A CONSTRUCTION FOR A SELF-POLAR DOUBLE-N ASSOCIATED WITH A PAIR OF NORMAL RATIONAL CURVES

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1. Introduction

In [2] the author introduced a self-polar double-N ("CD"): this double-N is associated with a pair of very specially related ("$\mathscr{S}$-related") normal rational curves, in that the spaces H_i of one row of the double-N are chordal to one of the curves while the spaces K_i of the other row are chordal to the other curve. The double-N might be said to be "associated with" the triple consisting of these two curves and the polarizing quadric.

We introduce in the present paper a double-N ("$C\bar{D}^{p,q}_{b,c}$") associated with a triple which is slightly less specialised than the triple associated with $CD^{p,q}_{b,c}$. This double-N has been shown by the author (in a recently submitted thesis) to be the most general determinantal double-N of Π_{p-2}'s and Π_{q-2}'s in Π_n ($3 \leqq p \leqq q$, $n = p+q-3$) associated with a triple whose curves have $n+2$ distinct common points and whose quadric is inpolar to each of the curves.

The main purpose of the paper is to establish a construction for $C\bar{D}^{p,q}_{b,c}$. Room [3] has given a construction for Coble's self-polar double $-\binom{n+1}{2}$ of lines and secunda in Π_n (a special case of $C\bar{D}^{p,q}_{b,c}$, obtained by fixing $p = 3$). We start with the two curves and a linear series on one of them, but do not use the quadric.

No construction is known for the general self-polar determinantal double-N.

2. $\mathscr{C}$-Triples

Definition. A pair of distinct normal rational curves (n.r.c.'s) of order n is called "a $\mathscr{C}$-related pair" if both curves lie on the same conical sheet CR_2^{n-1} (the locus of joins of a fixed point to the points of a normal rational curve of order $n-1$ lying in a Π_{n-1} not incident with the fixed point).

The $\mathscr{S}$-related pairs considered in [2] are (special) $\mathscr{C}$-related pairs.

A n.r.c. of order n on CR_2^{n-1} must pass through the vertex P and cut each generator once more.

The generators determine a natural $1-1$ correspondence between the points of any two such curves r^n and ρ^n. The cone may be regarded as the locus of joins of pairs in this correspondence, whence it follows (cf. for example [1] p. 18) that r^n and ρ^n have $n+1$ common points, apart from P, unless they touch at P (as they do when r^n, ρ^n is an $\mathscr{S}$-related pair).

We call the vertex of the cone "the $\mathscr{C}$-point of the pair r^n, ρ^n"; and the simplex whose vertices are the remaining points $P_0, \cdots, P_n$ common to r^n and ρ^n we call "the $\mathscr{C}$-simplex".

Let r^n be a n.r.c. on CR_2^{n-1}. Then there is at most one n.r.c. of order n on CR_2^{n-1} which passes through $n+1$ given points $P_0, \cdots, P_n$ (apart from P) on r^n and whose tangent line at P is a given generator ([1] p. 18).

Choose a coordinate system by taking as A_0 the vertex P, as A_n the point (assuming it is not P) in which the given generator meets r^n again, as unit point any point on r^n except A_0 and A_n, and as $A_1, \cdots, A_{n-1}$ the points determined (cf. [4] p. 220) by the condition that the equations

$$\left\|\begin{matrix} x_0 \cdots x_{n-1} \\ x_1 \cdots x_n \end{matrix}\right\|_1 = 0$$

are to represent r^n. Then CR_2^{n-1} is given by the equations

$$\left\|\begin{matrix} x_1 \cdots x_{n-1} \\ x_2 \cdots x_n \end{matrix}\right\|_1 = 0.$$

The tangent line to r^n at A_0 is the generator A_0A_1.

The curve r^n may be represented parametrically by $\kappa x_\delta = \theta^\delta$ $(\delta = 0, \cdots, n)$. Let $\theta^{n+1} - b_\delta \theta^\delta$ [1] be the monic polynomial whose roots are the parameters $\theta_0, \cdots, \theta_n$ of $P_0, \cdots, P_n$. Then the n.r.c. ρ^n given by

$$\left\|\begin{matrix} x_1 \cdots x_{n-1} & x_n \\ x_2 \cdots x_n & b_\delta x_\delta \end{matrix}\right\|_1 = 0$$

lies on CR_2^{n-1}, passes through $P_0, \cdots, P_n$, and has A_0A_n as its tangent at A_0.

Thus a $\mathscr{C}$-related pair can in general be represented by equations of the form

$$(1) \qquad \left\|\begin{matrix} x_0 \cdots x_{n-1} \\ x_1 \cdots x_n \end{matrix}\right\|_1 = 0, \qquad \left\|\begin{matrix} x_1 \cdots x_{n-1} & x_n \\ x_2 \cdots x_n & b_\delta x_\delta \end{matrix}\right\|_1 = 0$$

(with $b_0 \neq 0$). Conversely, such equations always represent a $\mathscr{C}$-related pair.

The coordinate system in which the pair r^n, ρ^n may be represented by equations of the form (1), with $b_0 = 1$, we call "the $\mathscr{C}$-system".

[1] Repetition of the same Greek suffix in one term indicates summation over the range of the suffix.

DEFINITIONS. A triple consisting of a pair of n.r.c.'s and a non-singular tangential quadic is called "a $\mathscr{C}$-triple" if the pair of n.r.c.'s is $\mathscr{C}$-related, with $n+2$ distinct common points, and the quadric polarizes the $\mathscr{C}$-simplex.

An "$\mathscr{H}_m$-space" of a $\mathscr{C}$-triple r^n, ρ^n, S is a chordal Π_m of r^n whose polar space is chordal to ρ^n; while a "$\mathscr{K}_{m'}$-space" is a chordal $\Pi_{m'}$ of ρ^n whose polar space is chordal to r^n.

The reasoning used in the proof of Theorem V in [2], § 1, establishes

THEOREM 1. *The $\mathscr{H}_m$-spaces $(m = 0, \cdots, n-1)$ of a $\mathscr{C}$-triple r^n, ρ^n, S are precisely the m-edges of the simplexes determined by a certain linear series of dimension one and order $n+1$ on r^n.*

We call these simplexes "the simplexes $\mathscr{A}$ of the $\mathscr{C}$-triple", and their polar reciprocals "the simplexes $\mathscr{A}'$." The simplexes $\mathscr{A}'$ are inscribed in ρ^n.

Denote by $\mathscr{T}$ the projectivity which maps each point A of r^n to the point A' in which the line joining A to the $\mathscr{C}$-point meets ρ^n again.

$\mathscr{T}$ is given (in the $\mathscr{C}$-system for r^n, ρ^n) by

$$\lambda x'_0 = x_n - b_{\gamma+1}x_\gamma \qquad (\gamma = 0, \cdots, n-1),$$
$$\lambda x'_{\gamma+1} = x_\gamma.$$

THEOREM 2. *Let H be any m-edge of any simplex $\mathscr{A}$, say $\mathscr{A}_H$. Let K be the polar space of H. Then K is the image under $\mathscr{T}$ of the $(n-m-1)$-edge of $\mathscr{A}_H$ opposite the m-edge H.*

PROOF. Let $\mathscr{S}$ (mapping primes to points) be the polarity determined by S. We seek first the quadric which polarizes the ∞^1 simplexes $\mathscr{A}$.

If $\lambda_0, \cdots, \lambda_n$ are the faces of any simplex $\mathscr{A}$ then $\mathscr{S}(\lambda_0), \cdots, \mathscr{S}(\lambda_n)$ are the vertices of a simplex $\mathscr{A}'$. Thus $\mathscr{T}^{-1}\mathscr{S}(\lambda_0), \cdots, \mathscr{T}^{-1}\mathscr{S}(\lambda_n)$ are $n+1$ points on r^n, say $L_0, \cdots, L_n$. Taking the $\mathscr{C}$-simplex as simplex of reference and the $\mathscr{C}$-point as unit point, $\mathscr{S}$ and $\mathscr{T}$ are given by diagonal matrices, say D and T. $T^{-1}D$ is diagonal, i.e. the correlation $\mathscr{T}^{-1}\mathscr{S}$ is the polarity determined by a non-singular tangential quadric, say S_1, which polarizes the $\mathscr{C}$-simplex. S_1 is inpolar to r^n and therefore polarizes ∞^1 simplexes inscribed in r^n. Two points of r^n are conjugate w.r.t. S_1 if and only if one is a vertex of the simplex determined by the other. It follows that $L_0, \cdots, L_n$ are the vertices of the simplex $\mathscr{A}$ whose faces are $\lambda_0, \cdots, \lambda_n$. Thus S_1 is the quadric which polarizes the simplexes $\mathscr{A}$.

Write $\mathscr{S}_1 = \mathscr{T}^{-1}\mathscr{S}$ and let λ be any face of $\mathscr{A}_H$. Then the vertex L of $\mathscr{A}_H$ opposite λ is $\mathscr{S}_1(\lambda)$. But $\mathscr{S}(\lambda)$ is the image of L under $\mathscr{T}$, since $\mathscr{S} = \mathscr{T}\mathscr{S}_1$. The theorem follows immediately.

3. Configurations $C\bar{D}^{p,\,q}_{b,\,c}$

Denote by $\bar{D}^{p,\,q}_{b,\,c}$ the locus given by the equations

$$\left\|\begin{matrix} x_1 & \cdots & x_{q-1} & x_q \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & x_n \\ x_{p-1} & \cdots & x_n & b_\delta x_\delta \\ x_0 & \cdots & x_{q-2} & c_\delta x_\delta \end{matrix}\right\|_{p-1} = 0 \qquad (b_0 = 1);$$

and denote by $C\bar{D}^{p,\,q}_{b,\,c}$ the associated double-N $\left[N = \binom{p+q-2}{p-1} = \binom{n+1}{p-1}\right]$.

It is easily verified, using Room's criterion for self-polarity ([3] p. 66) that the configuration $C\bar{D}^{p,\,q}_{b,\,c}$ is self-polar. The polarizing quadric S is inpolar to each of the curves r^n and ρ^n, where r^n is given by

$$\left\|\begin{matrix} x_0 \cdots x_{n-1} \\ x_1 \cdots x_n \end{matrix}\right\|_1 = 0$$

and ρ^n by

$$\left\|\begin{matrix} x_1 \cdots x_{n-1} & x_n \\ x_2 \cdots x_n & b_\delta x_\delta \end{matrix}\right\|_1 = 0.$$

S is determined by a matrix $[k_{\alpha\beta}] = [k_{\alpha+\beta-2}]$ $(\alpha, \beta = 1, \cdots, n+1)$ with

$$\text{(2)} \qquad \begin{cases} b_\delta k_{\delta+t} = k_{n+t+1} \\ c_\delta k_{\delta+t+1} = k_{q+t} \end{cases} \qquad (t = 0, \cdots, n-1).$$

The pair r^n, ρ^n is $\mathscr{C}$-related.

Assume that $\theta^{n+1} - b_\delta \theta^\delta$ has no repeated roots, that is r^n and ρ^n have $n+2$ distinct common points. Then r^n, ρ^n, S is a $\mathscr{C}$-triple: for S is inpolar to r^n, ρ^n and the quadric $b_\delta x_\delta x_0 = x_1 x_n$; and the $\mathscr{C}$-point A_0 is the pole of the prime $c_\delta x_\delta = x_{q-1}$ [by equations (2)].

From the form of the equations of $\bar{D}^{p,\,q}_{b,\,c}$, it is evident that $C\bar{D}^{p,\,q}_{b,\,c}$ is (cf. [2] p. 216) associated with the $\mathscr{C}$-triple r^n, ρ^n, S.

LEMMA 1. *Let r^n, ρ^n be a $\mathscr{C}$-related pair whose common points are distinct, H a chordal Π_{p-2} of r^n and K a chordal Π_{q-2} of ρ^n (where $3 \leqq p \leqq q$ and $n = p+q-3$). Suppose that neither H nor K passes through the $\mathscr{C}$-point. Then H, K is a pair in exactly one of the configurations $C\bar{D}^{p,\,q}_{b,\,c}$ associated with the pair r^n, ρ^n.*

PROOF. H is given (in the $\mathscr{C}$-system) by say $\lambda_\alpha x_{\alpha+\varepsilon} = 0$ $(\alpha = 0, \cdots, p-1;$ $\varepsilon = 0, \cdots, q-2)$ and K by say $\mu_\beta x_{\beta+\phi} = 0$ $(\beta = 0, \cdots, q-1; \phi = 1, \cdots, p-1)$, where $x_{n+1} \equiv b_\delta x_\delta$ $(\delta = 0, \cdots, n)$ (an identity in $x_0, \cdots, x_n$).

H, K is a pair in the configuration $C\bar{D}^{p,\,q}_{b,\,c}$ determined by $c_0, \cdots, c_n$ if and only if

$$\mu_\delta \lambda_\alpha x_{\alpha+\varepsilon} + \mu_{q-1} (\lambda_0 c_\delta x_\delta + \lambda_1 x_q + \cdots + \lambda_{p-1} x_{n+1}) \equiv 0 \begin{cases} (\alpha = 0, \cdots, p-1) \\ (\varepsilon = 0, \cdots, q-2) \end{cases}$$

(cf. [4] p. 72). Since neither H nor K passes through the $\mathscr{C}$-point A_0, we can suppose that $\lambda_0\mu_{q-1} = 1$. So the identity determines (uniquely) a suitable set of constants $c_0, \cdots, c_n$.

LEMMA 2. *Let r^n, ρ^n, S be a $\mathscr{C}$-triple. Suppose that H is an $\mathscr{H}_{p-2}$-space such that neither H nor its polar space K passes through the $\mathscr{C}$-point. Then S polarizes the configuration $C\bar{D}^{p,q}_{b,c}$ determined by r^n, ρ^n, H and K.*

PROOF. $C\bar{D}^{p,q}_{b,c}$ is determined by a matrix

$$\begin{bmatrix} \lambda_\alpha x_\alpha & \cdots & \lambda_\alpha x_{\alpha+q-2} & 0 \\ x_1 & \cdots & x_{q-1} & \mu_\beta x_{\beta+1} \\ \vdots & & \vdots & \vdots \\ x_{p-1} & \cdots & x_n & \mu_\beta x_{\beta+p-1} \end{bmatrix},$$

with $x_{n+1} \equiv b_\delta x_\delta$.

S is inpolar to each of r^n and ρ^n (since it polarizes the $\mathscr{C}$-simplex), and H is the polar of K. So, applying Room's criterion, we deduce that S polarizes $C\bar{D}^{p,q}_{b,c}$.

LEMMA 3. *Suppose that $C\bar{D}^{p,q}_{b,c}$ (with $c_\delta x_\delta \not\equiv x_{q-1}$) is associated with a $\mathscr{C}$-triple r^n, ρ^n, S. Then a configuration $C\bar{D}^{p,q}_{b,c}$ is associated with the same triple if and only if $a_\delta x_\delta \equiv c_\delta x_\delta + k(c_\delta x_\delta - x_{q-1})$ for some constant k.*

PROOF. This result follows immediately from the consideration of equations (2), using the fact that S is non-singular (since r^n, ρ^n, S is a $\mathscr{C}$-triple).

4. The $\mathscr{K}_{q-2}$-spaces which meet a fixed $\mathscr{H}_{p-2}$-space

Let r^n, ρ^n, S be a $\mathscr{C}$-triple, H_1 an $\mathscr{H}_{p-2}$-space of the triple, and K_1 its polar space. By Theorem 1, H_1 is a $(p-2)$-edge of one of the simplexes $\mathscr{A}$ of the triple, say $\mathscr{A}_{H_1}$. By Theorem 2, K_1 is the $(q-2)$-edge, of the simplex $\mathscr{A}'$ which is the polar reciprocal of $\mathscr{A}_{H_1}$, opposite the $(p-2)$-edge $\mathscr{T}(H_1)$. Write

$$\mathscr{A}'_{K_1} = \mathscr{T}(\mathscr{A}_{H_1}).$$

If $H_1 \not\ni$ the $\mathscr{C}$-point and $H_1 \not\subset$ the polar of the $\mathscr{C}$-point, then H_1 and K_1 are paired in a configuration $C\bar{D}^{p,q}_{b,c}$ associated with the $\mathscr{C}$-triple. If also H_1 is not a $(p-2)$-edge of the $\mathscr{C}$-simplex, it is readily verified that $c_\delta x_\delta \not\equiv x_{q-1}$.

The spaces K_i $(i \neq 1)$ of $C\bar{D}^{p,q}_{b,c}$ are all $\mathscr{K}_{q-2}$-spaces (of the $\mathscr{C}$-triple) which meet H_1.

THEOREM 3. *Let H_1 be an $\mathscr{H}_{p-2}$-space of a given $\mathscr{C}$-triple r^n, ρ^n, S, and K_1 its polar. Let K be any $\mathscr{K}_{q-2}$-space of the triple, excepting the $(q-2)$-edges* *of $\mathscr{A}'_{K_1}$, which meets H_1. Then K is one of the spaces K_i of the configuration $C\bar{D}^{p,q}_{b,c}$ (associated with the $\mathscr{C}$-triple) in which H_1, K_1 is a pair, provided that H_1 is not a member of a certain finite subset (to be specified) of the set of $\mathscr{H}_{p-2}$-spaces.*

PROOF. Suppose H_1 is not an edge of: (1) the simplex $\mathscr{A}$, say $\mathscr{A}_1$, one of whose vertices is the $\mathscr{C}$-point, (2) the simplex $\mathscr{A}$, say $\mathscr{A}_2$, one of whose faces is the polar of the $\mathscr{C}$-point, and (3) the $\mathscr{C}$-simplex; and that H_1 does not meet any $(q-2)$-edge of: (1) the simplex $\mathscr{T}(\mathscr{A}_1)$, (2) the simplex $\mathscr{T}(\mathscr{A}_2)$, and (3) any simplex $\mathscr{A}'$ which has a pair of coincident vertices.

Since the $\mathscr{H}_{p-2}$-spaces constitute an irreducible algebraic family of dimension one, any given Π_{q-2} either meets all the $\mathscr{H}_{p-2}$-spaces or else meets only finitely many of them. No Π_{q-2} meets all the $\mathscr{H}_{p-2}$-spaces since every $(p-2)$-edge of the $\mathscr{C}$-simplex is an $\mathscr{H}_{p-2}$-space and the $\mathscr{C}$-simplex is proper.

There are only finitely many simplexes $\mathscr{A}'$ having a pair of coincident vertices; for the $\mathscr{C}$-simplex is proper, inscribed in ρ^n and self-polar, so that ρ^n does not lie on the point quadric defined by S.

Thus the conditions on H_1 exclude the choice of only a finite number of $\mathscr{H}_{p-2}$-spaces.

We now show that $\dim (K \cap \bar{D}^{p,q}_{b,c}) > p-3$. The following lemma will be useful.

LEMMA. *Let M be an m-edge $(0 \leqq m \leqq q-3)$ of a simplex $\mathscr{A}'$. Suppose M meets H_1. Then M is an edge of $\mathscr{A}'_{K_1}$.*

PROOF. Since $\mathscr{A}_{H_1} \neq \mathscr{A}_1$, $H_1 \not\ni$ the $\mathscr{C}$-point. So $\{H_1$, the $\mathscr{C}$-point$\}$ is a Π_{p-1}, say N. N is chordal to ρ^n: it contains the $\mathscr{C}$-point and $\mathscr{T}(H_1)$; the $\mathscr{C}$-point $\not\subset \mathscr{T}(H_1)$ since $\mathscr{A}_{H_1} \neq \mathscr{A}_2$. M meets N, since $N \supset H_1$. $M \cap N$ is a chordal Π_s $(s \geqq 0)$, since $\dim M + \dim N \leqq p+q-4 = n-1$. $M \not\ni$ the $\mathscr{C}$-point, since H_1 does not meet any $(q-2)$-edge of $\mathscr{T}(\mathscr{A}_2)$; so $M \cap N \subset \mathscr{T}(H_1)$, which implies that $M \cap N$, and therefore M, is an edge of $\mathscr{A}'_{K_1}$.

It follows from this lemma that none of the chordal Π_{p-3}'s of ρ^n which lies in K meets H_1.

Moreover, $K \cap H_1 \not\subset$ the polar of the $\mathscr{C}$-point. For suppose the contrary. Let H be the polar of K. Then $\{H, K_1\} \supset \mathscr{T}(H)$. Also $\dim \{H, K_1\} \leqq n-1$. But $n-1 = p+q-4$, so that $\mathscr{T}(H) \cap K_1$ is a chordal Π_r $(r \geqq 0)$ of ρ^n. So $\mathscr{T}(H)$, and therefore K, is an edge of $\mathscr{A}'_{K_1}$. But K is, by hypothesis, not an edge of $\mathscr{A}'_{K_1}$.

Now, because of the restrictions on H_1, K contains $q-1$ distinct points of ρ^n, and therefore $\binom{q-1}{p-2}$ chordal Π_{p-3}'s of ρ^n. It is easily verified that every chordal Π_{p-3} of ρ^n lies on $\bar{D}^{p,q}_{b,c}$.

Also, K contains $\binom{q-1}{p-1}$ chordal Π_{p-2}'s of ρ^n. Let A be one of these. A is a $\mathscr{K}_{p-2}$-space. $A \not\subset$ the polar of the $\mathscr{C}$-point, since K is not an edge of $\mathscr{T}(\mathscr{A}_1)$; in fact, A meets the polar of the $\mathscr{C}$-point in the same Π_{p-3} as does the $\mathscr{H}_{p-2}$-space $\mathscr{T}^{-1}(A)$, since dim {the $\mathscr{C}$-point, the polar of A, the polar of $\mathscr{T}^{-1}(A)\} = q-1$. However, (the polar of the $\mathscr{C}$-point) $\cap\, \mathscr{T}^{-1}(A)$ lies on $\bar{D}^{p,q}_{b,c}$. For $\mathscr{T}^{-1}(A) \not\ni$ the $\mathscr{C}$-point and also $\mathscr{T}^{-1}(A) \not\subset$ the polar of the $\mathscr{C}$-point. So, by Lemmas 1, 2, 3, $\mathscr{T}^{-1}(A)$ is a space H_i in a configuration $C\bar{D}^{p,q}_{b,a}$ with $a_\delta x_\delta \equiv c_\delta x_\delta + k(c_\delta x_\delta - x_{q-1})$ for some k; that is $\mathscr{T}^{-1}(A)$ lies on $\bar{D}^{p,q}_{b,a}$. But the polar of the $\mathscr{C}$-point, being given by $c_\delta x_\delta = x_{q-1}$, has identical intersections with $\bar{D}^{p,q}_{b,a}$ and $\bar{D}^{p,q}_{b,c}$. So (the polar of the $\mathscr{C}$-point) $\cap\, A$ lies on $\bar{D}^{p,q}_{b,c}$; but it is not a chordal Π_{p-3} of ρ^n, since K is not an edge of $\mathscr{T}(\mathscr{A}_1)$.

We have shown that K contains $\binom{q-1}{p-2} + \binom{q-1}{p-1}$, that is $\binom{q}{p-1}$, distinct Π_{p-3}'s lying on $\bar{D}^{p,q}_{b,c}$.

The dimension of $\bar{D}^{p,q}_{b,c}$ is $2p-4$, and its order is $\binom{q}{p-1}$; dim $K = q-2$; and $n = p+q-3$. Moreover, $K \cap \bar{D}^{p,q}_{b,c}$ includes not only the $\binom{q}{p-1}$ Π_{p-3}'s found above but also a point which does not lie in of any these; for, none of the chordal Π_{p-3}'s of ρ^n which lies in K meets H_1; and $K \cap H_1 \not\subset$ the polar of the $\mathscr{C}$-point. It follows that dim $(K \cap \bar{D}^{p,q}_{b,c}) > p-3$.

Using this information, together with the fact that K is a space K_i in a configuration $C\bar{D}^{p,q}_{b,a}$ with $a_\delta x_\delta \equiv c_\delta x_\delta + k(c_\delta x_\delta - x_{q-1})$ for some k, so that dim $(K \cap \bar{D}^{p,q}_{b,a}) > p-3$ (cf. [4] p. 40), we show that K is a space K_i in $C\bar{D}^{p,q}_{b,c}$.

Let us suppose $k \neq 0$. Then $K \cap \bar{D}^{p,q}_{b,c} = K \cap$ the locus

$$\left\| \begin{matrix} \lambda_\alpha x_\alpha & \cdots & \lambda_\alpha x_{\alpha+q-2} & \lambda_0 \mu_{q-1} k(x_{q-1} - c_\delta x_\delta) \\ x_1 & \cdots & x_{q-1} & 0 \\ \vdots & & \vdots & \vdots \\ x_{p-1} & \cdots & x_n & 0 \end{matrix} \right\|_{p-1} = 0,$$

where $\lambda_\alpha x_{\alpha+\varepsilon} = 0$ ($\alpha = 0, \cdots, p-1$; $\varepsilon = 0, \cdots, q-2$) are the equations of the polar H of K, and $\mu_\beta x_{\beta+\phi+1} = 0$ ($\beta = 0, \cdots, q-1$; $\phi = 0, \cdots, p-2$), with $x_{n+1} \equiv b_\delta x_\delta$, are the equations of K. Since K is not an edge of either $\mathscr{T}(\mathscr{A}_1)$ or $\mathscr{T}(\mathscr{A}_2)$, $\lambda_0 \mu_{q-1} \neq 0$. Also $c_\delta x_\delta \not\equiv x_{q-1}$, and $k \neq 0$. So $K \cap \bar{D}^{p,q}_{b,c} = K \cap$ the prime $c_\delta x_\delta = x_{q-1} \cap$ the locus $\mathscr{L}$ given by

$$\left\| \begin{matrix} x_1 & \cdots & x_{q-1} \\ \vdots & & \vdots \\ x_{p-1} & \cdots & x_n \end{matrix} \right\|_{p-2} = 0.$$

Since $\mathscr{L}$ is generated by those chordal Π_{p-2}'s of ρ^n which pass through the $\mathscr{C}$-point A_0, and $K \not\ni$ the $\mathscr{C}$-point, dim $(K \cap \bar{D}^{p,q}_{b,c}) \leqq p-3$. But dim $(K \cap \bar{D}^{p,q}_{b,c}) > p-3$. So $k = 0$.

5. The construction

Suppose we are given a $\mathscr{C}$-related pair r^n, ρ^n whose common points are distinct, and a g_1^{n+1}, on ρ^n, one of whose simplexes is the $\mathscr{C}$-simplex. Then, in general, the g_1^{n+1} determines a non-singular quadric S such that any set of the g_1^{n+1} gives the $n+1$ vertices of a simplex $\mathscr{A}'$ of the $\mathscr{C}$-triple r^n, ρ^n, S (cf. the proof of Theorem 2 and [4] pp. 227—8). S is not needed in the construction.

Let K_1 be a $(q-2)$-edge of one of the simplexes $\mathscr{A}'$, say $\mathscr{A}'_{K_1}$. Then we can construct the edge, say H_1, of $\mathscr{T}^{-1}(\mathscr{A}'_{K_1})$ which lies opposite the $(q-2)$-edge $\mathscr{T}^{-1}(K_1)$. For if Q is any point on ρ^n then $\mathscr{T}^{-1}(Q)$ is the point in which the line joining Q to the $\mathscr{C}$-point meets r^n again.

We have shown (cf. Theorem 3) that, for general choice of K_1, any $\mathscr{K}_{q-2}$-space (i.e. edge of a simplex of the given g_1^{n+1}) which meets H_1, but is not an edge of $\mathscr{A}'_{K_1}$, is a space K_i $(i \neq 1)$ of the configuration $C\bar{D}^{p,q}_{b,c}$ (associated with r^n, ρ^n) in which H_1, K_1 is a pair.

Given K_i, H_i can be constructed in the manner used to construct H_1 (given K_1).

We leave unsettled the question: is any space K_i $(i \neq 1)$ an edge of $\mathscr{A}'_{K_1}$?

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