $N_i$  is contained in the simple ramification), and  $M_i$  and  $M'_i$  are disjoint. So  $f^*(N_i) = M_i + 2M'_i$ .

Since  $f$  is a 1 : 1 map restricted to  $M_i$  and  $M'_i$ , one obtains  $f_*(M_i) = f_*(M'_i) = N_i$  and, by the projection formula,

$$M_i^2 = (M_i + 2M'_i)M_i = f^*(N_i)M_i = N_i f_*(M_i) = N_i^2 = -2 \Rightarrow M_i^2 = -2,$$

$$2(M'_i)^2 = (M_i + 2M'_i)M'_i = f^*(N_i)M'_i = N_i f_*(M'_i) = N_i^2 = -2 \Rightarrow (M'_i)^2 = -1.$$

The branch locus of the cover  $\alpha$  consists of the curves  $M_i$ , that is, with the previous notation

$$J = \sum_{i=1}^k M_i \text{ so } I = \left( \sum_{i=1}^k M_i \right) / 2 \text{ and } I^2 = -\frac{k}{2}.$$

By  $-k = K_X I + 2I^2$ , it follows that  $K_X I = 0$ .

Hence,

$$\chi(Z) = \frac{48 - 3k}{4} + 10h = 2\chi(X) + \frac{1}{2}K_X I + \frac{1}{2}I^2 = 2\chi(X) - \frac{k}{4},$$

$$K_Z^2 = -3k + 48h = 2K_X^2 + 4K_X I + 2I^2 = 2K_X^2 - k,$$

$$e(Z) = 144 - 6k + 72h = 2e(X) + 2K_X I + 4I^2 = 2e(X) - 2k.$$

So the invariants of the surface  $X$  are

$$e(X) = 72 + 36h - 2k, \quad K_X^2 = -k + 24h, \quad \chi(X) = (24 - k)/4 + 5h.$$

The surface  $X$  is nonminimal, since it contains at least  $k(-1)$ -curves. We observe that in any case  $K_X^2 > 0$  and  $\kappa(X) \geq 0$ , and hence  $X$  is of general type.

We notice that  $h^{2,0}(Z) \geq 11$  and by choosing  $h$  big enough  $h^{2,0}(Z)$  and  $h^{2,0}(X)$  are arbitrarily big.

**§6. Triple covers of K3 surfaces: The nonsplit case**

The most general situation for a triple cover  $f: X \rightarrow S$  is when the Tschirnhausen bundle  $\mathcal{E}$  is indecomposable, in particular the cover is non-Galois. The main question that one has to address is the existence of the Tschirnhausen bundle  $\mathcal{E}$ ; and this boils down to the study of rank 2 indecomposable vector bundle  $\mathcal{E}$  on a K3 surface which satisfies the further condition  $H^0(Y, S^3 \mathcal{E}^\vee \otimes \wedge^2 \mathcal{E}) \neq 0$  given in Theorem 2.1.

A standard approach (see, e.g., [PP, §2]) is to construct the Tschirnhausen bundle  $\mathcal{E}$  exploiting the Cayley–Bacharach property (CB) of some zero-subscheme (see also [Fr], p. 36 and [Ca]), which we recall for simplicity.

**THEOREM 6.1** [HL, Th. 5.1.1]. *Let  $Z \subset S$  be a local complete intersection of codimension 2, and let  $\mathcal{L}$  and  $\mathcal{M}$  be line bundles on  $S$ . Then there exists an extension*

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{M} \otimes \mathcal{I}_Z \rightarrow 0$$

such that  $\mathcal{E}$  is locally free if and only if the pair  $(\mathcal{L}^{-1} \otimes \mathcal{M} \otimes K_S, Z)$  has the Cayley–Bacharach property.

**(CB)** *If  $Z' \subset Z$  is a subscheme with  $\ell(Z') = \ell(Z) - 1$  and  $s \in H^0(S, \mathcal{L}^{-1} \otimes \mathcal{M} \otimes K_S)$  with  $s|_{Z'} = 0$ , then  $s|_Z = 0$ .*

We consider

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{M} \otimes \mathcal{I}_Z \rightarrow 0, \quad (6.1)$$

where  $\mathcal{L}$  and  $\mathcal{M}$  are line bundles on a K3 surface  $S$  and  $Z$  a zero-cycle.

Notice that if  $Z = \emptyset$  and the sequence (6.1) splits, then  $\mathcal{E}^\vee = \mathcal{L} \oplus \mathcal{M}$  and we are back to the cases treated in the previous sections. Therefore, we would like to assume that  $Z \neq \emptyset$  and that the sequence (6.1) does not split.

First, we discuss some criteria which assure the existence of the triple cover associated with (6.2), then we apply them to some possible choices of the triple  $(\mathcal{L}, \mathcal{M}, Z)$ .

The following proposition gives a conditions on  $\mathcal{L}^\vee \otimes \mathcal{M}$  which assure the existence of the sequence (6.1).

**PROPOSITION 6.2.** *If  $h^0(S, \mathcal{L}^\vee \otimes \mathcal{M}) = 0$  and  $h^1(S, \mathcal{L}^\vee \otimes \mathcal{M}) \neq 0$ , then  $\text{Ext}^1(\mathcal{M} \otimes \mathcal{I}_Z, \mathcal{L}) \neq 0$  and the extension (6.1) exists. In particular, if  $h^1(S, \mathcal{L}^\vee \otimes \mathcal{M}) = 1$ , the extension is unique.*

*Proof.* Let  $\mathcal{G} := \mathcal{L}$  and  $\mathcal{F} = \mathcal{L}^\vee \otimes \mathcal{M} \otimes \mathcal{I}_Z$ . We want to prove that  $\text{Ext}^1(\mathcal{F} \otimes \mathcal{G}, \mathcal{G}) \neq 0$ . By Serre duality, we have

$$\text{Ext}^1(\mathcal{F} \otimes \mathcal{G}, \mathcal{G}) = \text{Ext}^1(\mathcal{F}, \mathcal{O}) = H^1(\mathcal{F}) = H^1(\mathcal{L}^\vee \otimes \mathcal{M} \otimes \mathcal{I}_Z).$$

Now, consider the fundamental exact sequence of the scheme  $Z$

$$0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_Z \rightarrow 0.$$

Tensorized by  $\mathcal{L}^\vee \otimes \mathcal{M}$ , we get

$$0 \rightarrow \mathcal{I}_Z \otimes \mathcal{L}^\vee \otimes \mathcal{M} \rightarrow \mathcal{L}^\vee \otimes \mathcal{M} \rightarrow \mathcal{O}_Z \otimes \mathcal{L}^\vee \otimes \mathcal{M} \rightarrow 0.$$

Since  $H^0(S, \mathcal{L}^\vee \otimes \mathcal{M}) = 0$ , one obtains  $h^0(Z, \mathcal{O}_Z \otimes \mathcal{L}^\vee \otimes \mathcal{M}) = 0$ , and subsequently the long exact sequence in cohomology gives

$$0 \rightarrow H^1(\mathcal{I}_Z \otimes \mathcal{L}^\vee \otimes \mathcal{M}) \rightarrow H^1(\mathcal{L}^\vee \otimes \mathcal{M}) \rightarrow 0.$$

So,

$$\dim \text{Ext}^1(\mathcal{F} \otimes \mathcal{G}, \mathcal{G}) = h^1(\mathcal{I}_Z \otimes \mathcal{L}^\vee \otimes \mathcal{M}) = h^1(\mathcal{L}^\vee \otimes \mathcal{M}),$$

and the claim follows.  $\square$

We observe that by Serre duality, on a K3 surface,

$$h^1(S, \mathcal{L}^\vee \otimes \mathcal{M}) = h^1(S, \mathcal{L} \otimes \mathcal{M}^\vee),$$

so one can substitute the hypothesis  $h^1(\mathcal{L}^\vee \otimes \mathcal{M}) \neq 0$  with  $h^1(\mathcal{L} \otimes \mathcal{M}^\vee) \neq 0$ .

**THEOREM 6.3.** *Let  $S$  be a K3 surface, let  $Z$  be a nonempty zero-dimensional scheme on  $S$ , and let  $\mathcal{L}, \mathcal{M}$  be two line bundles such that:*

- $h^0(S, \mathcal{L}^\vee \otimes \mathcal{M}) = 0$ ;
- $h^1(S, \mathcal{L}^\vee \otimes \mathcal{M}) = h^1(S, \mathcal{L} \otimes \mathcal{M}^\vee) \neq 0$ ; and
- $h^0(S, \mathcal{L}^{\otimes 2} \otimes \mathcal{M}^\vee) \geq 1$ .

*Then there exists a triple cover  $X \rightarrow S$  with Tschirnhausen  $\mathcal{E}$ , defined by (6.1) for any possible choice of  $Z$ .*

*Proof.* The condition (CB) is automatically satisfied if  $h^0(\mathcal{L}^\vee \otimes \mathcal{M}) = 0$  (see [Fr, Th. 12]), and by Proposition 6.2,  $\mathcal{E}$  exists and is locally free, and its dual as well. To assure the existence of the triple cover, we have to prove that

$$h^0(S, S^3 \mathcal{E}^\vee \otimes \bigwedge^2 \mathcal{E}) \neq 0.$$

We apply the Eagon–Northcott complex (see, e.g., [E, Appendix 2] and [CT, Lem. 4.7]) to the sequence (6.1), and we obtain

$$0 \rightarrow \mathcal{L} \otimes S^2 \mathcal{E}^\vee \rightarrow S^3 \mathcal{E}^\vee \rightarrow \mathcal{M}^3 \otimes \mathcal{I}_Z^3 \rightarrow 0.$$

Now, let us tensorize the previous sequence by  $\Lambda^2 \mathcal{E} \cong \mathcal{L}^{-1} \otimes \mathcal{M}^{-1}$ , and we get

$$0 \rightarrow S^2 \mathcal{E}^\vee \otimes \mathcal{M}^{-1} \rightarrow S^3 \mathcal{E}^\vee \otimes \Lambda^2 \mathcal{E} \rightarrow \mathcal{M}^2 \otimes \mathcal{L}^{-1} \otimes \mathcal{I}_Z^3 \rightarrow 0.$$

So, if we prove that  $S^2 \mathcal{E}^\vee \otimes \mathcal{M}^{-1}$  has global section, we are done. To do so, we apply the Eagon–Northcott complex to the sequence (6.1), and we tensorize it by  $\mathcal{M}^{-1}$ . We have

$$0 \rightarrow \mathcal{L} \otimes \mathcal{E}^\vee \otimes \mathcal{M}^{-1} \rightarrow S^2 \mathcal{E}^\vee \otimes \mathcal{M}^{-1} \rightarrow \mathcal{M} \otimes \mathcal{I}_Z^2 \rightarrow 0.$$

As a last step, we show that  $\mathcal{L} \otimes \mathcal{E}^\vee \otimes \mathcal{M}^{-1}$  has global section. This is true by hypothesis and by the short exact sequence

$$0 \rightarrow \mathcal{L}^2 \otimes \mathcal{M}^{-1} \rightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{M}^{-1} \rightarrow \mathcal{L} \otimes \mathcal{I}_Z \rightarrow 0$$

obtained tensorizing the sequence (6.1) by  $\mathcal{L} \otimes \mathcal{M}^{-1}$ .

Therefore, we have  $h^0(S^3 \mathcal{E}^\vee \otimes \bigwedge^2 \mathcal{E}) \geq 1$ . □

Now, we give some explicit examples, choosing the line bundles  $\mathcal{L}$  and  $\mathcal{M}$ . Let  $C$  be a smooth genus 1 curve on  $S$ , and hence a fiber of an elliptic fibration on  $S$ . We pose

$$\mathcal{L} = \mathcal{O}_S(nC), \quad \mathcal{M} = \mathcal{O}_S(mC).$$

If  $-n + m < 0$ , then  $h^0(\mathcal{L}^\vee \otimes \mathcal{M}) = 0$ .

If  $n - m \geq 2$ , then  $h^1(\mathcal{L} \otimes \mathcal{M}^\vee) \neq 0$  and  $h^0(\mathcal{L}^{\otimes 2} \otimes \mathcal{M}) \geq 1$ . Hence is  $n \geq m + 2$  the hypothesis of Theorem 6.3 are satisfied and hence there exists the triple cover  $X \rightarrow S$ . If  $n = m + 2$ , then the extension given by (6.1) is unique.

So we now discuss the triple cover associated with the sequence

$$0 \rightarrow \mathcal{O}_S(nC) \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z(mC) \rightarrow 0 \quad n \geq m + 2, \tag{6.2}$$

and in particular, we analyze it according to the choice of zero-scheme  $Z$ .

LEMMA 6.4. *Let  $S$  be an elliptic K3 surface with elliptic fibration  $\varphi_{|C|} : S \rightarrow \mathbb{P}^1$ . Let us assume that  $Z$  is a zero-cycle on  $S$  such that  $\ell(Z) = 1$ .*

- If  $m \geq 2$ , then  $h^0(\mathcal{I}_Z(mC)) \neq 0$  and  $h^1(\mathcal{I}_Z(mC)) \neq 0$ .
- If  $m = 1$ , then  $h^0(\mathcal{I}_Z(C)) = 1$ , and  $h^1(\mathcal{I}_Z(C)) = 0$ .

*Proof.* Since  $\ell(Z) = 1$ , the subscheme  $Z$  consists in a single point  $p$ . By

$$0 \rightarrow \mathcal{I}_p(mC) \rightarrow \mathcal{O}_S(mC) \rightarrow \mathcal{O}_p(mC) \rightarrow 0,$$

one obtains the long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{I}_p(mC)) \rightarrow H^0(\mathcal{O}_S(mC)) \rightarrow H^0(\mathcal{O}_p(mC)) \rightarrow \\ \rightarrow H^1(\mathcal{I}_p(mC)) \rightarrow H^1(\mathcal{O}_S(mC)) \rightarrow 0, \end{aligned} \tag{6.3}$$

which is

$$0 \rightarrow H^0(\mathcal{I}_p(mC)) \rightarrow \mathbb{C}^{m+1} \rightarrow \mathbb{C} \rightarrow H^1(\mathcal{I}_p(mC)) \rightarrow \mathbb{C}^{m-1} \rightarrow 0.$$

This yields at once the first statement. For the second one, let us insert the value  $m = 1$  in (6.3) and get

$$0 \rightarrow H^0(\mathcal{I}_p(C)) \rightarrow H^0(\mathcal{O}(C)) \xrightarrow{\psi} H^0(\mathcal{O}_p(C)) \cong \mathbb{C} \rightarrow H^1(\mathcal{I}_p(C)) \rightarrow 0.$$

By [H, Prop. 3.10], the linear system  $|C|$  is base-point-free, and hence  $\psi$ , which is an evaluation map, is not the zero map, and we have to conclude the proof.  $\square$

LEMMA 6.5. *Let  $S$  be an elliptic K3 surface with elliptic fibration  $\varphi_{|C|} : S \rightarrow \mathbb{P}^1$ . Let us assume that  $Z$  is a zero-cycle on  $S$  such that  $\ell(Z) = 2$ .*

- If  $m \geq 2$ , then  $h^0(\mathcal{I}_Z(mC)) \neq 0$  and  $h^1(\mathcal{I}_Z(mC)) \neq 0$ .
- If  $m = 1$ , then  $h^0(\mathcal{I}_Z(C)) = h^1(\mathcal{I}_Z(C)) = 0$  if  $Z$  consists of two distinct smooth points  $z_1$  and  $z_2$  which do not lie on the same fiber of the fibration.
- If  $m = 1$ , then  $h^0(\mathcal{I}_Z(C)) = h^1(\mathcal{I}_Z(C)) = 1$  if  $Z$  consists of two distinct smooth points  $z_1$  and  $z_2$  which lie on the same fiber of the fibration or  $Z$  is a single point.

*Proof.* The first statement is proved exactly in the same way as in Lemma 6.4.

Let  $m = 1$ , then  $H^0(\mathcal{O}_S(C)) \cong \mathbb{C}^2$  and also  $H^0(\mathcal{O}_Z(C)) \cong \mathbb{C}^2$ . In the long exact sequence

$$0 \rightarrow H^0(\mathcal{I}_Z(C)) \rightarrow H^0(\mathcal{O}_S(C)) \xrightarrow{\psi} H^0(\mathcal{O}_Z(C)) \rightarrow H^1(\mathcal{I}_Z(C)) \rightarrow H^1(\mathcal{O}_S(C)) = 0,$$

the map  $\psi$  is the evaluation map  $ev : s \mapsto s(x)$  with  $x \in Z$ , which is the zero map if and only if  $Z$  is not the base locus of  $|C|$ . By [H, Prop. 3.10], the linear system  $|C|$  is base-point-free. This yields that  $h^0(\mathcal{I}_Z(C)) \leq 1$ .

There are two cases:

- (1) There is a section  $s \in H^0(\mathcal{O}_S(C))$  which passes through  $Z$ , and in this case,  $Z$  consists either of two distinct smooth points  $z_1$  and  $z_2$  which lie on the same fiber of  $\varphi_{|C|}$  or  $Z$

is a single double point. In this case,

$$H^0(\mathcal{I}_Z(C)) \cong \langle s \rangle.$$

- (2) No single section passes through  $Z$ , and in this case,  $Z$  consists of two distinct smooth points  $z_1$  and  $z_2$  which do not lie on the same fiber of the fibration and  $H^0(\mathcal{I}_Z(C)) = 0$ .

□

LEMMA 6.6. *The total Chern classes of the vector bundle  $\mathcal{E}$ , determined by the extension (6.2), is  $c(\mathcal{E}) = (1, (n + m)C, \ell(Z))$ .*

*Proof.* We use  $ch(C \otimes \mathcal{I}_Z) = ch(C)ch(\mathcal{I}_Z)$ .

Recalling that

$$ch(V) = (rk(V), c_1(V), \frac{1}{2}(c_1^2(V) - 2c_2(V))),$$

one has  $ch(C) = (1, C, 0)$ ,  $ch(\mathcal{I}_Z) = (1, 0, -\ell(Z))$ , and thus

$$ch(C \otimes \mathcal{I}_Z) = (1, C, -\ell(Z)), \text{ so } c(C \otimes \mathcal{I}_Z) = (1, C, \ell(Z)).$$

Since  $c(\mathcal{E}) = c(nC)c(mC \otimes \mathcal{I}_Z)$  (see, e.g., [HL, §5]), one obtains

$$c(\mathcal{E}) = (1, nC, 0)(1, mC, \ell(Z)) = (1, (n + m)C, \ell(Z)).$$

□

LEMMA 6.7. *Let  $S$  be an elliptic K3 surface with elliptic fibration  $\varphi_{|C|} : S \rightarrow \mathbb{P}^1$ .*

*Let  $(n, m, \ell(Z)) = (3, 1, 2)$ .*

*If  $Z$  is supported on two points  $z_1$  and  $z_2$  which do not lie on the same fiber of the fibration  $\varphi_{|C|} : S \rightarrow \mathbb{P}^1$ , then  $h^0(\mathcal{E}) = 4$  and  $h^1(\mathcal{E}) = 2$ .*

*Proof.* Suppose  $(n, m, \ell(Z)) = (3, 1, 2)$ . The computation of  $h^i(\mathcal{E})$  is based on the sequence

$$0 \rightarrow H^0(3C) \cong \mathbb{C}^4 \rightarrow H^0(\mathcal{E}) \rightarrow H^0(\mathcal{I}_Z(C)) \rightarrow H^1(3C) \cong \mathbb{C}^2 \rightarrow H^1(\mathcal{E}) \rightarrow H^1(\mathcal{I}_Z(C)) \rightarrow 0.$$

If  $Z$  consists of two distinct smooth points  $z_1$  and  $z_2$  which do not lie on the same fiber of the fibration, then by Lemma 6.5, we have  $H^0(\mathcal{I}_Z(C)) = H^1(\mathcal{I}_Z(C)) = 0$  and we have  $(h^0(\mathcal{E}), h^1(\mathcal{E})) = (4, 2)$ . □

PROPOSITION 6.8. *Let us suppose that  $S$  is an elliptic K3 surface with elliptic fibration  $\varphi_{|C|} : S \rightarrow \mathbb{P}^1$  and that  $(n, m, \ell(Z)) = (3, 1, 2)$ . Moreover, let us assume that  $Z$  is supported on two points  $z_1$  and  $z_2$  which do not lie on the same fiber of the fibration  $\varphi_{|C|} : S \rightarrow \mathbb{P}^1$ . Then there exists a properly elliptic surface  $X$  which is a non-Galois triple cover of an elliptic K3 surface such that  $(h^{1,0}(X), h^{2,0}(X)) = (3, 6)$ .*

*Proof.* The existence of the triple cover follows by Theorem 6.3, and hence the surface  $X$  exists. Moreover, the birational numerical invariants of  $X$  are given by Proposition 2.6(i) with the information given in Lemma 6.7.

By Proposition 2.2, the branch divisor of  $f : X \rightarrow S$  is given by  $\Lambda^2 \mathcal{E}^{-2} \simeq \mathcal{O}_S(8C)$ . Finally, by Proposition 2.9,  $X$  is a properly elliptic surface. □

REMARK 6.9. The triple cover  $X \rightarrow S$  is not induced by a base change  $g : C \rightarrow \mathbb{P}^1$  (for a certain curve  $C$ ) as in §5.2 and in Proposition 4.11 because otherwise  $\mathcal{E}$  would split.

Another possible choice of  $\mathcal{L}$  and  $\mathcal{M}$  in the sequence (6.1) is presented in the following corollary.

**COROLLARY 6.10.** *Let  $S$  be a K3 surface with an irreducible curve  $C$  of self-intersection  $2d > 0$  such that there exist  $h$  disjoint rational curves  $R_i$ 's, which are disjoint also from  $C$ . If  $h \leq 10$ , a K3 surface with this configuration of curves exists.*

*If  $2 \leq h \leq 10$  and  $(9h - 1)/4 \leq d \leq 4h - 3$ , there exists a non-Galois triple cover  $f : S \rightarrow X$  whose Tschirnhausen is determined by the sequence*

$$0 \rightarrow \mathcal{O}_S \left( C - \sum_{i=1}^h R_i \right) \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{I}_Z \left( \sum_{i=1}^h R_j \right) \rightarrow 0.$$

*The surface  $X$  is a surface of general type.*

*Proof.* The existence of  $S$  depends on the existence of a primitive embedding of the lattice spanned by  $C$  and the  $R_i$  in the K3 lattice, which is guaranteed if  $h \leq 10$ , because in this case the lattice has rank at most 11.

The genus of the curve  $C$  is  $g(C) = d + 1 > 1$ , by hypothesis. We pose

$$\mathcal{L} = \mathcal{O}_S \left( C - \sum_{i=1}^h R_i \right), \quad \mathcal{M} = \mathcal{O}_S \left( \sum_{i=1}^h R_i \right).$$

Since  $\mathcal{L}^\vee \otimes \mathcal{M} = \mathcal{O}_S \left( -C + 2 \sum_{i=1}^h R_i \right)$ , one has  $h^0(\mathcal{L}^\vee \otimes \mathcal{M}) = 0$ . Indeed,  $C$  is an irreducible divisor with positive self-intersection, and hence it would intersect nonnegatively every effective divisor, but  $(-C + 2 \sum_{i=1}^h R_i)C = -CC < 0$ .

Moreover,  $(-C + 2 \sum_{i=1}^h R_i)^2 = 2d - 8h$ , and if  $d \leq 4h - 3$ ,  $(-C + 2 \sum_{i=1}^h R_i)^2 \leq -6$ . By the Riemann–Roch theorem, one obtains  $h^1(\mathcal{L}^\vee \otimes \mathcal{M}) \neq 0$ . Moreover,

$$\mathcal{L}^{\otimes 2} \otimes \mathcal{M}^\vee = \mathcal{O}_S \left( 2C - 3 \sum_{i=1}^h R_i \right).$$

If  $d \geq (9h - 1)/4$ ,  $(2C - 3 \sum_{i=1}^h R_i)^2 = 8d - 18h \geq -2$ , and since  $(2C - 3 \sum_{i=1}^h R_i)C > 0$ , one has  $h^0(\mathcal{L}^{\otimes 2} \otimes \mathcal{M}^\vee) \geq 1$ . We conclude that the triple cover exists by Theorem 6.3.

Since the branch divisor is  $\wedge^2 \mathcal{E}^{-2} = (\mathcal{L} \otimes \mathcal{M})^{\otimes 2} = \mathcal{O}_S(2C)$ , if the branch locus was not reduced, it would be two times  $C_q$  where  $C_q$  is a specific curve in  $|C|$ . Moreover, every global section of  $S^3 \mathcal{E} \otimes \wedge^2 \mathcal{E}^\vee$  should vanish along  $C_q$ . In particular, also every global section of  $\mathcal{L}^2 \otimes \mathcal{M}^\vee$  would vanish along  $C_q$ . However, this implies that

$$\begin{aligned} H^0(\mathcal{O}_S(2C - 3 \sum R_i)) &= H^0(\mathcal{L}^2 \otimes \mathcal{M}^\vee) = H^0(\mathcal{L}^2 \otimes \mathcal{M}^\vee \otimes \mathcal{O}_S(-C_q)) \\ &= H^0(\mathcal{O}_S(C - 3 \sum R_i)), \end{aligned}$$

which is not the case.

In particular,  $X$  is normal and the branch locus contains curves with positive genus, so  $X$  is of general type. □

**Acknowledgments.** We thank R. Pignatelli and F. Polizzi for useful discussions and suggestions. We also wish to thank the referee for many comments and suggestions that improved the presentation of these results.

## Competing Interests. None.

## REFERENCES

- [Ba] W. Barth, *K3 surfaces with nine cusps*, Geom. Dedicata **72** (1998), 171–178.
- [BHPV] W. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact Complex Surfaces*, 2nd enlarged ed., Grundlehren Math. Wiss. **4**, Springer, Berlin, 2003.
- [BP] G. Bini and M. Penegini, *New fourfolds from F-theory*, Math. Nachr. **290** (2017), 699–709.
- [CE] G. Casnati and T. Ekedahl, *Covers of algebraic varieties I. A general structure theorem, covers of degree 3, 4 and Enriques surfaces*, J. Algebraic Geom. **5** (1996), 439–460.
- [Ca] F. Catanese, “Footnotes to a theorem of I. Reider” in *Algebraic Geometry (L’Aquila, 1988)*, Lecture Notes in Math. **1417**, Springer, Berlin, 1990, 67–74.
- [CD] F. Catanese and O. Debarre, *Surfaces with  $K^2 = 2, p_g = 1, q = 0$* , J. Reine Angew. Math. **395** (1989), 1–55.
- [CP] F. Catanese and F. Perroni, *Dihedral Galois covers of algebraic varieties and the simple cases*, J. Geom. Phys. **118** (2017), 67–93.
- [CT] F. Catanese and F. Tonoli, *Even sets of nodes on sextic surfaces*, J. Eur. Math. Soc. (JEMS) **9** (2007), 705–737.
- [E] D. Eisenbud, *Commutative Algebra. With a View Toward Algebraic Geometry*, Graduate Texts in Math. **150**, Springer, New York, 1995.
- [FPV] D. Faenzi, F. Polizzi, and J. Vallès, *Triple planes with  $p_g = q = 0$* , Trans. Amer. Math. Soc. **371** (2019), 589–639.
- [Fr] R. Friedman, *Algebraic Surfaces and Holomorphic Vector Bundles*, Universitext, Springer, New York, 1998.
- [Fu] A. Fujiki, *Finite automorphism groups of complex tori of dimension two*, Publ. Res. Inst. Math. Sci. **24** (1988), 1–97.
- [G1] A. Garbagnati, *Elliptic K3 surfaces with abelian and dihedral groups of symplectic automorphisms*, Comm. Algebra **41** (2013), 583–616.
- [G2] A. Garbagnati, *K3 surfaces quotients of K3 and Abelian surfaces*, Can. J. Math. **69** (2017), 338–372.
- [G3] A. Garbagnati, *Smooth double covers of K3 surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **19** (2019), 345–386.
- [GS1] A. Garbagnati and C. Salgado, *Elliptic fibrations on K3 surfaces with a non-symplectic involution fixing rational curves and a curve of positive genus*, Rev. Mat. Iberoam. **36** (2020), 1167–1206.
- [GS2] A. Garbagnati and A. Sarti, *Symplectic automorphisms of prime order on K3 surfaces*, J. Algebra **318** (2007), 323–350.
- [GS3] A. Garbagnati and A. Sarti, *Kummer surfaces and K3 surfaces with  $(\mathbb{Z}/2\mathbb{Z})^4$  symplectic action*, Rocky Mountain J. Math. **46** (2016), 1141–1205.
- [H] D. Huybrechts, *Lectures on K3 surfaces*, Cambridge Univ. Press, Cambridge, 2016, 496.
- [HL] D. Huybrechts and M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, 2nd ed., Cambridge Math. Libr., Cambridge Univ. Press, Berlin, 2010.
- [K] S. Kondo, *Automorphisms of algebraic K3 surfaces which act trivially on Picard groups*, J. Math. Soc. Japan **44** (1992), 75–98.
- [L1] R. Laterveer, *Algebraic cycles and triple K3 burgers*, Ark. Mat. **57** (2019), 157–189.
- [L2] R. Laterveer, *Zero-cycles on Garbagnati surfaces*, Tsukuba J. Math. **45** (2021), 1–12.
- [Mi1] R. Miranda, *Triple covers in algebraic geometry*, Amer. J. Math. **107** (1985), 1123–1158.
- [Mi2] R. Miranda, *The basic theory of elliptic surfaces*, Dip. di matematica, Univ. Pisa, <http://www.math.colostate.edu/~miranda/BTES-Mirand>.
- [Mo] D. R. Morrison, *Isogenies between algebraic surfaces with geometric genus one*, Tokyo J. Math. **10** (1987), 179–187.
- [N] V. V. Nikulin, *Integral symmetric bilinear forms and some applications*, Math. USSR Izvestija **14** (1980), 103–167.
- [Pa1] R. Pardini, *Triple covers in positive characteristic*, Ark. Mat. **27** (1989), 319–341.
- [Pa2] R. Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. **417** (1991), 191–213.
- [PZ] G. Pearlstein and Z. Zhang, *A generic global Torelli theorem for certain Horikawa surfaces*, Algebr. Geom. **6** (2019), 132–147.
- [PP] M. Penegini and F. Polizzi, *On surfaces with  $p_g = q = 2, K^2 = 5$  and Albanese map of degree 3*, Osaka J. Math. **50** (2013), 643–686.



- [R] M. Reid, “Young person’s guide to canonical singularities” in *Algebraic Geometry, Bowdoin, 1985, Part 1 (Brunswick, Maine, 1985)*, Proc. Sympos. Pure Math. **46**, Amer. Math. Soc., Providence, RI, 1987, 345–414.
- [RRS] C. Rito, X. Roulleau, and A. Sarti, *Explicit Schoen surfaces*, Algebr. Geom. **6** (2019), 410–426.
- [Sa] A. Sarti, Group actions, cyclic coverings and families of K3-surfaces, Canad. Math. Bull. **49** (2006), 592–608.
- [SS] M. Schütt and T. Shioda, “Elliptic surfaces” in *Algebraic Geometry in East Asia—Seoul 2008*, Adv. Stud. Pure Math. **60**, Math. Soc. Japan, Tokyo, 2010, 51–160.
- [Sh] I. Shimada, *On elliptic K3 surfaces*, Michigan Math. J. **47** (2000), 423–446.
- [T] S. L. Tan, “Triple covers on smooth algebraic surfaces” in *Geometry and Nonlinear Partial Differential Equations (Hangzhou, 2001)*, AMS/IP Stud. Adv. Math. **29**, Amer. Math. Soc., Providence, RI, 2002.
- [T91] H. Tokunaga, *Triple coverings of algebraic surfaces according to the Cardano formula*. J. Math. Kyoto Univ. **31**, 1991, no. 2, 359–375.
- [T94] H. Tokunaga, *On dihedral Galois coverings*. Canad. J. Math. **46**, 1994, no. 6, 1299–1317.
- [X] G. Xiao, *Galois covers between K3 surfaces*, Ann. Inst. Fourier (Grenoble) **46** (1996), 73–88.

Alice Garbagnati  
*Dipartimento di Matematica “Federigo Enriques”*  
*Università degli Studi di Milano*  
*I-20133 Milano,*  
*Italy*  
[alice.garbagnati@unimi.it](mailto:alice.garbagnati@unimi.it)

Matteo Penegini  
*DIMA Dipartimento di Matematica*  
*Università degli Studi di Genova*  
*I-16146 Genova,*  
*Italy*  
[penegini@dim.unige.it](mailto:penegini@dim.unige.it)