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# OVERPARTITIONS AND THE $q$-BAILEY IDENTITY

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*Abstract* Using the framework of overpartitions, we give a combinatorial interpretation and proof of the $q$-Bailey identity. We then deduce from this identity a couple of facts about overpartitions. We show that the method of proof of the $q$-Bailey identity also applies to the (first) $q$-Gauss identity.

*Keywords:* partition; overpartition; $q$-Bailey identity; generalized Durfee square

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## 1. Statement of results

In 1973 Andrews [**1**] established the $q$-series identity

$$\sum_{n\geqslant 0}\frac{(-a,-q/a)_n b^n q^{n(n+1)/2}}{(bq)_n(q^2;q^2)_n}=\frac{(-abq,-bq^2/a;q^2)_\infty}{(bq)_\infty}. \tag{1.1}$$

Here we employ the usual basic hypergeometric series notation [**9**]:

$$(x,y)_n := (x,y;q)_n := \prod_{k=0}^{n-1}(1-xq^k)(1-yq^k). \tag{1.2}$$

When $q=1$, equation (1.1) reduces to a result of Bailey [**13**, (III.7), p. 243] on ordinary hypergeometric series, and hence we call it the $q$-Bailey identity. While this identity gets mentioned from time to time (see, for example, [**9**–**11**]), not much has been written about it in the three decades since Andrews's paper appeared. As we shall see, however, there is indeed something more to be said about the $q$-Bailey identity, particularly in the context of overpartitions.

We begin with a combinatorial interpretation and proof of (1.1). Recalling that an overpartition is simply a partition wherein we may overline the first occurrence of a number, it is clear enough that the right-hand side is a generating function for overpartitions, the numerator generating the overlined parts and the denominator generating the non-overlined parts. But what about the left-hand side? It turns out that the summation variable $n$ can be viewed as the size of the *generalized Durfee square* of an overpartition, a generalization of the usual Durfee square which has recently arisen in combinatorial studies of Rogers–Ramanujan-type identities [**7**]. For our purposes, we need only know that the size of the generalized Durfee square of an overpartition $\lambda$, denoted $D(\lambda)$, is defined to be the largest number $n$ such that the number of overlined parts plus the number of non-overlined parts greater than or equal to $n$ is at least $n$. For example, the overpartition $\lambda = (10, 9, \bar{8}, 8, \bar{7}, 4, 3, 2, 2)$ has $D(\lambda) = 5$.

By arguing combinatorially that the right-hand side of (1.3), below, satisfies a recurrence and initial condition also satisfied by the product on the left-hand side, we will establish Theorem 1.1, from which the $q$-Bailey identity will then easily follow.

**Theorem 1.1.** *Let $f_n(r, m)$ be the number of overpartitions of $m$ into $n$ parts such that*

(i) *all non-overlined parts are at least $n$ and*

(ii) *$r$ is the number of odd overlined parts minus the number of even overlined parts.*

*Then*

$$\frac{(-a, -q/a)_n q^{n(n+1)/2}}{(q^2; q^2)_n} = \sum_{\substack{r\in\mathbb{Z},\\ m\geqslant 0}} f_n(r, m) a^r q^m. \tag{1.3}$$

Next we examine two Rogers–Ramanujan-type identities contained in the $q$-Bailey identity: the case when $a = \mathrm{i}\sqrt{q}$ and $b = 1$,

$$\sum_{n\geqslant 0} \frac{(-q; q^2)_n q^{n(n+1)/2}}{(q)_n (q^2; q^2)_n} = \frac{(-q^3; q^4)_\infty}{(q)_\infty}, \tag{1.4}$$

and the case when $q = q^2$, $a = \zeta_6 q$ and $b = 1$,

$$\sum_{n\geqslant 0} \frac{(q^3; q^6)_n q^{n^2+n}}{(q)_{2n} (q^4; q^4)_n} = \frac{(q^9; q^{12})_\infty}{(q^3; q^4)_\infty (q^2; q^2)_\infty}. \tag{1.5}$$

Here $\zeta_6$ is a primitive sixth root of unity.

We interpret these as overpartition identities by viewing the summation variable $n$ in a second way: as the number of columns in the Frobenius representation of an overpartition. We recall [**6**] that the Frobenius representation of an overpartition of $m$ is a two-row array,

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}, \tag{1.6}$$

where the top row is a partition into distinct parts, the bottom row is an overpartition into non-negative parts and the sum of all the entries is $m$. Using a bijection between

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an overpartition and its Frobenius representation [**8**], we also interpret (1.4) in terms of the standard representation of an overpartition. (This can also be done for (1.5), but the result does not have the desired elegance.)

Here and throughout we identify an overpartition $\lambda$ with a pair of partitions $(\rho, \delta)$, the first element of the pair containing the non-overlined parts and the second containing the overlined parts. We also employ the notation $\ell(\cdot)$ for the largest part and $\nu(\cdot)$ for the number of parts.

**Theorem 1.2.** *Let $A_1(m)$ denote the number of overpartitions $\lambda = (\rho, \delta)$ of $m$ such that*

(i) $\ell(\delta) \equiv \nu(\delta) \pmod 2$,

(ii) *the overlined parts alternate in parity,*

(iii) *the largest $D(\lambda) - \nu(\delta)$ non-overlined parts alternate in parity, and*

(iv) *if $D(\lambda) - \nu(\delta) > 0$, then $\ell(\rho) \equiv \nu(\delta) \pmod 2$.*

*Let $B_1(m)$ denote the number of overpartitions of $m$ whose Frobenius representations have a bottom row that is a partition without repeated odd parts. Let $C_1(m)$ denote the number of overpartitions whose overlined parts are congruent to $3 \bmod 4$. Then $A_1(m) = B_1(m) = C_1(m)$.*

**Theorem 1.3.** *Let $B_2(m)$ denote the number of overpartitions whose Frobenius representations have a top row in which the smallest part as well as the differences between successive parts are congruent to $2 \bmod 4$, and a bottom row in which*

(i) *odd parts are overlined and*

(ii) *if $\overline{2k}$ occurs, then $k$ is positive and $\overline{2k+2}, \overline{2k+1}$, $2k$, $\overline{2k-1}$ and $\overline{2k-2}$ do not occur.*

*Let $C_2(m)$ denote the number of ordinary partitions of $m$ where odd parts are congruent to $3 \bmod 4$ and occur at most twice.*

We take a moment to illustrate Theorem 1.2 by recording the overpartitions counted by $A_1(5)$,

$$(\bar{5}),\ (4,1),\ (\bar{4},\bar{1}),\ (\bar{2},2,\bar{1}),\ (\bar{3},1,1),$$
$$(3,\bar{1},1),\ (2,1,1,1),\ (\bar{2},\bar{1},1,1),\ (1,1,1,1,1),$$

the Frobenius representations of the overpartitions counted by $B_1(5)$,

$$\begin{pmatrix}5\\0\end{pmatrix},\ \begin{pmatrix}4\\1\end{pmatrix},\ \begin{pmatrix}3\\2\end{pmatrix},\ \begin{pmatrix}2\\3\end{pmatrix},\ \begin{pmatrix}1\\4\end{pmatrix},$$
$$\begin{pmatrix}2&1\\2&0\end{pmatrix},\ \begin{pmatrix}3&1\\1&0\end{pmatrix},\ \begin{pmatrix}3&2\\0&0\end{pmatrix},\ \begin{pmatrix}4&1\\0&0\end{pmatrix},$$

and the overpartitions counted by $C_1(5)$,

$$(5),\ (4,1),\ (3,2),\ (\bar{3},2),\ (3,1,1),$$
$$(\bar{3},1,1),\ (2,2,1),\ (2,1,1,1),\ (1,1,1,1,1).$$

In each case there are nine overpartitions.

Finally, since there have recently [**4**, **5**, **14**] been some simple and straightforward overpartition-theoretic proofs of the (first) $q$-Gauss identity,

$$\sum_{n\geqslant 0} \frac{(-1/a,-1/b)_n(abcq)^n}{(q,cq)_n} = \frac{(-acq,-bcq)_\infty}{(cq,abcq)_\infty}, \tag{1.7}$$

we will show that the method of proof used in establishing the $q$-Bailey identity may also be applied to prove this identity.

## 2. Proof of the $q$-Bailey identity

### 2.1. Proof of Theorem 1.1

Let $F_n(a)$ denote the right-hand side of (1.3). Clearly, $F_0(a) = 1$. For $n \geqslant 1$, we shall establish that

$$F_n(a) = \frac{q^n}{1-q^{2n}}((a+q^{n-1})F_{n-1}(1/a) + q^n(1/a+q^{n-1})F_{n-1}(a)). \tag{2.1}$$

Start with an overpartition $\lambda$ counted by $F_n(a)$. If neither $\bar{1}$ nor $n$ (non-overlined) occurs in $\lambda$, then remove 1 from each part of $\lambda$. If there is still no $\bar{1}$ or $n$, then remove 1 from each part again, continuing this process until either there is a $\bar{1}$ or $n$ (or both).

Call the resulting overpartition $\lambda'$. If $\bar{1}$ occurs in $\lambda'$, then remove it and then subtract 1 from each remaining part. The result $\lambda''$ is an overpartition into $n-1$ parts whose non-overlined parts are all at least $n-1$. If we subtracted 1 from each part an even number of times in passing from $\lambda$ to $\lambda'$, then $\lambda''$ is an overpartition counted by

$$\frac{aq^n}{1-q^{2n}}F_{n-1}(1/a);$$

otherwise it is counted by

$$\frac{q^{2n}}{a(1-q^{2n})}F_{n-1}(a).$$

Now if $\bar{1}$ does not occur in $\lambda'$, then it has at least one occurrence of $n$. We remove one part of size $n$ and then subtract 1 from each remaining part to obtain $\lambda''$, which is again an overpartition into $n-1$ parts whose non-overlined parts are at least $n-1$. If we subtracted 1 from each part an even number of times in passing from $\lambda$ to $\lambda'$, then $\lambda''$ is an overpartition counted by

$$\frac{q^{2n-1}}{1-q^{2n}}F_{n-1}(1/a),$$

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otherwise it is counted by

$$\frac{q^{3n-1}}{1-q^{2n}} F_{n-1}(a).$$

Putting the four cases together gives (2.1). A simple computation shows that the product on the left-hand side of (1.3) also satisfies the recurrence in (2.1):

$$\begin{aligned}
\frac{q^n}{1-q^{2n}}&\left(\frac{(a+q^{n-1})(-1/a,-aq)_{n-1}q^{n(n-1)/2}}{(q^2;q^2)_{n-1}} + \frac{q^n(a+q^{n-1})(-a,-q/a)_{n-1}q^{n(n-1)/2}}{(q^2;q^2)_{n-1}}\right)\\
&= \frac{q^{n(n+1)/2}}{(q^2;q^2)_n}\left(a(-1/a)_n(-aq)_{n-1} + \frac{q^n}{a}(-a)_n(-q/a)_{n-1}\right)\\
&= \frac{q^{n(n+1)/2}}{(q^2;q^2)_n}\left(\frac{a(-a)_n}{(1+a)}\frac{(-q/a)_n(1+1/a)}{(1+q^n/a)} + \frac{q^n(-a)_n}{a}\frac{(-q/a)_n}{(1+q^n/a)}\right)\\
&= \frac{q^{n(n+1)/2}}{(q^2;q^2)_n}\frac{(-a,-q/a)_n}{(1+q^n/a)}(1+q^n/a)\\
&= \frac{(-a,-q/a)_n q^{n(n+1)/2}}{(q^2;q^2)_n}.
\end{aligned}$$

Together with the initial condition

$$q^{0(0+1)/2}(-a,-q/a)_0/(q^2;q^2)_0 = 1,$$

this implies Theorem 1.1.

### 2.2. The $q$-Bailey identity from Theorem 1.1

Clearly, the right-hand side of the $q$-Bailey identity is the generating function for the number of overpartitions, where the exponent of $b$ counts the number of parts and the exponent of $a$ counts the number of odd overlined parts minus the number of even overlined parts. For the left-hand side, suppose that the generalized Durfee square of an overpartition $\lambda$ has size $n$. Then $\lambda$ may be decomposed into an overpartition $\mu_1$ into exactly $n$ parts whose non-overlined parts are at least $n$ and an ordinary partition $\mu_2$ into (non-overlined) parts at most $n$. Letting the exponent of $a$ keep track of the difference between the number of odd overlined parts and the number of even overlined parts and letting the exponent of $b$ count the number of parts, Theorem 1.1 tells us that the generating function for the overpartitions $\mu_1$ is

$$\frac{b^n q^{n(n+1)/2}(-a,-q/a)_n}{(q^2;q^2)_n}.$$

Of course, the generating function for the partitions $\mu_2$ is $1/(bq)_n$. Putting these together and summing over all $n$ gives (1.1).

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| | | | | |
|---|---|---|---|---|
| 2 | 2 | 2 | 2 | 1 |
| 2 | 2 | 2 | 1 | |
| 2 | 2 | 2 | | |
| 2 | 2 | 2 | | |
| 2 | 2 | | | |
| 2 | 1 | | | |

Figure 1. The 2-modular diagram of $(9, 7, 6, 6, 4, 3)$.

## 3. Overpartition identities from the $q$-Bailey identity

### 3.1. Proof of Theorem 1.2

The right-hand side of (1.4) is clearly the generating function for $C_1(m)$, those overpartitions of $m$ whose overlined parts are congruent to $3 \bmod 4$. To discover the function $B_1(m)$, we decompose the summand into two pieces, corresponding to $q^{n(n+1)/2}/(q)_n$ and $(-q; q^2)_n/(q^2; q^2)_n$. The first piece is the generating function for partitions into $n$ distinct positive parts. The second piece is the generating function for partitions without repeated odd parts and whose parts are at most $2n$, or equivalently, partitions into exactly $n$ non-negative parts without repeated odd parts. This equivalence may be deduced by reading the columns of the 2-modular diagram of a partition without repeated odd parts and whose parts are at most $2n$. For example, take $n = 6$ and the partition $(9, 7, 6, 6, 4, 3)$, whose 2-modular diagram is displayed in Figure 1. Reading the columns, we obtain a partition into six non-negative parts, $(12, 11, 8, 3, 1, 0)$.

Evidently, the two pieces generate Frobenius representations with $n$ columns counted by $B_1(m)$.

Now to see that the coefficient of $q^m$ on the left-hand side is $A_1(m)$, we use a simple bijection (generalizations of which were presented in [**8**]). Let us call $\mu_1$ the partition into distinct parts contributed by the top row of the Frobenius symbol and $\mu_2$ (respectively, $\mu_3$) the partition into even parts (respectively, odd parts) coming from the bottom row. Notice that the number of parts in $\mu_2$ plus the number of parts in $\mu_3$ is necessarily equal to the number of parts in $\mu_1$.

We now make a diagram with $\mu_1$, $\mu_2$ and $\mu_3$. This is illustrated in Figure 2, wherein $n = 5$, $\mu_1 = (9, 8, 5, 3, 1)$, $\mu_2 = (6, 6)$ and $\mu_3 = (7, 5, 3)$. First, draw the Ferrers diagram for $\mu_1$ in the normal way, except that each part is shifted one unit to the right of the preceding part. This creates a diagonal $(d_1, d_2, \dots, d_n)$ with $n$ boxes, running from northwest to southeast. Second, add the $i$th largest part of $\mu_2$ as a row to the left of the diagonal entry $d_i$. Third, add the $j$th smallest part of $\mu_3$ as a column under the diagonal entry $d_{n-j+1}$. Finally, draw a vertical line just to the left of the column containing the largest part of $\mu_3$ (if $\mu_3$ is empty, this line goes just to the right of the diagonal entry $d_n$). The rows to the left of the line form a partition $\delta$ into distinct parts and the columns to the right of the line form an ordinary partition $\rho$. Together these give an overpartition $\lambda = (\rho, \delta)$. In our example, we obtain $(10, 9, \overline{8}, 8, \overline{7}, 4, 3, 2, 2)$.

It is clear that $n$ is now the size of the generalized Durfee square. Verifying conditions (i)–(iv) in the statement of the theorem is routine. For example, the fact that the parts

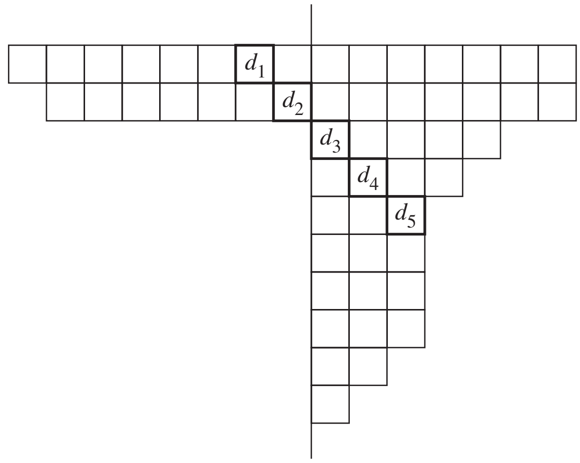


Figure 2. A bijection between an overpartition and its Frobenius representation.

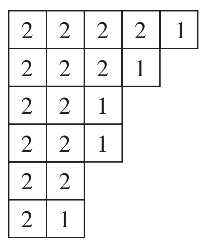


Figure 3. The 2-modular diagram of $(9, 7, 5, 5, 4, 3)$.

added in the rows to the left of the diagonal entries are even together with the fact that the vertical line cuts out a staircase between itself and these even rows gives conditions (i) and (ii).

### 3.2. Proof of Theorem 1.3

We proceed as in the proof of Theorem 1.2, this time examining the identity (1.5). First, we write the product side as

$$\frac{1}{(q^2;q^2)_\infty} \prod_{k \equiv 3 \,(\mathrm{mod}\, 4)} (1 + q^k + q^{2k}). \tag{3.1}$$

This is clearly the generating function for $C_2(m)$.

For $B_2(m)$, we break the summand on the left-hand side into two pieces:

$$q^{n^2+n}/(q^4;q^4)_n \quad \text{and} \quad (q^3;q^6)_n/(q)_{2n}.$$

The first piece is certainly the generating function for the top row of a Frobenius representation counted by $B_2(m)$. Now rewrite the second piece as

$$\frac{1}{(q^2;q^2)_n} \prod_{k=1}^{n} (1 + q^{2k-1} + q^{4k-2}).$$

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This is the generating function for partitions into parts at most $2n$, where odd parts occur at most twice. By reading the columns of the 2-modular diagram of such a partition and overlining any part whose column contains a 1, we see that this piece is the generating function for those overpartitions into $n$ non-negative parts such that odd parts are overlined and if $\overline{2k}$ occurs, then $k$ is positive and $\overline{2k+2}$, $\overline{2k+1}$, $2k$, $\overline{2k-1}$ and $\overline{2k-2}$ do not occur. For example, take $n=6$ and the partition $(9,7,5,5,4,3)$. Its 2-modular diagram is displayed in Figure 3. Reading the columns, we obtain the overpartition $(12,\overline{11},\overline{6},\overline{3},\overline{1})$. This is the bottom row.

## 4. The $q$-Gauss identity

We now prove (1.7) using the same kind of argument as in § 2. To begin, the right-hand side of (1.7) is clearly the generating function for overpartition pairs $(\lambda,\mu)$ such that the exponent of $c$ is $\nu(\lambda)+\nu(\mu)$, the exponent of $a$ is the number of overlined parts of $\lambda$ plus the number of non-overlined parts of $\mu$ and the exponent of $b$ is the number of parts of $\mu$.

We shall also interpret the summand of the left-hand side as this same generating function for overpartitions pairs $(\lambda,\mu)$, with a restriction introduced by the summation variable. This summation variable $n$ will be the largest $n$ such that $\nu(\mu)$ plus the number of overlined parts in $\lambda$ plus the number of non-overlined parts in $\lambda$ which are greater than or equal to $n$ is at least $n$.

Let us write the summand as

$$\frac{G_n(a,b)c^n}{(cq)_n}$$

with

$$G_n(a,b)=\frac{(-1/a)_n(-1/b)_n}{(q)_n}(abq)^n.$$

Now define $F_n(a,b)$ to be the generating function for overpartition pairs $(\lambda,\mu)$ such that $\nu(\lambda)+\nu(\mu)=n$, the exponent of $a$ is the number of overlined parts of $\lambda$ plus the number of non-overlined parts of $\mu$, the exponent of $b$ is the number of parts of $\mu$, and the non-overlined parts of $\lambda$ are greater than or equal to $n$.

Since $1/(cq)_n$ is the generating function for partitions into parts at most $n$, with the exponent of $c$ tracking the number of parts, we will be done if we can show that $G_n(a,b)=F_n(a,b)$. Clearly, $G_0(a,b)=F_0(a,b)=1$. For $n\geqslant 1$, we shall establish that

$$F_n(a,b)=\frac{abq}{1-q^n}(1+q^{n-1}/a)(1+q^{n-1}/b)F_{n-1}(a,b). \tag{4.1}$$

This is obviously true with $F$ replaced by $G$.

We require an auxiliary function. Let $\tilde{F}_n(a,b)$ be the generating function for overpartition pairs $(\lambda,\mu)$ counted by $F_n(a,b)$ with the extra condition that the non-overlined parts of $\lambda$ are *greater than* $n$. This definition implies that

$$\tilde{F}_n(a,b)=\begin{cases}F_n(a,b)-q^n\tilde{F}_{n-1}(a,b), & \text{if } n\geqslant 0,\\ 0 & \text{otherwise.}\end{cases} \tag{4.2}$$

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Now take an overpartition pair $(\lambda, \mu)$ counted by $F_n(a, b)$. If neither $\bar{1}$ nor $n$ (non-overlined) occurs in $\lambda$ and neither $\bar{1}$ nor 1 (non-overlined) occurs in $\mu$, then remove 1 from each part of $\lambda$ and 1 from each part of $\mu$. Continue this process until one of the above conditions is filled. Call the resulting overpartition pair $(\lambda', \mu')$.

If 1 (non-overlined) occurs in $\mu'$, then remove it from $\mu'$. The result $(\lambda'', \mu'')$ is an overpartition pair counted by

$$A_1 = \frac{abq}{1-q^n}\tilde{F}_{n-1}(a,b).$$

If $\bar{1}$ occurs in $\lambda'$ and neither 1 nor $\bar{1}$ occur in $\mu'$, then remove $\bar{1}$ from $\lambda'$ and then subtract 1 from each remaining part of $\lambda'$ and $\mu'$. The result $(\lambda'', \mu'')$ is an overpartition pair counted by

$$A_2 = \frac{aq^n}{1-q^n}F_{n-1}(a,b).$$

If $\bar{1}$ occurs in $\mu'$ while 1 (non-overlined) does not occur in $\mu'$ and $\bar{1}$ does not occur in $\lambda$, then remove $\bar{1}$ from $\mu'$ and then subtract 1 from each remaining part of $\lambda'$ and $\mu'$. The result $(\lambda'', \mu'')$ is an overpartition pair counted by

$$A_3 = \frac{bq^n}{1-q^n}F_{n-1}(a,b).$$

If $\bar{1}$ occurs in $\mu'$ and $\lambda'$ and 1 (non-overlined) does not occur in $\mu'$, then remove the $\bar{1}$ from $\mu'$ and $\lambda'$. Then subtract 1 from each remaining part of $\lambda'$ and $\mu'$. The result $(\lambda'', \mu'')$ is an overpartition pair counted by

$$A_4 = \frac{abq^n}{1-q^n}\tilde{F}_{n-2}(a,b).$$

(Notice that $n$ must be at least 2 for the above to happen.)

Finally, if $n$ occurs in $\lambda'$, $\bar{1}$ does not occurs in $\lambda'$ and 1 or $\bar{1}$ do not occur in $\mu$, then remove $n$ from $\lambda'$ and then subtract 1 from each remaining part of $\lambda'$ and $\mu'$. The result $(\lambda'', \mu'')$ is an overpartition pair counted by

$$A_5 = \frac{q^{2n-1}}{1-q^n}F_{n-1}(a,b).$$

Now all the cases have been covered, and putting them all together gives

$$\begin{aligned} F_n(a,b) &= A_1 + A_2 + A_3 + A_4 + A_5 \\ &= \frac{1}{1-q^n}(aq^n + bq^n + q^{2n-1})F_{n-1}(a,b) + abq(\tilde{F}_{n-1} + q^{n-1}\tilde{F}_{n-2}) \\ &= \frac{abq}{1-q^n}(1 + q^{n-1}/a)(1 + q^{n-1}/b)F_{n-1}(a,b), \end{aligned}$$

by an application of (4.2). This is the recurrence (4.1), completing the proof.

## 5. Concluding remarks

We close with a few questions. First, can the combinatorial method employed in the proofs of (1.1) and (1.7) be applied to other basic hypergeometric series identities? Second, is it possible to prove families of overpartition identities generalizing Theorems 1.2 or 1.3 by using the Bailey machinery [**2**] to embed (1.4) or (1.5) in an infinite family of $q$-series identities? Finally, what about other identities coming from the $q$-Bailey identity? Drew Sills has kindly pointed out to us that some instances of (1.1) occur in Ramanujan's lost notebook (e.g. [**3**, (5.35), (5.38), (5.39)]) and in Slater's list of identities of the Rogers–Ramanujan type (e.g. [**12**, (110), corrected]); undoubtedly some of these have nice consequences for overpartitions as well.

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