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Transversals with Residue in Moderately Overlapping $T(k)$-Families of Translates

Dedicated to Ted Bisztriczky, on his sixtieth birthday.

Aladár Heppes

Abstract. Let K denote an oval, a centrally symmetric compact convex domain with non-empty interior. A family of translates of K is said to have property $T(k)$ if for every subset of at most k translates there exists a common line transversal intersecting all of them. The integer k is the stabbing level of the family. Two translates $K_i = K + c_i$ and $K_j = K + c_j$ are said to be σ-disjoint if $\sigma K + c_i$ and $\sigma K + c_j$ are disjoint. A recent Helly-type result claims that for every $\sigma > 0$ there exists an integer $k(\sigma)$ such that if a family of σ-disjoint unit diameter discs has property $T(k)|k \geq k(\sigma)$, then there exists a straight line meeting all members of the family. In the first part of the paper we give the extension of this theorem to translates of an oval K. The asymptotic behavior of $k(\sigma)$ for $\sigma \longrightarrow 0$ is considered as well.

Katchalski and Lewis proved the existence of a constant r such that for every pairwise disjoint family of translates of an oval K with property $T(3)$ a straight line can be found meeting all but at most r members of the family. In the second part of the paper σ-disjoint families of translates of K are considered and the relation of σ and the residue r is investigated. The asymptotic behavior of $r(\sigma)$ for $\sigma \longrightarrow 0$ is also discussed.

1 Introduction

Helly type problems for line transversals of families of convex domains have been studied by a number of authors. The interested reader is referred to the survey papers [9, 11, 17, 23]. In the present note finite families of translated copies of a compact convex domain are considered in the Euclidean plane.

1.1 Definitions and Notations

Throughout the paper the term *oval* will be used for a centrally symmetric compact convex domain in the plane with non-empty interior. (It is well known that the assumption of central symmetry is no restriction when transversal problems are considered in the plane.) Let K be an oval centered at the origin O and

$$\mathcal{F} = \{K_i = K + c_i, i = 1, \dots, n\}$$

a family of a finite number of translated copies, *translates*, of K. Then $\mathcal{C} = \{c_i, i = 1, \dots, n\}$ denotes the set of the centers. The term K-*distance* and notation $|p - q|_K$ will be used for the distance of points p and q measured in the norm induced by K.

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The *width* $w(X)$, (K-*width* $w_K(X)$) of a closed set is the minimum of the distances (K-distances) of pairs of parallel support lines of X. The *diameter* (K-*diameter*) of a closed set is the maximum of the distances (K-distances) between pairs of points in the set.

Family $\mathcal{F}$ is a T-*family* (or it has property T) if there exists a straight line, a *transversal*, intersecting all members of $\mathcal{F}$. We say $\mathcal{F}$ is a $T(k)$-*family* (it has property $T(k)$) for some integer $k \geq 3$, if every subfamily of at most k members of $\mathcal{F}$ has property T. We also say that the family has *stabbing level* k. $\mathcal{F}$ is a T-family if and only if the set of the centers can be covered with a strip of K-width 2.

If for families of some type $T(k)$ implies T, but $T(k-1)$ does not, we say that families of that type have *transversal Helly number* k. A family is said to have property $T-r$ if a straight line exists that intersects all but r members of the family. The (non-negative) integer r is called the *transversal residue* or *residue* for short.

For a real number $\sigma > 0$ let σX denote a set obtained by scaling set X by factor σ. Here, σK is also called the σ-*core* of K. (For $\sigma > 1$ the core is larger than K.) Two translates $K' = K + c'$ and $K'' = K + c''$ are σ-disjoint if $\sigma K + c'$ and $\sigma K + c''$ are disjoint, and family $\mathcal{F}$ is a *packing* if its members are mutually disjoint. $\mathcal{F}$ is a σ-*packing* if the σ-cores of its members form a packing, *i.e.,* if its members are mutually σ-disjoint. If two σ-cores are disjoint, then the K-distance of their centers is more than 2σ.

The notion of σ-packing is closely related to that of the t-disjointness of families, used in [3] for families of discs of unit diameter. Two discs are said t-disjoint, $t > 0$, if their centers are more than t apart [3]. Leaving the realm of Euclidean norm, the notion of σ-packing seems to serve better the general case discussed here.

In Section 2 a few related known theorems are listed. The goals of the investigation and the attained new results can be found in Section 3. Section 4 is devoted to the proofs.

2 Known Results

From the many results connected with transversal Helly numbers we cite only those which are most related to the topic discussed here.

2.1 Families of Disjoint Translates

At first the research was directed to families of disjoint special objects, like congruent discs and translates of a square. In 1957, Danzer [7] proved that the transversal Helly number of families consisting of disjoint congruent discs is 5.

Theorem 2.1 *In a finite family of $n \geq 5$ pairwise disjoint congruent discs $T(5) \Rightarrow T$.*

In the following year, Grünbaum proved an analogous theorem for squares [12].

Theorem 2.2 *In a finite family of $n \geq 5$ pairwise disjoint translates of a square $T(5) \Rightarrow T$.*

Both theorems are sharp in terms of the stabbing level, *i.e.,* $T(4) \not\Rightarrow T$ (see Figure 1(a)). Simple examples show that the requirement of disjointness cannot be

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(fully) dropped from these theorems without losing property T. (For the case of discs, see Figure 1(b)). However, families of overlapping objects have not been investigated for a long period of time.

(a) (b)

Figure 1

In his 1958 paper, Grünbaum conjectured that these results could be extended to families of disjoint translates of an *arbitrary oval* [12].

Grünbaum's Transversal Conjecture In a finite family of $n \geq 5$ pairwise disjoint congruent copies of an oval $T(5) \Rightarrow T$.

As there was no progress reported for a long time, a relaxed form of this conjecture was also published later [21].

The Weak Conjecture of Grünbaum There exists a universal integer $k_0 \geq 5$ such that $T(k_0) \Rightarrow T$ in any finite family consisting of at least k_0 mutually disjoint copies of an oval K.

The weak conjecture of Grünbaum was first verified by Katchalski [19] in 1986.

Theorem 2.3 *In a finite family of $n \geq 128$ disjoint translates of an oval K, $T(128)$ implies T.*

Finally the original conjecture, *i.e.*, the $k_0 = 5$ case, was proved by Tverberg [22].

Theorem 2.4 *In a finite family of $n \geq 5$ disjoint translates of an oval K, $T(5)$ implies T.*

2.2 σ-Packings of Translates

Recently, while investigating families of moderately overlapping unit diameter discs ($\sigma < 1$), more exactly the relation between t-disjointness and property $T(k)$ in families consisting of unit diameter discs, Bezdek, Bisztriczky, Csikós, and Heppes [3] proved that for every positive real number $t > 0$, t-disjoint families of unit diameter discs have a finite transversal Helly number $k(t)$. In other words, it has been proved that to ensure property T a "lower level of disjointness" can be compensated by the requirement of a higher stabbing level and vice versa. The decreasing step function $k(t)$ is fully described by the sequence of its points of discontinuity $t_k = \inf(t|k(t) \leq k), k \geq 3$. The cited result [3, Theorem 4.3], rephrased in our terminology, is the following.

Theorem 2.5 *Let $0 < \sigma < 1$ and $\mathcal{F}$ a finite σ-packing of unit discs. If $k \geq 5$ and $2\sigma(k-3) > \nu_1 = \sqrt{8} + 4\sqrt{\sqrt{3}-1} = 6.2508\cdots$, then $T(k) \Rightarrow T$.*

Using the notation $\sigma_k = \inf(\sigma|k(\sigma) \leq k)$, a simple lower bound for σ, based on regular $(k+1)$-gonal arrangements (see Figures 1(a) and 1(b)), can also be given [3, Theorem 2.1].

Theorem 2.6 *If $k \geq 3$ is odd, then*

$$\sigma_k \geq \frac{\sin \frac{\pi}{k+1}}{1 + \cos \frac{2\pi}{k+1}},$$

and if $k \geq 4$ is even, then

$$\sigma_k \geq \frac{\sin \frac{\pi}{k+1}}{\cos \frac{\pi}{k+1} + \cos \frac{2\pi}{k+1}}.$$

These results also clarify the asymptotic behavior of σ_k since the rate of the upper bound and the lower bound tends to a value smaller than $\nu_1/\pi < 2$. Theorems 2.5 and 2.6 imply the following.

Corollary 2.7 *We have $k(\sigma) = O(1/\sigma)$ as $\sigma \to 0$ and $\sigma_k = O(1/k)$ as $k \to \infty$.*

Unfortunately, the exact value of σ_k is not known for any $k \geq 5$.

2.3 Transversals with Given Residue

In Theorems 2.1, 2.2 and 2.4 (due to Danzer, Grünbaum and Tverberg, respectively) the stabbing level $k = 5$ cannot be lowered without losing property T. However, Katchalski and Lewis proved the existence of a universal (K-independent) integer r such that to every $T(3)$-family of disjoint translates of an arbitrary oval a straight line can be found meeting all but at most r members of the family [20].

Theorem 2.8 *There exists an integer r such that in any $T(3)$-family of disjoint translates of an oval K, $T(3)$ implies $T - r$.*

The first estimates for r were rather rough (starting with $r = 603$ in [20]), but gradual improvement led to the present bound given by Holmsen in 2003 [16].

Theorem 2.9 *In any $T(3)$-family of disjoint translates of an oval K, $T(3)$ implies $T - 22$.*

In the special case when K is a disc much more is known. In 1991 A. Bezdek supplied the lower bound $r \geq 2$ for discs by construction [2].

Theorem 2.10 *In a family of $n \geq 6$ pairwise disjoint congruent discs, $T(3)$ does not imply $T - 1$.*

Recently, expanding Kaiser's method [18] to show that $r \leq 12$, the author succeeded in proving the conjectured missing upper bound $r = 2$ [14, 15].

Theorem 2.11 *In a family of $n \geq 6$ pairwise disjoint congruent discs, $T(3)$ implies $T - 2$.*

The gap between the stabbing levels $k = 3$ and $k = 5$ has also been attacked. The lower bound $r \geq 1$ for $T(4)$-families of discs is due to Aronov, Goodman, Pollack, and Wenger [1].

Theorem 2.12 *In a family of $n \geq 6$ pairwise disjoint congruent discs, $T(4)$ does not imply T.*

The k-sharp upper bound was given by Bisztriczky, Fodor, and Oliveros [4–6].

Theorem 2.13 *In a finite family of $n \geq 6$ pairwise disjoint congruent discs, $T(4)$ implies $T - 1$.*

Theorems 2.1, 2.10, 2.11, 2.12, and 2.13 provide a complete set of k-sharp Katchalski–Lewis type results for the residue in families of disjoint discs with any reasonable stabbing level.

Corollary 2.14 *In a family $\mathcal{F}$ of $n \geq 6$ pairwise disjoint congruent discs, $T(5 - r)$ implies $T - r$, and $T(5 - r - 1)$ does not imply $T - r$, $0 \leq r \leq 2$.*

Let it be mentioned that none of the three upper bound theorems are proved to be sharp considering disjointness, and at least one of them, Theorem 2.11, keeps holding for families of slightly overlapping discs indicating that disjointness might be an artificial condition in terms of transversal Helly-numbers.

3 New Results

Our goal in the present paper is to study the relation between the three essential parameters: the stabbing level $k \geq 3$, the (packing) factor $\sigma \leq 1$ and the residue $r \geq 0$ in σ-packings of translates of an oval K. To any feasible pair of the integer parameters k and r, a critical factor value $\sigma(k, -r)$ can be defined such that if $\sigma > \sigma(k, -r)$, then $T(k) \Rightarrow T - r$ (for any oval K) and if $\sigma < \sigma(k, -r)$, then $T(k) \not\Rightarrow T - r$ (for at least one particular oval). The first question, of course, is whether such a universal K-independent critical value really exists for every pair of $k \geq 3$ and $r \geq 0$. If for some triplet σ, k, and r the relation $T(k) \Rightarrow T - r$ holds, then an increase in any of the three parameters improves the conditions and thereby might admit some relaxation in one or both of the other parameters.

We wish to give lower and upper bounds on the values in this "$\sigma(k, -r)$-table". In the following the two boundary lines of the table, the $r = 0$, $k \geq 3$ row and the $k = 3$, $r \geq 0$ column will be considered. In the first row of the table $\sigma(k, -0) \leq 1$ for $k \geq 5$ (Theorem 2.4), and in the first column $\sigma(3, -r) \leq 1$ for $r \leq 22$ (Theorem 2.9). Our goal is to improve these bounds in the $\sigma \le 1$ range. (By setting the upper bound $\sigma \le 1$, the method we are following excludes from further study a finite part of the first row ($k \le 26$) and a finite part of the first column ($r \le 35$).)

Theorem 3.1 is a generalization of Theorem 2.5 from σ-packings of discs to σ-packings of translates of an oval.

Theorem 3.1 *Let $0 < \sigma \le 1$, $\mathcal{F}$ a finite σ-packing of translates an oval K, and let w denote the width of the narrowest strip covering the centers in $\mathcal{F}$. Let k be an integer fulfilling the inequality*

$$(k-7)\sigma \ge 5w. \tag{3.1}$$

If $\mathcal{F}$ has more than k members and has property $T(k)$, then it has a transversal.

This theorem has two interesting corollaries. According to a result of Eckhoff [8] for K-width of a $T(3)$-family of translates holds

$$w_K \le 4. \tag{3.2}$$

The combination of this bound with Theorem 3.1 yields the following.

Corollary 3.2 *Let $0 < \sigma \le 1$, K an oval, and $\mathcal{F}$ a finite σ-packing of more than k translates of K. If k fulfills the inequality $(k-7)\sigma \ge 20$ and $\mathcal{F}$ has $T(k)$, then it has a transversal.*

(In the above terminology this theorem deals with the $k \ge 27$ part of the first, $r = 0$ row of the table.)

To understand the asymptotic behavior of $\sigma(k,-0)$, asymptotic lower and upper bounds are needed. Corollary 3.2 provides an upper bound and Theorem 2.6 a lower bound for $\sigma(k,-0)$, both of the same order of magnitude. Asymptotically the lower bound is $\pi/(2k)$, while Corollary 3.2 gives $20/k$. Thus we have the following corollary.

Corollary 3.3 *$\sigma(k,-0) = O(1/k)$ as $k \to \infty$.*

Next the $r > 35$ part of the other boundary line of the table, the $\sigma(3,-r)$ column is considered. To lay a foundation for an asymptotic analysis we give an upper bound on the residue r in the following result.

Theorem 3.4 *Let $0 < \sigma \le 1$, K an oval, $\mathcal{F}$ a finite σ-packing of translates of K, and r a non negative integer fulfilling the inequality $r \le (4/\sigma + 2)^2$. Then $T(3)$ implies $T - r$.*

Remark. Although the existence of a σ-dependent upper bound for the residue directly follows from [20, Corollary 3], it does not give the order of magnitude. The required bound with the correct order can be derived, however, from [20, Theorem 2]. Nevertheless, the above theorem has been included in this paper for completeness sake and to provide a concrete (and stronger) bound attained in a different way.

A lower bound for r is given in the following.

Theorem 3.5 *Let K be a unit disc. There exists a number $\sigma^* > 0$ such that if $\sigma < \sigma^*$, then $r(\sigma) \geq c^*/\sigma^2$, where $c^* = 0.077$.*

Theorems 3.4 and 3.5 describe the asymptotic as k tends to infinity.

Corollary 3.6 *We have $r = O(1/\sigma^2)$ as $\sigma \to 0$ and $\sigma_k(-r) = O(1/\sqrt{r})$ as $r \to \infty$.*

Remark. It is worth comparing Corollary 2.7, Corollary 3.3 and Corollary 3.6. It is little wonder that the lower and upper boundaries of σ_k have the same order of magnitude when σ-packing $T(k)$-families of discs are studied. It is more surprising that a similar phenomenon is experienced when, instead of a single shape, common bounds are derived for the set of *all ovals*. This shows that the shape of an oval basically does not influence the behavior of the investigated parameters, at least asymptotically.

To further illustrate the nature of the problem, we mention that, while among $T(3)$-families of disjoint discs the smallest residue is 2 (Theorem 2.11), for $T(3)$-families of translates of a square the residue can be as large as 4.[1] On the one hand, we do not know if an oval exists for which a $T(3)$-family of disjoint translates has residue $r > 4$ (see Theorem 2.9), on the other hand, we do not know if an oval exists for which the guaranteed smallest residue is $r = 3$ or $r = 1$. What we do know is that the smallest r depends on the shape of K. As a matter of fact, instead of a $\sigma(k, -r)$ table of *values* it would be more correct to speak of a table of *intervals*. It would be desirable to determine the best and worst *shape* for at least a single entry of our table.

4 Proofs

First some useful tools are established to serve the proofs of Theorems 3.1 and 3.4.

Referring to standard reduction, it can be assumed that the centers of $\mathcal{F}$ are in general position (for details, see [22]). In particular, all centers have different x-coordinates and all pairs of centers determine different directions.

We can suppose without loss of generality that strip S, one of the strips of smallest K-width covering all centers in $\mathcal{F}$, is horizontal with boundary lines $y = w/2$ and $y = -w/2$. As affinity does not effect the claims of these theorems we can assume throughout the proofs that K is centered at O, inscribed into the axis-parallel square Q of side length 2 centered at O (see Figure 2), and has the additional property that point $(1, 0)$ and points $(a, 1)$, $-1 \leq a \leq 1$, as well as their symmetric counterparts, are common boundary points of K and Q (for details, see [13]).

The first lemma is about the relation of the width and the K-width of strips.

Lemma 4.1 *Let $p(x_p, y_p)$ and $q(x_q, y_q)$, $x_p < x_q$, be two points such that the segment going from p to q has a finite slope s, and let S_1 and S_2 be two strips, each with its lower boundary line through $p(x_p, y_p)$ and its upper boundary line through $q(x_q, y_q)$, and s_1 and s_2 denote their respective slopes.*

If $|x_q - x_p| > w_K(S_1)$ and either $-1 \leq s_1 < s_2 < s \leq 1$ or $-1 \leq s < s_2 < s_1 \leq 1$, then $w_K(S_2) < w_K(S_1)$.

Proof Let us consider the translate $K^* = K + (p + q)/2$ and the axis-parallel square $Q^* = Q + (p + q)/2$ circumscribed about K^*. Clearly, we have $w_K(S_i) = 2\alpha_i$, where

[1] A. Holmsen, *A transversal theorem for rectangles in the plane.* Unpublished manuscript.

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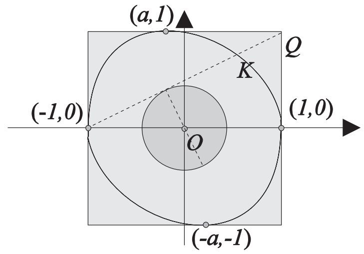

Figure 2: Oval K inscribed into square Q.

$\alpha_i K^*$ is inscribed into S_i, $i = 1, 2$. The side length of $\alpha_1 Q^*$ is $2\alpha_1$. The bounds on the values of the slopes ensure that the line through p and q intersects the vertical sides of $\alpha_1 Q^*$. If, in addition, the inequality $|x_q - x_p| > 2\alpha_1$ holds, then p and q lie outside of $\alpha_1 Q^*$. Consequently, changing the slope of S_1 by tilting the boundaries of S_1 about p and q toward the center $(p+q)/2$ of K^*, brings the slope of the strip closer to s. The boundary lines of S_2 cut off symmetrical parts of $\alpha_1 K^*$ together with the points of $\alpha_1 K^*$ lying on the boundary lines of S_1 (see Figure 3). Hence $\alpha_2 < \alpha_1$ holds and $w_K(S_2) < w_K(S_1)$ as claimed. ■

Since the centers of $\mathcal{F}$ are in general position, there are at most three centers on the boundary lines of S, say, a single center, $c_0(0, w/2)$ on the upper line and at most two centers on the lower one.

Lemma 4.1 has the following corollary.

Corollary 4.2 *Suppose that $c_1(x_1, -w/2)$ and $c_2(x_2, -w/2)$, $x_1 \leq x_2$, are the only centers on the lower boundary of strip S. Then $x_1 \leq w$ and $x_2 \geq -w$. (The two centers might coincide.)*

Proof Suppose that the corollary is false and either $w < x_1 \leq x_2$ or $x_1 \leq x_2 < -w$. The case symmetry allows us to consider only the second case (Figure 4).

There is no center on the open half lines $x > x_2$, $y = -w/2$ and $x < 0$, $y = w/2$, respectively. Consequently, the set of all centers can be covered by a strip S' of slightly

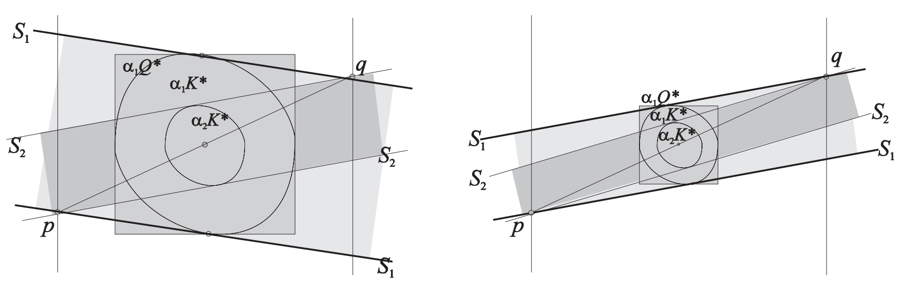

Figure 3: Illustration of the case $-1 \leq s_1 \leq s_2 \leq s \leq 1$.

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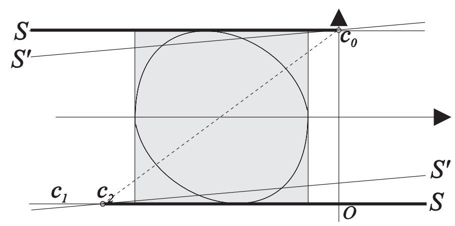

Figure 4: The case $x_2 < -w$.

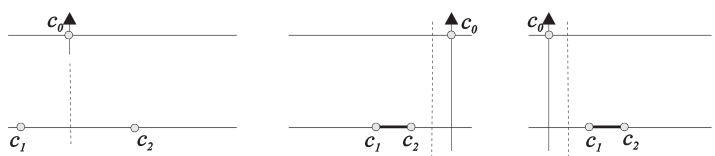

Figure 5: The spine of a family.

positive slope with its boundary lines invariably going through c_0 and c_2, respectively. The application of Lemma 4.1 to c_0, c_2, S, and S' as p, q, S_1, and S_2, respectively, implies, with reference to $|x_0 - x_2| > w$, that $w_K(S') < w = w_K(S)$. This would, however, contradict the assumption that S is a strip of minimal K-width covering all centers. ■

Proof of Theorem 3.1 Let $\sigma < 1$ be a positive scalar and, accordingly, k an integer fulfilling condition (3.1).

We assume that contrary to the claim of the theorem, a counterexample exists consisting of $n \geq k + 1$ translates. Moreover, let n be the smallest such integer and $\mathcal{F}$ an n-member counterexample, *i.e.*, $\mathcal{F}$ has no transversal but each of its subsets of at most $n - 1$ translates has one. It will be proved that such a family $\mathcal{F}$ cannot exist.

Since $\mathcal{F}$ is an assumed counterexample, for the width $w = w(S)$, this time identical with the K-width $w_K(S)$, we have $w_K(S) = w(S) > 2$.

The n centers of $\mathcal{F}$ are convexly independent, else a center could be removed without reducing the covering width of the set of all centers. Let us now order the centers by increasing abscissas and consider the subset $\mathcal{C}'$ of the "middle" $n - 4$ centers received by removing the first two centers and the last two ones and let R be the smallest horizontal rectangle of height w and length l, a slice of strip S, covering $\mathcal{C}'$.

Next, we define the *spine* of the family (see Figure 5). If c_1 and c_2 are separated by the y-axis then the spine is the y-axis, else it is the vertical line half way between c_0 and segment c_1c_2. In the second case the spine is the vertical line $x = c_1/2$ or $x = c_2/2$. By Corollary 4.2 the distance to the spine and the y-axis is never larger than $w/2$.

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Proposition 4.3 *The length of R is at least $3w$.*

Proof By exploiting the fact that the points of $\mathcal{C}$ are convexly independent, it is easy to see that the points of $\mathcal{C}'$ can be shifted outwards up to the boundary of R so that the K-distances of the points do not decrease (see Figure 6). Let us denote the new set of $(n-4)$ points, the vertices of a weakly convex $(n-4)$-gon, by $\mathcal{C}''$.

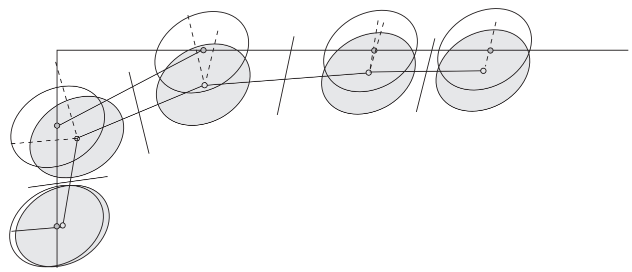

Figure 6: Length estimate for R.

The distance between two consecutive new vertices on a vertical and horizontal sides of R is at least σ and 2σ, respectively. The new $(n-4)$-gon, the convex hull of $\mathcal{C}''$ will have at most 4 sides cutting off the vertices of R, at most $2w/\sigma$ vertical sides, subsets of the two vertical sides of R, and at most l/σ horizontal sides on the two horizontal sides of R.

The inequality $4 + 2w/\sigma + l/\sigma \geq n-4$ and condition (3.1) imply

$$l \geq 3w \tag{4.1}$$

as stated in Proposition 4.3. ∎

Without loss of the generality we may assume that the distance of the spine and the right-most point of set $\mathcal{C}'$ is at least $l/2 \geq 3w/2$. Going from left to right, let us denote the last three points of $\mathcal{C}$ in order of their position on the boundary of the convex hull of $\mathcal{C}$ by $c'(x', y')$, $c'''(x''', y''')$, and $c''(x'', y'')$. (The middle center c''' is not necessarily the right-most one.) It follows partly from (4.1), partly from the possible position of the spine, limited by Corollary 4.2, that in all cases

$$x' - x_0 \geq w, \quad x' - x_1 \geq w, \quad x'' - x_0 \geq w, \quad \text{and} \quad x'' - x_1 \geq w. \tag{4.2}$$

Our goal is to show that the K-width of the "reduced" set $\mathcal{C}^{\text{red}}$ received by removing c''' from the original set $\mathcal{C}$ is larger than 2, *i.e.*, property $T(n-1)$ does not hold in $\mathcal{C}$. (Notice that neither c_0 nor c_1 can be identical with the removed center c''' since $x_0 = 0$ and $x_1 \leq w$, and $x''' > \min(x', x'')$, *i.e.*, $c_0 \in \mathcal{C}^{\text{red}}$ and $c_1 \in \mathcal{C}^{\text{red}}$.)

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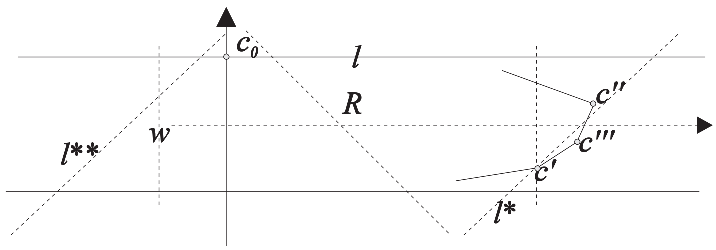

Figure 7: The reduced set of centers.

Let strip S^{red} be the strip of minimal K-width covering $\mathcal{C}^{\mathrm{red}}$ and denote its slope by s^{red}. If property $T(n-1)$ is valid in the original family, then, necessarily, the K-width of S^{red} is fulfilling the inequality

$$w_K(S^{\mathrm{red}}) \leq 2. \tag{4.3}$$

Clearly, strip S^{red} cannot include the removed center c''' as c''' is cut off by a boundary line l^* of S^{red}, actually by a chord of the convex hull of $\mathcal{C}$, together with at least a part of triangle $c'c''c'''$. Moreover, l^* passes through at least one center, c' and/or c'', since it has minimal width (Figure 7).

Next we prove the following.

Proposition 4.4 $|s^{\mathrm{red}}| \geq 1$.

Proof Suppose first that S^{red} is not "very steep", more exactly $s^{\mathrm{red}} \in (-1, 1)$. Evidently, S^{red} cannot be horizontal, because it must be different from S. Let us assume first that s^{red} is positive and let us denote by $c_u(x_u, y_u)$ the center on l^*, or that of lower position if both of c' and c'' are lying on l^*, and by $c_v(x_v, y_v)$ the center of highest position on the other boundary line l^{**} of S^{red}. Notice that by this choice there is no center on l^* below c_u and there is no center on l^{**} above c_v. Clearly, l^{**} must not cut off c_0, thus $x_v \leq 0$ and we also have (4.2) whence in all cases

$$x_u - x_v \geq w. \tag{4.4}$$

Then, as on the other hand $|y_u - y_v| \leq w$ also holds, c_u and c_v are weakly separated by a line orthogonal to S^{red} and passing through $(c_u + c_v)/2$. Consequently, l^* and l^{**} can be slightly rotated toward the midpoint of the segment c_uc_v, (to reduce the slope of the covering strip), without excluding any center from the strip. By (4.3) and (4.4) the conditions of applying Lemma 4.1 are fulfilled: the rotated strip has smaller K-width than $\mathcal{C}^{\mathrm{red}}$ does. However, this result is in conflict with our assumption that the K-width is minimal for S^{red}.

If the slope of l^* is *negative*, then the argument of the proof is the same with the only exception that the selection of centers c_u and c_v on the lines l^* and l^{**} follows reverse direction. ■

Assume now that S^{red} is "steep", *i.e.*, $s^{\text{red}} \notin (-1, 1)$. Then $\mathcal{C}^{\text{red}}$ has at least one point on each vertical side of rectangle R (of height w and length at least $3w$). These two points must be covered by S^{red}. This single condition implies that the (common) width $w(S^{\text{red}})$ of a "steep" S^{red} is at least $\sqrt{2}w$. The side length of an axis parallel square inscribed into S^{red} is necessarily at least w.

Hence $w_K(S^{\text{red}}) \geq w > 2$ as claimed in Theorem 3.1. ∎

Proof of Theorem 3.4 Suppose $\sigma < 1$ and accordingly $r > 36$. Let $c_i(x_i, y_i)$ be the vertices on the lower half of the boundary of the convex hull H of the centers. Let $l(\alpha)$ denote the lower support line of H at some vertex c_i having (signed) angle $-\pi/2 \leq \alpha \leq \pi/2$ measured from the horizontal direction to $l(\alpha)$. Further let $l'(\alpha)$ and $l''(\alpha)$ be the parallel lines tangent to K_i and $2K_i$ from above, respectively. Clearly, $l'(\alpha)$ is a transversal line to all translates except to those whose centers are strictly above $l''(\alpha)$. This part of strip S is cut into (at most) three parts by two vertical lines: by definition, $L(\alpha)$ is the part for which $x < x_i - 2$, $R(\alpha)$ is the one for which $x > x_i + 2$, and $M(\alpha)$ is the part $x_i - 2 \leq x \leq x_i + 2$ (Figure 8).

Following Kaiser [18], we establish that the simultaneous existence of a center in $L(\alpha)$ and another center in $R(\alpha)$ would contradict property $T(3)$, since each of the three translates about these two centers and c_i could be separated from the other two by a line: two vertical lines (sufficiently close to the vertical tangents of K_i) and a parallel line above l'' (sufficiently close to it).

The next observation is that the number k_{left} (k_{right}) of centers in $L(\alpha)$ ($R(\alpha)$) is a left-continuous increasing (right-continuous decreasing) integer function of α. (For $\alpha = -\pi/2$ $k_{\text{left}} = 0$ and, with increasing α, k_{left} changes value when a new center becomes an inner point of the expanding domain $L(\alpha)$. This expansion is continuous while the support line is rotated about the same vertex and discontinuous when the center of rotation changes.)

In view of these two observations, we can rely in the rest of the proof on the validity of the following.

Proposition 4.5 *There exists a value α^* for which $k_{\text{left}} = k_{\text{right}} = 0$.*

Now let line $l'(\alpha^*)$ be selected as a potential transversal line. Two cases will be

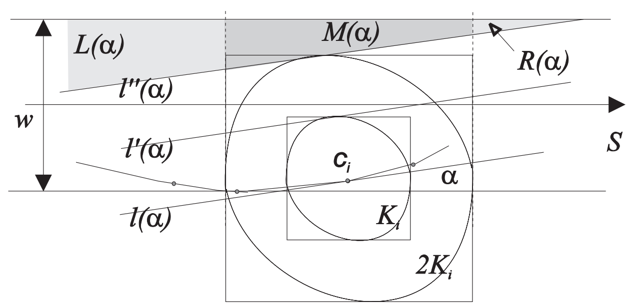

Figure 8: Definition of domains $L(\alpha)$, $M(\alpha)$ and $R(\alpha)$.

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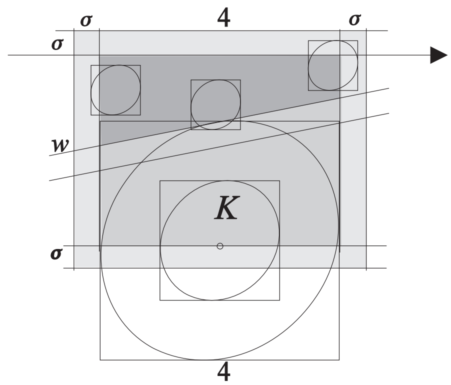

Figure 9: Upper bound on $n(\sigma)$.

distinguished.

Assume first that the support line $l(\alpha^*)$ is tangent to H at a single vertex c_i.

Since $M(\alpha^*)$ is containing the centers of all translates not met by $l'(\alpha^*)$, we are left to give an upper bound on the number $n(\sigma)$ of centers in $M(\alpha^*)$. To give an estimate on $n(\sigma)$, observe that this number cannot be more than the number of non-overlapping cores in a rectangle of height $w + 2\sigma$ and length $4 + 2\sigma$ (Figure 9). Using the trivial lower bound $\sigma^2/2$ for the area of a core and the upper bound (3.2) for the width w we get for the residue (in this case)

$$n(\sigma) \leq (4 + 2\sigma)^2/(2\sigma^2) = (4/\sigma + 2)^2/2. \tag{4.5}$$

In the second case the support line $l(\alpha^*)$ is tangent to H along a side $c_i c_j$.

Now there are (at most) three parts of $M(\alpha^*)$ to consider. Apart from the two domains E_i and E_j above $2K_i$ and $2K_j$ there can be a third one, E_{ij}, between E_i and E_j (Figure 10) that might contain exceptional centers, centers of translates avoided by the potential transversal line $l(\alpha^*)$. However, this cannot happen since any translate about such a center could be separated from K_i and K_j by a line parallel and close to $l'(\alpha^*)$ and properly chosen vertical lines could separate any of K_i and K_j from the other two translates of this triple, contradicting the assumed property $T(3)$ of the family.

Then in this second case, since the residue cannot be larger than the double of $n(\sigma)$ in (4.5), we have $r \leq 2n(\sigma) = (4/\sigma + 2)^2$ as stated.

This concludes the proof of Theorem 3.4. ■

Proof of Theorem 3.5 Consider a square of side length 2 and circles of radius $R = (1 - 1/\sqrt{2}) = 0.292\cdots$ about its vertices (Figure 11).

For an arbitrary positive $\sigma < R/2$, let $n(\sigma)$ denote the maximum number of discs of radius σ which can be packed into a circle of radius R. As is well known (see [10]), the packing density $n(\sigma)\sigma^2/R^2$ of at least two σ-radius discs in a circle of radius R

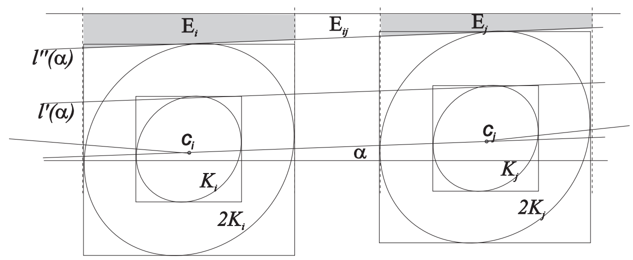

Figure 10: The subdivision of $M(\alpha^*)$.

is smaller than $\pi/\sqrt{12}$ but its value is arbitrarily close to $\pi/\sqrt{12}$ if σ is sufficiently small.

Let $R^2\pi/\sqrt{12} = 0.0777\cdots > c^* = 0.077$ and $\sigma^* > 0$ a value for which $n(\sigma)\sigma^2 > c^*$ whenever $0 < \sigma < \sigma^*$. Consider a packing P of $n(\sigma)$ discs of radius σ in a circle of radius R and copy P by translation into each of the four circles centered at the vertices of the square. The family of the unit discs concentric with the $4n(\sigma)$ small discs defined above is a σ-packing on one hand and has property $T(3)$ on the other, since every selected triple of centers is lying in the union of (at most) three of the circles of radius R and such a triple can be covered by a strip of width 2. It is easy to see that every strip of width 2 leaves at least one quarter of the union of the four circles of radius R uncovered: we can assume that the angle of the strip and the vertical diagonal of the square is at least $\pi/4$. Then the vertical chord of the strip is not longer than the diagonal of the square. Consequently, the strip cannot cover more than half of the union of the two circles at the ends of the vertical diagonal. Hence the residue of every "near-transversal" of this family is at least as large as the number of centers in P: $r(\sigma) > c^*/\sigma^2$, as claimed. ∎

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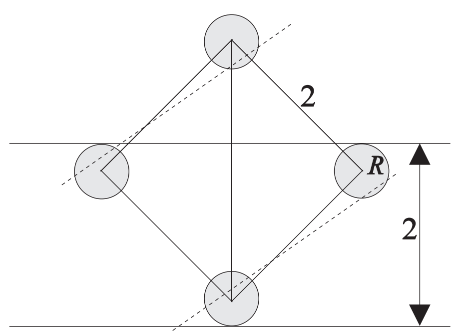

Figure 11: $T(3)$-family of discs with large residue.

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