relativistic field theories. One very good chapter is devoted to the theory of a many-component field. The existence of a canonical energy-momentum tensor is deduced directly from Lorentz-invariance, and the Belinfante-Rosenfeld symmetrization method is added. As examples, the author discusses the scalar field and the Proca vector field.

The last and longest chapter deals with spinors and the Lorentz group. The unimodular representation and the anti-linear representation of temporal and spatial reflexions are discussed in some detail. Other irreducible representations of the full Lorentz groups are considered as well. There is, of course, something to be said for the inclusion of a single chapter on spinors in a textbook of special relativity. However, the author neither provides nor (apparently) expects much understanding of group theory for its own sake, which seems unsatisfactory in view of the close connexion between spinors and the Lorentz group. In the reviewer's opinion, some account of the abstract theory of groups and their representations would do much to explain the rather elaborate notation which spinors require. Possible the happiest alternatives would be either the deliberate omission of spinors, or a full treatment of the Lorentz group on the same level as, say, Wigner's book.

Apart from an appendix on the Michelson-Morley experiment, not much attention has been paid to the experimental foundations of the theory. Furthermore, it seems odd that in a book whose tone is formal throughout there should be several patches of rather inelegant notation. No problems are provided, but a bibliography at the end lists a score of titles.

R.H. Boyer, McGill University

Formes sesquilinéaires et formes quadratiques, by N. Bourbaki, Eléments de Mathématique, Livre II, Algèbre, Chapitre IX. Actualités scientifiques et industrielles 1272, Hermann, Paris, 1959. 211 pages.

Early in 1953, when chapitres I-II, livre V of the Eléments de Mathématique came out, the readers were informed (p. 41 Espaces vectoriels topologiques, chapitre II) that for affine spaces they should look in the book on Algebra (chapitre II, livre II). But in 1955 the second edition on Linear Algebra (chapitre II, livre II) introduced affine and projective spaces respectively in Appendice II and Appendice III. Appendice I called Applications semi-linéaires was a prelude (in a certain sense) to the present chapitre IX.

It is in fact very natural to change what had been planned many years ago for the livre II. As a result Bourbaki's treatment of algebra can still be considered as the most suitable training in this field, for anybody who looks for a solid and deep knowledge regardless of the length of the text.

The meaning of the prefix "sesqui" is "one and a half". Let us see what the sesquilinear transformations are and we will agree that the name was chosen very suggestively. A sesquilinear map satisfies all but the "last half" of the conditions asked (in terms of two variables) for a bilinear map.

If A, B are rings with units and E is a right A-module, F a left B-module and G an (A, B)-bimodule, then a map $\phi: E \times F \to G$ is bilinear if

(1) for each $y \in F$, the map $x \to \phi(x, y)$ is A-linear,

(2) for each $x \in E$, the map $y \to \phi(x, y)$ is B-linear.

These two conditions imply the existence of two maps

(1') $d_\phi: F \to \mathcal{L}_A(E, G)$

(2') $s_\phi: E \to \mathcal{L}_B(F, G)$

which are respectively B-linear and A-linear, $\mathcal{L}_A(E, G)$ being the B-module of all the A-linear maps of E into G and $\mathcal{L}_B(F, G)$ the A-module of all the B-linear maps of F into G. Conversely two such maps d_ϕ, s_ϕ define just a bilinear map of $E \times F$ into the (A, B)-bimodule G.

If both d_ϕ, s_ϕ are injective, ϕ is called non-degenerate.

The bilinear maps are completely defined if we know the values for the generators of E and F. If E, F, G have finite bases then we can associate a matrix with every bilinear map.

A semi-linear map $u: E \to F$, where E is a right (left) A-module and F is a right (left) B-module, is said to be associated with an isomorphism $\lambda: A \to B$, if u is an homomorphism of the two additive underlying groups E and F which satisfies also the following diagram

$$(3) \qquad \begin{array}{ccc} A \times E & \xrightarrow{\sigma_E} & E \\ \lambda \downarrow \quad \downarrow u & & \downarrow u \\ B \times F & \xrightarrow[\sigma_F]{} & F \end{array} \quad \text{where} \begin{cases} \sigma_E(a, x) = ax \\ \sigma_F(b, y) = by \end{cases}.$$

If A and B are the same ring and λ the identity isomorphism, then u will be an A-linear map. The knowledge of the isomorphism $\lambda: A \to B$ gives the possibility of knowing u completely if the values of u for a set of generators are given.

Suppose now that, in the definition of the bilinear maps, we replace the condition (2) by something which can be considered as "half as strong" and we will obtain the sesquilinear maps. This is done as follows. Take a fixed anti-automorphism $J: B \to B$; then, instead of each map $y \to \phi(x, y)$ being a B-linear map, assume that it is just a semi-linear map, where the J-anti-automorphism is considered as an isomorphism of B onto the opposite ring structure of B. If these conditions are satisfied then $\phi: E \times F \to G$ is called a sesquilinear right map associated with J.

We can get the J-right sesquilinear maps also by requiring that $d_\phi: F \to \mathcal{L}_A(E, G)$ should be a semilinear map with respect to J, considered as an isomorphism of B onto the opposite ring structure of B. A similar definition will give the J-left sesquilinear maps.

I believe that the foregoing is sufficient justification of this name, a sesquilinear map being "one and a half" linear (but not necessarily twice linear as the bilinear maps). Also it is conceivable that many of the techniques related to linear algebra may be used for the sesquilinear maps. A matrix can be associated with a sesquilinear map $\phi: E \times F \to G$ if E, F, G have finite bases.

Taking the particular case of a left A-module E, a right A-module F, and the (A, A)-bimodule structure on A, we will get the right J-sesquilinear forms $\phi: E \quad F \to A$. In this case $d_\phi: F \to E^*$ (dual of E) and $s_\phi: E \to F^*$. Therefore we have

(4) $$\phi(x, y) = \langle x, d_\phi(y) \rangle = \langle y, s_\phi(x) \rangle .$$

Rank, tensor product, orthogonality and calculus of matrices can easily be adapted to sesquilinear maps.

The discriminant of a sesquilinear form is introduced in §2.

§3 deals with hermitian and quadratic forms; if A is a ring with an involutory anti-automorphism $\alpha \to \overline{\alpha}$ (as for instance $z \to \overline{z}$ for complex numbers) and ε a central element of A, a sesquilinear form ϕ (for this anti-automorphism) is called a ε-hermitian form if $\phi(x, y) = \varepsilon \phi(y, x)$. The 1-hermitian form is simply called hermitian and the (-1)-hermitian form is called anti-hermitian.

If A is a commutative ring, a quadratic form Q on E is a map $Q: E \to A$ such that

(i) $Q(\alpha x) = \alpha^2 Q(x)$

(ii) $\phi: (x, y) \to Q(x + y) - Q(x) - Q(y)$ is a bilinear symmetric (alternating) form if A is of characteristic $\neq 2$ (characteristic 2). ϕ is called the associated bilinear form of Q and $\phi(x, x) = 2Q(x)$.

Conversely, for each symmetric bilinear form ϕ on E, $x \to \phi(x, x)$

is a quadratic form on E. There is therefore a 1-1 correspondence between the symmetric bilinear forms and the quadratic forms, if the scalar 1+1 has an inverse in A(therefore if characteristic A $\neq$ 2).

§4 introduces the isotropic elements of a module (an element is called isotropic for an hermitian form ϕ if $\phi(x, x) = 0$, that is if it is orthogonal to itself for ϕ) and gives the Witt decomposition of a vector space and the theorem of E. Witt.

§5 is concerned with the reduction of the bilinear alternating forms defined for a free A-module where A is a domain of integrity having only principal ideals. Then the symplectic group Sp(ϕ) is defined for an alternating form ϕ on E, as the group of all the automorphisms of E which leave ϕ invariant.

§6 gives more information on hermitian forms. The group U(ϕ) of all the automorphisms of E which leave invariant the hermitian form ϕ on E is called the unitary group associated with ϕ. For a quadratic form $Q \neq 0$ on E the orthogonal group O(Q) associated with Q is defined in the same way. Then the group of similarities associated with a hermitian form is introduced.

The structure of hermitian geometry is given as follows: an additional structure is put on the underlying affine space L by considering a non-degenerate hermitian form ϕ (with respect to an anti-automorphism $J: A \to A$) on the A-vector space T of the translations of L; ϕ is called the defining metric form of the hermitian geometry on L. If $J: A \to A$ is the identity and A is commutative then L is called a euclidean space (for ϕ); if the characteristic of A $\neq$ 2 the "Theorem of Pythagoras" holds for the euclidean case. For this paragraph, 29 exercises, which are left to the reader who wants to check his ability, contain a large amount of important classical results.

In §7 the field of scalars A is restricted to the cases :

(i) a maximal ordered field (therefore commutative and of characteristic zero).

(ii) a quadratic extension K(i) of a maximal ordered field K with $i^2 = -1$.

(iii) the quaternions on a maximal ordered K.

The J-anti-automorphism is the identity in (i) and $J: \alpha \to \overline{\alpha}$ for (ii) and (iii).

Then the positive hermitian forms are introduced and the theorem known as the "Law of inertia" is proved. Also much information is given in the 27 exercises left to the reader.

§8, Types de formes quadratiques, is based on E. Witt's paper

(J. Reine Angew. Math. 176 (1937), 31-44). An equivalence relation is defined for the non-degenerate quadratic forms on an A-vector space E, where A is commutative. The equivalence classes for this relation are called "types" (set-theoretical precautions are taken in the Bourbaki way so that the "all" difficulties are avoided). A group structure, called the Witt group, is available for the set of types of non-degenerate quadratic forms defined for the finite dimensional vector spaces on A. If characteristic A $\neq 2$ then the Witt group can become the underlying additive group of a commutative ring with unit, where the multiplication is obtained by passing through a tensor product of two quadratic forms. An unusual exception; no exercise is given.

In §9 we have the description of Clifford Algebras and their classification. The Clifford group and the special Clifford group are introduced, with a good deal of material in the exercises.

§10 is called "Angles". As "space" is taken a "plane", i.e., an A-vector space E of dimension 2, where A is commutative, of characteristic $\neq 2$, with an associated bilinear, non-degenerate, symmetric form ϕ on E. The group of similarities, the group S^+ of direct similarities, the group O of rotations, the group H of homotheties are introduced, and a sketch is given of plane trigonometry in this general setting. For the definition of angles the field A is restricted to a maximal ordered field (therefore of characteristic 0) and ϕ is positive. The angles are equivalence classes of straight lines (1-dimensional vector spaces), for a relation involving direct similarities (or rotations). A canonical 1-1 map of S^+/H onto the set $\mathfrak{A}_0$ of all the angles gives the possibility of transporting the group structure on $\mathfrak{A}_0$. The right angle is then the only element of order two in this group. Finally trigonometric functions are introduced.

The historical note has 14 pages and is a masterpiece. As usual, the bibliography is associated with the historical note but not (sometimes unfortunately) with the text itself.

148 exercises containing minor and major results will insure a good training on all these topics.

An index of notations and of terminology is included and at the end there is a fold-out with a summary of the most important definitions.

In conclusion I would adopt this formula of advertising: "a new Bourbaki product of high quality".

Philip Obreanu, Queen's University