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ON SUMS INVOLVING THE EULER TOTIENT FUNCTION

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Abstract

Let $\gcd(n_1, \ldots, n_k)$ denote the greatest common divisor of positive integers $n_1, \ldots, n_k$ and let ϕ be the Euler totient function. For any real number $x > 3$ and any integer $k \geq 2$, we investigate the asymptotic behaviour of $\sum_{n_1 \ldots n_k \leq x} \phi(\gcd(n_1, \ldots, n_k))$.

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1. Introduction and main results

Let $s = \sigma + it$ be the complex variable and let $\zeta(s)$ denote the Riemann zeta-function. For any positive integer $k \geq 2$, let τ_k denote the k-factors divisor function defined by $\mathbf{1} * \mathbf{1} * \cdots * \mathbf{1}$ and $\tau = \tau_2$. Here $*$ denotes the Dirichlet convolution of arithmetic functions and $\mathbf{1}$ is given by $\mathbf{1}(n) = 1$ for any positive integer n. We define the error term $\Delta_k(x)$ in the generalised divisor problem by

$$\sum_{n \leq x} \tau_k(n) = Q_k(\log x)x + \Delta_k(x), \tag{1.1}$$

where $Q_k(\log x) = \mathrm{Res}_{s=1}\, \zeta^k(s)x^{s-1}/s$ is a polynomial in $\log x$ of degree $k-1$. The order of magnitude of $\Delta_k(x)$ as $x \to \infty$ is an open problem called the Piltz divisor problem and it has attracted much interest in analytic number theory. It has been conjectured that

$$\Delta_k(x) = O(x^{(k-1)/2k+\varepsilon}) \tag{1.2}$$

for any integer $k \geq 2$ and any $\varepsilon > 0$ (see Ivić [7, Chapter 13] or Titchmarsh [14]). Let μ denote the Möbius function defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is squarefree and } n = p_1 p_2 \ldots p_k, \\ 0 & \text{if } n \text{ is not squarefree,} \end{cases}$$

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and let $\gcd(n_1, \ldots, n_k)$ denote the greatest common divisor of the positive integers $n_1, \ldots, n_k$ for any integer $k \geq 2$. For a real number $x > 3$, let $S_{f,k}(x)$ denote the summatory function

$$S_{f,k}(x) := \sum_{n_1 \ldots n_k \leq x} f(\gcd(n_1, \ldots, n_k)), \tag{1.3}$$

where f is any arithmetic function, by summing over the hyperbolic region $\{(n_1, \ldots, n_k) \in \mathbb{N}^k : n_1 \ldots n_k \leq x\}$. In 2012, Krätzel *et al.* [10] showed that

$$S_{f,k}(x) = \sum_{n \leq x} g_{f,k}(n),$$

where

$$g_{f,k}(n) = \sum_{n = m^k l} (\mu * f)(m) \tau_k(l) \tag{1.4}$$

(see also Heyman and Tóth [3], Kiuchi and Saad Eddin [8]). If f is multiplicative, then (1.4) is multiplicative. We use (1.4) to get the formal Dirichlet series

$$\sum_{n=1}^{\infty} \frac{g_{f,k}(n)}{n^s} = \frac{\zeta^k(s)}{\zeta(ks)} \sum_{n=1}^{\infty} \frac{f(n)}{n^{ks}}, \tag{1.5}$$

which converges absolutely in the half-plane $\sigma > \sigma_0$, where σ_0 depends on f and k.

When $f = \mathrm{id}$, it follows from (1.5) that

$$\sum_{n=1}^{\infty} \frac{g_{\mathrm{id},k}(n)}{n^s} = \frac{\zeta^k(s)\zeta(ks-1)}{\zeta(ks)} \quad \text{for } \operatorname{Re} s > 1.$$

Here the symbol id is given by $\mathrm{id}(n) = n$ for any positive integer n. For $k = 2$, Krätzel *et al.* [10] used the following three methods:

(1) the complex integration approach (see [7, 14]);
(2) a combination of fractional part sums and the theory of exponent pairs (see [2, 9]); and
(3) Huxley's method (see [4–6]),

to prove

$$\sum_{ab \leq x} \gcd(a, b) = P_2(\log x)x + O(x^{\theta}(\log x)^{\theta'}). \tag{1.6}$$

Here θ satisfies $\frac{1}{2} < \theta < 1$, θ' is some real number and P_2 is a certain quadratic polynomial with

$$P_2(\log x) = \operatorname*{Res}_{s=1} \frac{\zeta^2(s)\zeta(2s-1)}{\zeta(2s)} \frac{x^{s-1}}{s}.$$

They showed that methods (1), (2) and (3) imply the results $\theta = \frac{2}{3}$ and $\theta' = 16/9$, $\theta = 925/1392$ and $\theta' = 0$, and $\theta = 547/832$ and $\theta' = 26947/8320$, respectively. Let ϕ

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denote the Euler totient function defined by $\phi = \mathrm{id} * \mu$. The Dirichlet series (1.5) with $f = \phi$ implies that

$$\sum_{n=1}^{\infty} \frac{g_{\phi,k}(n)}{n^s} = \frac{\zeta^k(s)\zeta(ks-1)}{\zeta^2(ks)} \quad \text{for } \mathrm{Re}\, s > 1. \tag{1.7}$$

We consider some properties of the hyperbolic summation for the Euler totient function involving the gcd. The first purpose of this paper is to investigate the asymptotic behaviour of (1.3) with $f = \phi$ for $k = 2$. Applying fractional part sums and the theory of exponent pairs, we obtain the following result.

THEOREM 1.1. *For any real number $x > 3$,*

$$\begin{aligned}\sum_{ab\le x} \phi(\gcd(a,b)) &= \frac{1}{4\zeta^2(2)} x\log^2 x + \frac{1}{\zeta^2(2)}\Big(2\gamma - \frac{1}{2} - 2\frac{\zeta'(2)}{\zeta(2)}\Big) x\log x \\ &+ \frac{1}{2\zeta^2(2)}\Big(5\gamma^2 + 6\gamma_1 - 4\gamma + 1 - 4(4\gamma-1)\frac{\zeta'(2)}{\zeta(2)} - 4\frac{\zeta''(2)}{\zeta(2)} + 12\Big(\frac{\zeta'(2)}{\zeta(2)}\Big)^2\Big)x \\ &+ O(x^{55/84+\varepsilon}),\end{aligned} \tag{1.8}$$

where γ and γ_1 are the Euler constant and the first Stieltjes constant, respectively.

We note that the main term of (1.8) is given by (7.2) below.

REMARK 1.2. Note that $\frac{1}{2} < 55/84 = \frac{1}{2} + 13/84 < 547/832 = \frac{1}{2} + 131/832$.

The summation for the arithmetic functions $h(n)$ in the Dirichlet series

$$F_h(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \zeta^2(s)\zeta(2s-1)\zeta^M(2s)$$

for $\mathrm{Re}\, s > 1$ and any fixed integer M was considered by Kühleitner and Nowak [11] in 2013. We use their results to show that the error term on the right-hand side of (1.8) is $\Omega(x^{1/2}\log^2 x/\log\log x)$ as $x \to \infty$. This suggests the following conjecture.

CONJECTURE 1.3. The order of magnitude of the error term on the right-hand side of (1.8) is $O(x^{1/2}(\log x)^A)$ with $A > 2$.

When $k = 3$, Krätzel *et al.* [10] also derived the formula

$$\sum_{abc\le x} \gcd(a,b,c) = M_3(x) + O(x^{1/2}(\log x)^5),$$

where

$$\begin{aligned}M_3(x) &= \sum_{s_0=1,2/3} \operatorname*{Res}_{s=s_0}\Big(\frac{\zeta^3(s)\zeta(3s-1)}{\zeta(3s)}\frac{x^s}{s}\Big) \\ &= x(0.6842\ldots\log^2 x - 0.6620\ldots\log x + 4.845\ldots) - 4.4569\ldots x^{2/3}.\end{aligned}$$

For $k = 3$, we derive an asymptotic formula for (1.3) with $f = \phi$ by using the complex integration approach.

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THEOREM 1.4. *For any real number $x > 3$,*

$$\sum_{abc\le x}\phi(\gcd(a,b,c))$$
$$= \frac{\zeta(2)}{2\zeta^2(3)}x\log^2 x + \frac{\zeta(2)}{\zeta^2(3)}\Big(3\gamma - 1 + 3\frac{\zeta'(2)}{\zeta(2)} - 6\frac{\zeta'(3)}{\zeta(3)}\Big)x\log x$$
$$+ \frac{\zeta(2)}{\zeta^2(3)}\Big(3\gamma^2 + 3\gamma_1 - 3\gamma + 1 + 3(3\gamma-1)\frac{\zeta'(2)}{\zeta(2)} - 6(3\gamma-1)\frac{\zeta'(3)}{\zeta(3)}\Big)x$$
$$+ \frac{\zeta(2)}{\zeta^2(3)}\Big(27\Big(\frac{\zeta'(3)}{\zeta(3)}\Big)^2 + \frac{9}{2}\frac{\zeta''(2)}{\zeta(2)} - 9\frac{\zeta''(3)}{\zeta(3)} - 18\frac{\zeta'(2)}{\zeta(2)}\frac{\zeta'(3)}{\zeta(3)}\Big)x$$
$$+ \frac{\zeta^3(\frac{2}{3})}{2\zeta^2(2)}x^{2/3} + O(x^{1/2}\log^5 x), \tag{1.9}$$

where γ and γ_1 are the Euler constant and the first Stieltjes constant, respectively.

We note that the main term of (1.9) is given by (7.3) below.

For $k = 4$, we use the complex integration approach to calculate the asymptotic formula for (1.3) with $f = \phi$.

THEOREM 1.5. *For any real number $x > 3$,*

$$\sum_{abcd\le x}\phi(\gcd(a,b,c,d)) = xP_{\phi,4}(\log x) + O(x^{1/2}\log^{17/3} x), \tag{1.10}$$

where $P_{\phi,4}(u)$ is a polynomial in u of degree three depending on ϕ.

For $k = 5$, from Ivić [7, Theorem 13.2], the error term $\Delta_5(x)$ is estimated by

$$\Delta_5(x) = O(x^{11/20+\varepsilon}) \tag{1.11}$$

for any $\varepsilon > 0$. We use an elementary method and (1.1) to obtain the following result.

THEOREM 1.6. *For any real number $x > 3$,*

$$\sum_{abcde\le x}\phi(\gcd(a,b,c,d,e)) = xP_{\phi,5}(\log x) + \sum_{n\le x^{1/5}}(\mu*\mu)(n)\sum_{m\le x^{1/5}/n} m\Delta_5\Big(\frac{x}{m^5n^5}\Big), \tag{1.12}$$

where $P_{\phi,5}(u)$ is a polynomial in u of degree four depending on ϕ. In particular, it follows from (1.11) that the error term on the right-hand side of (1.12) is $O(x^{11/20+\varepsilon})$.

REMARK 1.7. If we can use Conjecture (1.2) with $k = 5$, then the error term on the right-hand side of (1.12) becomes $O(x^{2/5+\varepsilon})$.

Assuming Conjecture (1.2), it is easy to obtain an asymptotic formula for $S_{\phi,k}(x)$ for any integer $k \ge 5$.

PROPOSITION 1.8. *Assume Conjecture (1.2). With the previous notation,*

$$\sum_{n_1n_2\dots n_k\le x}\phi(\gcd(n_1,n_2,\dots,n_k)) = xP_{\phi,k}(\log x) + O(x^{(k-1)/2k+\varepsilon})$$

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for any real number $x > 3$, where $P_{\phi,k}(u)$ $(k \geq 5)$ is a polynomial in u of degree $k-1$ depending on ϕ.

Notation. We denote by ε an arbitrary small positive number which may be different at each occurrence.

2. Auxiliary results

We will need the following lemma.

LEMMA 2.1. *For $t \geq t_0 > 0$, uniformly in σ,*

$$\zeta(\sigma+it) \ll \begin{cases} t^{(3-4\sigma)/6}\log t & \text{if } 0 \leq \sigma \leq 1/2, \\ t^{(1-\sigma)/3}\log t & \text{if } 1/2 \leq \sigma \leq 1, \\ \log t & \text{if } 1 \leq \sigma < 2, \\ 1 & \text{if } \sigma \geq 2. \end{cases}$$

PROOF. The lemma follows from Tenenbaum [13, Theorem II.3.8]; see also Ivić [7] or Titchmarsh [14]. □

3. Proof of Theorem 1.1

Our main work is to evaluate the sum $A(x) = \sum_{mnl^2 \leq x,\, m,n,l>0} l$. We utilise [10, Section 3.2] to derive the formula

$$A(x) = M_1(x) + \Delta(x),$$

where the error term $\Delta(x)$ is estimated by $O(x^{1/4+(\alpha+\beta)/2})$. Here (α,β) is an exponent pair (see [2, 7]) and $M_1(x)$ is the main term given by

$$M_1(x) = \operatorname*{Res}_{s=1} \zeta^2(s)\zeta(2s-1)\frac{x^s}{s}.$$

From (1.7), this gives

$$\sum_{ab\leq x}\phi(\gcd(a,b)) = \sum_{l\leq\sqrt{x}}(\mu*\mu)(l)A\Big(\frac{x}{l^2}\Big) = M_2(x) + O(x^{1/4+(\alpha+\beta)/2+\varepsilon}), \tag{3.1}$$

where the main term $M_2(x)$ is given by

$$\begin{aligned} M_2(x) &= \operatorname*{Res}_{s=1} \frac{\zeta^2(s)\zeta(2s-1)}{\zeta^2(2s)}\frac{x^s}{s} \\ &= \frac{1}{4\zeta^2(2)}x\log^2 x + \frac{1}{\zeta^2(2)}\Big(2\gamma - \frac{1}{2} - 2\frac{\zeta'(2)}{\zeta(2)}\Big)x\log x \\ &\quad + \frac{1}{2\zeta^2(2)}\Big(5\gamma^2 + 6\gamma_1 - 4\gamma + 1 - 4(4\gamma-1)\frac{\zeta'(2)}{\zeta(2)} - 4\frac{\zeta''(2)}{\zeta(2)} + 12\Big(\frac{\zeta'(2)}{\zeta(2)}\Big)^2\Big)x \end{aligned}$$

by (7.2) below. Choosing, in particular, the exponent pair

$$(\alpha,\beta) = (\tfrac{13}{84}+\varepsilon, \tfrac{55}{84}+\varepsilon),$$

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discovered by Bourgain [1, Theorem 6], we obtain the order of magnitude $O(x^{55/84+\varepsilon})$ of the error term on the right-hand side of (3.1). This completes the proof of Theorem 1.1.

4. Preparations for the proof of Theorems 1.4 and 1.5

In order to derive the formulas (1.9) and (1.10), we use the following notation. Let k be any integer such that $k \ge 3$ and let $\sigma_0 = 1 + 1/k + \varepsilon$. Consider the estimation of the error terms of Perron's formula (see [12, Theorem 5.2 and Corollary 5.3]) for (1.4) with $f = \phi$. The estimation of $g_{\phi,k}(n)$ is given by

$$g_{\phi,k}(n) = \sum_{n=m^k l} (\mathrm{id} * \mu * \mu)(m)\tau_k(l) = \sum_{n=d^k m^k l} d(\mu * \mu)(m)\tau_k(l)$$
$$\ll n^{1/k} \sum_{n=d^k m^k l} \tau(m)\tau_k(l) \ll n^{1/k+\varepsilon}.$$

In Perron's formula,

$$R \ll x^{1/k+\varepsilon}\Big(1 + \frac{x}{T}\sum_{1\le k\le x}\frac{1}{k}\Big) + \frac{(4x)^{\sigma_0}}{T}\left|\frac{\zeta^k(\sigma_0)\zeta(k\sigma_0 - 1)}{\zeta^2(k\sigma_0)}\right| \ll \frac{x^{\sigma_0}}{T}$$

for $T \le x$. Hence, from Perron's formula and (1.7),

$$S_{\phi,k}(x) = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta^k(s)\zeta(ks-1)}{\zeta^2(ks)}\frac{x^s}{s}\,ds + O\Big(\frac{x^{\sigma_0}}{T}\Big)$$

for any real number $x > 3$. When $k = 3$, we move the line of integration to $\operatorname{Re} s = \frac{1}{2}$ and consider the rectangular contour formed by the line segments joining the points $c_0 - iT$, $c_0 + iT$, $\frac{1}{2} + iT$, $\frac{1}{2} - iT$ and $c_0 - iT$ in the anticlockwise sense. We observe that the integrand has a triple pole at $s = 1$ and a simple pole at $s = \frac{2}{3}$. Thus, we obtain the main term from the sum of the residues coming from the poles at $s = 1$ and $\frac{2}{3}$. Hence, using the Cauchy residue theorem,

$$S_{\phi,3}(x) = J_3(x,T) + I_{3,1}(x,T) + I_{3,2}(x,T) - I_{3,3}(x,T) + O\Big(\frac{x^{4/3+\varepsilon}}{T}\Big), \tag{4.1}$$

where

$$J_3(x,T) = \Big(\operatorname*{Res}_{s=1} + \operatorname*{Res}_{s=\frac{2}{3}}\Big)\frac{\zeta^3(s)\zeta(3s-1)}{\zeta^2(3s)}\frac{x^s}{s}. \tag{4.2}$$

Here the integrals are given by

$$I_{3,1}(x,T) = \frac{1}{2\pi i}\int_{1/2+iT}^{4/3+\varepsilon+iT}\frac{\zeta^3(s)\zeta(3s-1)}{\zeta^2(3s)}\frac{x^s}{s}\,ds, \tag{4.3}$$

$$I_{3,2}(x,T) = \frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT}\frac{\zeta^3(s)\zeta(3s-1)}{\zeta^2(3s)}\frac{x^s}{s}\,ds, \tag{4.4}$$

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$$I_{3,3}(x,T) = \frac{1}{2\pi i}\int_{1/2-iT}^{4/3+\varepsilon-iT} \frac{\zeta^3(s)\zeta(3s-1)}{\zeta^2(3s)}\frac{x^s}{s}\,ds.$$

Similarly,

$$S_{\phi,4}(x) = J_4(x,T) + I_{4,1}(x,T) + I_{4,2}(x,T) - I_{4,3}(x,T) + O\Big(\frac{x^{5/4+\varepsilon}}{T}\Big),$$

where

$$J_4(x,T) = \operatorname*{Res}_{s=1} \frac{\zeta^4(s)\zeta(4s-1)}{\zeta^2(4s)}\frac{x^s}{s}, \tag{4.5}$$

$$I_{4,1}(x,T) = \frac{1}{2\pi i}\int_{1/2+a+iT}^{5/4+\varepsilon+iT} \frac{\zeta^4(s)\zeta(4s-1)}{\zeta^2(4s)}\frac{x^s}{s}\,ds, \tag{4.6}$$

$$I_{4,2}(x,T) = \frac{1}{2\pi i}\int_{1/2+a-iT}^{1/2+a+iT} \frac{\zeta^4(s)\zeta(4s-1)}{\zeta^2(4s)}\frac{x^s}{s}\,ds,$$

$$I_{4,3}(x,T) = \frac{1}{2\pi i}\int_{1/2+a-iT}^{5/4+\varepsilon-iT} \frac{\zeta^4(s)\zeta(4s-1)}{\zeta^2(4s)}\frac{x^s}{s}\,ds$$

with $a = 1/\log T$ for any large number $T(> 5)$.

5. Proofs of Theorems 1.4 and 1.5

5.1. Proof of the formula (1.9). Consider the estimate $I_{3,1}(x,T)$. We use Lemma 2.1 and (4.3) to deduce the estimation

$$\begin{aligned}
I_{3,1}(x;T) &= \frac{1}{2\pi i}\int_{1/2}^{4/3+\varepsilon} \frac{\zeta(\sigma+iT)^3\zeta(3\sigma-1+3iT)}{\zeta(3\sigma+3iT)^2(\sigma+iT)}x^{\sigma+iT}\,d\sigma\\
&\ll \frac{1}{T}\Big(\int_{1/2}^{2/3}+\int_{2/3}^{1}+\int_{1}^{4/3+\varepsilon}\Big)|\zeta(\sigma+iT)|^3|\zeta(3\sigma-1+3iT)|x^\sigma\,d\sigma\\
&\ll T^{2/3}\log^4 T\int_{1/2}^{2/3}\Big(\frac{x}{T^2}\Big)^\sigma d\sigma + \log^4 T\int_{2/3}^{1}\Big(\frac{x}{T}\Big)^\sigma d\sigma + \frac{\log^4 T}{T}\int_1^{4/3+\varepsilon} x^\sigma\,d\sigma\\
&\ll \frac{x^{4/3+\varepsilon}}{T}\log^4 T.
\end{aligned}$$

Similarly, the estimation of $I_{3,3}(x,T)$ is of the same order. Hence, taking $T = x$ in the estimations of $I_{3,1}(x,T)$ and $I_{3,3}(x,T)$, we find that the total contribution of the horizontal lines in absolute value is

$$\ll x^{1/3+\varepsilon}. \tag{5.1}$$

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Now we estimate $I_{3,2}(x,T)$. We use (4.4), the estimate $\zeta(\frac{3}{2}+it) \asymp 1$ for $t \ge 1$, the well-known estimate

$$\int_1^T \frac{|\zeta(\frac{1}{2}+iu)|^4}{u}\,du \ll \log^5 T \tag{5.2}$$

for any large T and the Hölder inequality to obtain the estimate

$$\begin{aligned} I_{3,2}(x,T) &= \frac{1}{2\pi}\int_{-T}^{T} \frac{\zeta^3(\frac{1}{2}+it)\zeta(\frac{1}{2}+3it)}{\zeta^2(\frac{3}{2}+3it)} \frac{x^{1/2+it}}{\frac{1}{2}+it}\,dt \\ &\ll x^{1/2} + x^{1/2}\int_1^T \frac{|\zeta(\frac{1}{2}+it)|^3}{|\zeta(\frac{3}{2}+3it)|^2}\cdot\frac{|\zeta(\frac{1}{2}+3it)|}{t}\,dt \\ &\ll x^{1/2}\Big(\int_1^T \frac{|\zeta(\frac{1}{2}+it)|^4}{t}\,dt\Big)^{3/4}\Big(\int_1^T \frac{|\zeta(\frac{1}{2}+3it)|^4}{3t}\,dt\Big)^{1/4} \\ &\ll x^{1/2}\log^5 T. \end{aligned} \tag{5.3}$$

Taking $T = x$ in (5.1), (5.3) and (4.1) with $k = 3$, and substituting the above and the residue (4.2) into (4.1) with $k = 3$, we obtain the formula (1.9).

5.2. Proof of the formula (1.10). Let $a = 1/\log T$ ($T \ge 5$). From (4.5) with $k = 4$,

$$J_4(x,T) = xP_{\phi,4}(\log x), \tag{5.4}$$

since $s = 1$ is a pole of $\zeta^4(s)$ of order four, where $P_{\phi,4}(u)$ is a polynomial in u of degree three depending on ϕ. Consider the estimate $I_{4,1}(x,T)$. From (4.6) and Lemma 2.1,

$$\begin{aligned} I_{4,1}(x,T) &\ll \frac{1}{T}\Big(\int_{1/2+a}^{1} + \int_1^{5/4+\varepsilon}\Big)|\zeta(\sigma+iT)|^4|\zeta(4\sigma-1+4iT)|x^\sigma\,d\sigma \\ &\ll T^{1/3}\log^5 T\int_{1/2+a}^{1}\Big(\frac{x}{T^{4/3}}\Big)^{\sigma}\,d\sigma + \frac{\log^5 T}{T}\int_1^{5/4+\varepsilon} x^\sigma\,d\sigma \\ &\ll x^{1/2+a}\frac{\log^5 T}{T^{1/3}} + x^{5/4+\varepsilon}\frac{\log^5 T}{T}. \end{aligned}$$

Similarly, the estimation of $I_{4,3}(x,T)$ is of the same order. Hence, taking $T = x$ in the estimations $I_{4,1}(x,T)$ and $I_{4,3}(x,T)$, we find that the total contribution of the horizontal lines in absolute value is

$$\ll x^{1/4+\varepsilon}. \tag{5.5}$$

We use (5.2) and the estimation $\zeta(1+it) \ll \log^{2/3} t$ for $t \ge t_0$ (see [7, Theorem 6.3]) to obtain the estimation

$$I_{4,2}(x,T) \ll x^{1/2+a} + x^{1/2+a}\log^{2/3} T\int_1^T \frac{|\zeta(\frac{1}{2}+it)|^4}{t}\,dt \ll x^{1/2}\log^{17/3} T. \tag{5.6}$$

We take $T = x$ in (5.4), (5.5), (5.6) and (4.1) with $k = 4$ to complete the proof of the formula (1.10).

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6. Proof of Theorem 1.6

We use (1.4) with $k = 5$ to deduce that

$$\sum_{abcde \le x} \phi(\gcd(a,b,c,d,e)) = \sum_{lm^5n^5 \le x} (\mu * \mu)(n) m\tau_5(l) = \sum_{n \le x^{1/5}} (\mu * \mu)(n) B\Big(\frac{x}{n^5}\Big), \tag{6.1}$$

where $B(x) := \sum_{lm^5 \le x} m\tau_5(l)$. From (1.1),

$$\begin{aligned} B(x) &= \sum_{m \le x^{1/5}} m \sum_{n \le x/m^5} \tau_5(n) \\ &= \sum_{m \le x^{1/5}} m\Big(A_1 \frac{x}{m^5} \log^4 \frac{x}{m^5} + \cdots + A_5 \frac{x}{m^5} + \Delta_5\Big(\frac{x}{m^5}\Big)\Big) \\ &= \widetilde{Q_4}(\log x)x + \sum_{m \le x^{1/5}} m\Delta_5\Big(\frac{x}{m^5}\Big), \end{aligned} \tag{6.2}$$

where $\widetilde{Q_4}(u)$ is a polynomial in u of degree four and $A_1, A_2, \ldots, A_5$ are computable constants. Inserting (6.2) into (6.1),

$$\sum_{abcde \le x} \phi(\gcd(a,b,c,d,e)) = xP_{\phi,5}(\log x) + \sum_{n \le x^{1/5}} (\mu * \mu)(n) \sum_{m \le x^{1/5}/n} m\Delta_5\Big(\frac{x}{m^5 n^5}\Big).$$

Hence, we obtain the formula (1.12). □

7. Appendix

To calculate the main terms of Theorems 1.1 and 1.4, we use the Laurent expansion of the Riemann zeta-function at $s = 1$: that is,

$$\zeta(s) = \frac{1}{s-1} + \gamma + \gamma_1(s-1) + \gamma_2(s-1)^2 + \gamma_3(s-1)^3 + \cdots \tag{7.1}$$

as $s \to \infty$, where γ is the Euler constant and γ_k $(k = 1, 2, 3, \ldots)$ are the Stieltjes constants,

$$\gamma_k := \frac{(-1)^k}{k!} \lim_{N \to \infty} \Big(\sum_{m \le N} \frac{\log^k m}{m} - \frac{\log^{k+1} N}{k+1} \Big).$$

We need the following residues.

$$\begin{aligned} M_2(x) &:= \operatorname*{Res}_{s=1} \frac{\zeta^2(s)\zeta(2s-1)}{\zeta^2(2s)} \frac{x^s}{s} \\ &= \frac{1}{4\zeta^2(2)} x \log^2 x + \frac{1}{\zeta^2(2)}\Big(2\gamma - \frac{1}{2} - 2\frac{\zeta'(2)}{\zeta(2)}\Big) x \log x \\ &\quad + \frac{1}{2\zeta^2(2)}\Big(5\gamma^2 + 6\gamma_1 - 4\gamma + 1 - 4(4\gamma - 1)\frac{\zeta'(2)}{\zeta(2)} - 4\frac{\zeta''(2)}{\zeta(2)} + 12\Big(\frac{\zeta'(2)}{\zeta(2)}\Big)^2\Big)x, \end{aligned} \tag{7.2}$$

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and

$$
\begin{aligned}
J_3(x,T) &:= \Big(\operatorname*{Res}_{s=1} + \operatorname*{Res}_{s=2/3}\Big) \frac{\zeta^3(s)\zeta(3s-1)}{\zeta^2(3s)} \frac{x^s}{s} \\
&= \frac{\zeta(2)}{2\zeta^2(3)} x \log^2 x + \frac{\zeta(2)}{\zeta^2(3)}\Big(3\gamma - 1 + 3\frac{\zeta'(2)}{\zeta(2)} - 6\frac{\zeta'(3)}{\zeta(3)}\Big) x \log x \\
&\quad + \frac{\zeta(2)}{\zeta^2(3)}\Big(3\gamma^2 + 3\gamma_1 - 3\gamma + 1 + 3(3\gamma-1)\frac{\zeta'(2)}{\zeta(2)} - 6(3\gamma - 1)\frac{\zeta'(3)}{\zeta(3)}\Big) x \\
&\quad + \frac{\zeta(2)}{\zeta^2(3)}\Big(27\Big(\frac{\zeta'(3)}{\zeta(3)}\Big)^2 + \frac{9}{2}\frac{\zeta''(2)}{\zeta(2)} - 9\frac{\zeta''(3)}{\zeta(3)} - 18\frac{\zeta'(2)}{\zeta(2)}\frac{\zeta'(3)}{\zeta(3)}\Big) x + \frac{\zeta(\frac{2}{3})^3}{2\zeta^2(2)} x^{2/3}.
\end{aligned}
\tag{7.3}
$$

PROOF. Suppose that $g(s)$ is regular in the neighbourhood of $s = 1$ and $f(s)$ has only a triple pole at $s = 1$. Then the Laurent expansion of $f(s)$ implies that

$$
f(s) := \frac{a}{(s-1)^3} + \frac{b}{(s-1)^2} + \frac{c}{s-1} + h(s),
$$

where $h(s)$ is regular in the neighbourhood of $s = 1$ and a, b, c are computable constants. We use the residue calculation to deduce that

$$
\operatorname*{Res}_{s=1} f(s)g(s) = \frac{a}{2} g''(1) + b g'(1) + c g(1).
$$

To prove (7.3), we use (7.1) to deduce that

$$
\zeta^3(s) = \frac{1}{(s-1)^3} + \frac{3\gamma}{(s-1)^2} + \frac{3\gamma^2 + 3\gamma_1}{s-1} + O(1) \quad \text{as } s \to 1.
$$

Setting

$$
g(s) := \frac{\zeta(3s-1)}{\zeta^2(3s)} \cdot \frac{x^s}{s},
$$

we have

$$
g(1) = \frac{\zeta(2)}{\zeta^2(3)} x, \quad g'(1) = \frac{\zeta(2)}{\zeta^2(3)}\Big(\log x + 3\frac{\zeta'(2)}{\zeta(2)} - 6\frac{\zeta'(3)}{\zeta(3)} - 1\Big) x,
$$

$$
\begin{aligned}
g''(1) &= \frac{\zeta(2)}{\zeta^2(3)} x \log^2 x + \frac{2\zeta(2)}{\zeta^2(3)}\Big(3\frac{\zeta'(2)}{\zeta(2)} - 6\frac{\zeta'(3)}{\zeta(3)} - 1\Big) x \log x \\
&\quad + \frac{2\zeta(2)}{\zeta^2(3)}\Big(1 + 6\frac{\zeta'(3)}{\zeta(3)} - 3\frac{\zeta'(2)}{\zeta(2)} + 27\Big(\frac{\zeta'(3)}{\zeta(3)}\Big)^2\Big) x \\
&\quad + \frac{2\zeta(2)}{\zeta^2(3)}\Big(\frac{9}{2}\frac{\zeta''(2)}{\zeta(2)} - 9\frac{\zeta''(3)}{\zeta(3)} - 18\frac{\zeta'(2)}{\zeta(2)}\frac{\zeta'(3)}{\zeta(3)}\Big) x.
\end{aligned}
$$

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Hence,

$$\Big(\operatorname*{Res}_{s=1} + \operatorname*{Res}_{s=2/3}\Big) \frac{\zeta^3(s)\zeta(3s-1)}{\zeta^2(3s)} \frac{x^s}{s} = \frac{1}{2} g''(1) + 3\gamma g'(1) + 3(\gamma_1 + \gamma^2) g(1) + \frac{\zeta^3(\frac{2}{3})}{2\zeta^2(2)} x^{2/3}.$$

Hence, we obtain the stated identity. To prove (7.2), we use

$$\zeta^2(s)\zeta(2s-1) = \frac{\frac{1}{2}}{(s-1)^3} + \frac{2\gamma}{(s-1)^2} + \frac{\frac{5}{2}\gamma^2 + 3\gamma_1}{s-1} + O(1) \quad \text{as } s \to 1.$$

The proof of (7.2) is similar to that of (7.3). □

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