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ON CHARACTERS IN THE PRINCIPAL 2-BLOCK

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Abstract

Let k be a complex number and let u be an element of a finite group G. Suppose that u does not belong to $O(G)$, the maximal normal subgroup of G of odd order. It is shown that G satisfies $X(1)-X(u)=k$ for every complex nonprincipal irreducible character X in the principal 2-block of G if and only if $G/O(G)$ is isomorphic either to C_2, a cyclic group of order 2, or to $PSL(2,2^n)$, $n \geqq 2$.

Let G be a finite group. It was shown by Kwok (1975) that if $u \in G^{\#}$ satisfies

$$X(1)-X(u)=k \tag{1}$$

for every complex nonprincipal irreducible character X of G, then a Sylow 2-subgroup of G is elementary abelian. Moreover, if G is simple, then $G \cong PSL(2,2^n)$. A complex characterization of such groups is given in Herzog (1976). A more general equality is analyzed in Herzog (to appear).

The aim of this paper is to classify groups satisfying (1) for every complex nonprincipal irreducible character in the principal 2-block of G. We prove:

THEOREM. *Let G be a finite group, u an element of G and k a complex number. Suppose that* (1) *is satisfied by every complex nonprincipal irreducible character of G belonging to B, the principal 2-block of G.*

Then one of the following statements holds:

(a) $u \in O(G)$,

(b) $|G/O(G)|=2$, $u \notin O(G)$, *or*

(c) $G/O(G) \cong PSL(2,2^n)$, $n \geqq 2$, $u \notin O(G)$ *but* $u^2 \in O(G)$.

Conversely, if G and u satisfy (a), (b) or (c), then (1) *holds.*

PROOF. It is well known for the principal block that

$$O(G) = \cap\{\ker X \mid X \in B^{\#}\}$$

where $B^{\#}$ denotes $B \backslash 1_G$. Thus (1) holds with $k=0$ if and only if $u \in O(G)$.

It is easy to check that if G and u satisfy (b) or (c), then (1) holds. Consequently, it suffices to show that if

(i) $O(G)=1$, $u \neq 1$ (hence $k \neq 0$) and (1) holds, then either (b) or (c) is satisfied.

From now on we denote by Σ or Σ^* the summation over all $X \in B$ or $X \in B^*$, respectively.

By (i) $g=|G|$ is even. If z is an involution in G, then, by the orthogonality relations in blocks (O.R.B.), $\Sigma X(1)X(z)=0$. Since $X(z) \equiv X(1) \pmod 2$, $1_G(1)1_G(z)=1$, and $X(1)^2 \equiv X(1) \pmod 2$, it follows that

(ii) $\Sigma^* X(1)$ is odd.

As $\Sigma^* X(u)$ is a rational integer, (1) implies that k is both rational and an algebraic integer, therefore,

(iii) k is a positive integer.

Suppose that $y \in G$ lies outside the 2-sections of 1 and u. Then by the O.R.B. we get, in view of (1);

$$0=\Sigma X(y)(X(1)-X(u))=k\Sigma^* X(y),$$

hence

$$\Sigma^* X(y)=0. \tag{2}$$

Let w be a 2-element of G of maximal order and let z be the involution in $\langle w \rangle$. Let $\mathscr{P}$ be a prime ideal lying over 2 in $\mathcal{O}$, the integers in $Q(\sqrt[g]{1})$. Since each of $X(w)$, $X(z)$, and $X(1)$ is a sum of $X(1)$ 2-power roots of unity, we have

$$X(w) \equiv X(z) \equiv X(1) \pmod{\mathscr{P}},$$

hence by (ii)

$$\Sigma^* X(w) \equiv \Sigma^* X(z) \equiv \Sigma^* X(1) \equiv 1 \pmod{\mathscr{P}}.$$

Thus, by (2) w and z belong to the 2-section of u. Consequently, if S denotes a Sylow 2-subgroup of G, then:

(iv) S is elementary abelian,

(v) G has one class of involutions, and

(vi) $|u|=2d$, d odd.

Choose H, a minimal normal subgroup in G. Then, by (iv) and (v), $|G:H|$ is odd and H is characteristically simple. Suppose that $H = H_1 \times \cdots \times H_t$, where H_i is nonabelian simple and $t>1$. Let $x \in H_1$, $y \in H_2$ be involutions. By the Krull–Schmidt Theorem the components of H are unique, so that, by conjugation, G acts to permute the components H_i. Hence x and xy are nonconjugate involutions in G, contradicting (v). Thus we have proven that H is either an elementary abelian 2-group or a nonabelian simple group with an elementary abelian Sylow 2-subgroup S. By Walter (1969), in the latter case H is isomorphic to one of the following groups: $PSL(2,q)$, $q>3$, $q\equiv 0$, 3 or 5 (mod 8), J (Janko's smallest group) or $\mathrm{Re}(q)$ (group of Ree type).

If $G=H$, then it is easy to check that either (b) or (c) holds. Thus assume, from now on, that

(vii) G/H is a nontrivial solvable group of odd order.

Let Y be a nonprincipal linear character of G/H and suppose that $Y\in B$. Then as $Y(1)=1$, by (1) and (iii) $Y(u)=-1$, in contradiction to (vii). Thus:

(viii) no nonprincipal linear character of G/H belongs to B.

By the Frattini argument $G=N(S)H$, hence $C_G(S)H\lhd G$. Suppose that $G\supset C_G(S)H$; then by the solvability of $G/C_G(S)H$, $G'C_G(S)H\subset G$. Let M be a maximal (hence normal of prime index) subgroup of G containing $G'C_G(S)H$ and let Y be a nonprincipal linear character of G/M. By (viii) $Y\notin B$, hence by Brauer's criterion for block membership for some $x\in G$

$$cY(x)\not\equiv c1_G(x)(\mathrm{mod}\,\mathcal{P})$$

where $c=|G:C_G(x)|$ and $\mathcal{P}$ is a prime ideal over 2 in $\mathcal{O}$, the integers of $Q(\sqrt[g]{1})$. We conclude that c is odd and $x\notin\ker Y=M$. Thus $G=\langle x\rangle M$ and $x\in C_G(S_1)$ for some Sylow 2-subgroup S_1 of G. As $M\supseteq C_G(S_1)$, $G=M$, a contradiction. We have shown that

(ix) $G=C_G(S)H$.

If H is a 2-group, then by Lemma 1.2.3 of Hall and Higman

$$C_G(S)\subseteq S=H=G$$

contradicting (vii). So assume, from now on, that

(x) H is a nonabelian simple group.

Suppose that $x\in C_G(H)$; then x is of odd order, hence $C_G(H)\subseteq O(G)=1$. Thus

(xi) $G/H\mathrel{\tilde{\subset}} Out(H)$.

As $Out(J)=1$ (Janko, 1966), by (vii) $H\not\cong J$. If $H\cong PSL(2,2^n)$, $n\geqq 2$, then G is generated by H and odd order field automorphisms of H (Carter (1972), p. 211). Therefore these field automorphisms may be chosen to normalize and act faithfully on S, in contradiction to (ix). Since $PSL(2,4)\cong PSL$ (2, 5), it remains to deal with the cases: $H\cong PSL(2,q)$, $q>5$, $q\equiv 3$ or 5 (mod 8) and $H\cong\mathrm{Re}(q)$.

First we prove, denoting by $\mathrm{Irr}(G/H)$ the set of the irreducible characters of G/H, that

(xii) If $Y \in \mathrm{Irr}(G/H) \cap B$, then $Y = 1_G$.

Suppose that $Y \neq 1_G$. Since $\bar{G} = G/H$ is of odd order $\bar{g}$, Y does not belong to the principal 2-block of $\bar{G}$ and there exists $x \in C_G(S)$ such that, denoting xH by $\bar{x}$, and using Brauer's criterion for block membership,

$$\frac{Y(\bar{x})\bar{c}}{Y(1)} \not\equiv \bar{c}\, 1_{\bar{G}}(\bar{x}) = \bar{c} \pmod{\mathcal{P}},$$

where $\bar{c} = |\bar{G} : C_{\bar{G}}(\bar{x})|$ and $\mathcal{P}$ is a prime ideal over 2 in $\mathcal{O}$, the integers of $Q(\sqrt[\bar{g}]{1})$. As $Y(1)$ and $\bar{c}$ are odd integers and $Y(\bar{x}) = Y(x)$, it follows that

$$Y(x) \not\equiv Y(1) \quad \text{hence} \quad \frac{Y(x)c}{Y(1)} \not\equiv c \pmod{\mathcal{P}},$$

where c is the odd integer $|G : C_G(x)|$. Thus $Y \notin B$, a contradiction.

Suppose that $H \cong \mathrm{Re}(q)$. Simple groups of Ree type were described in Ward (1966), where their character table is given on pp. 87–88. We shall use his notation for H. Since $|G : H|$ is odd, it is easily seen from the character table of $\mathrm{Re}(q)$ that each of the 8 irreducible characters ξ_i, $i = 1, \cdots, 8$ belonging to $B_0(H)$ (the principal 2-block of H), is stable in G. Thus if $X \in B$, then $X|_H = e_x Y_x$, where $Y_x \in B_0(H)$ and e_x is a positive integer. Consequently, if h and f denote elements of H of even and odd order, respectively, we get by the O.R.B.:

$$0 = \Sigma X(h)X(f) = \Sigma e_x^2 Y_x(h) Y_x(f) = \sum_{i=1}^{8} n_i \xi_i(h) \xi_i(f). \tag{3}$$

Clearly $n_i \geqq 1$ for $i = 1, \cdots, 8$ and since by (xii) 1_G is the only element of B with H in its kernel, $n_1 = 1$. Again in the notation of Ward, choose $h = JR^a \neq J$ and $f = V$, so that by (3) $0 = n_1 - n_3$, hence $n_3 = n_1 = 1$. Subsequent choices of $h = JR^a \neq J, JR^a \neq J, JT^{-1}, JT^{-1}$ and $f = Y, X, V, W$, respectively, yield $n_2 = 1$, $n_4 = 1$, $n_5 = n_7 = 1$ and $n_6 = n_8 = 1$, respectively, since 1, $im\sqrt{3/2}$ are rationally independent. It follows that B consists of 8 characters X_i, $i = 1, \cdots, 8$ such that $X_i|_H = \xi_i$, $i = 1, \cdots, 8$. Thus, by the O.R.B. and (1)

$$\begin{aligned} 0 &= \Sigma X(u)X(W) = 1 - X_3(u) + X_6(u) + X_8(u) \\ &= 1 - X_3(1) + k + X_6(1) + X_8(1) - 2k, \end{aligned}$$

hence

$$k = 1 - q^3 + (q-1)m(q+1-3m) < 0$$

a contradiction.

Finally, suppose that $H \cong PSL(2, q)$, $q > 5$, $q \equiv 3$ or $5 \pmod 8$. The character table of H is given in Ward (1966), p. 65. Since $|G : H|$ is odd, it is easy to see from his character table that each of the 4 irreducible characters θ_i, $i = 1, \cdots, 4$ belonging to $B_0(H)$ is stable in G. Thus we get a formula similar to (3) and by choosing $h = S_0^{(q-e)/4}$, $S_0^{(q-e)/4}$ and $f = R$, T, respectively, we get $1 = n_1 = n_4$ and $n_2 = n_3 = 1$ since $(q-e)/4$ is odd and 1, $\sqrt{eq}$ are rationally independent. Hence B consists of 4 characters X_i, $i = 1, \cdots, 4$, such that $X_i|_H = \theta_i$, $i = 1, \cdots, 4$. Thus, by the O.R.B.,

$$0 = \Sigma X(R^a)X(u) = 1 - eX_4(u)$$

hence $X_4(u) = e$ and by (1) $k = q - e$. As $q > 5$, a contradiction is then reached by considering the equality $\Sigma X(1)X(u) = 0$, completing the proof of the theorem.

REFERENCES

Roger W. Carter (1972), *Simple Groups of Lie Type* (John Wiley, New York).

Marcel Herzog (1976), 'On groups with extremal blocks', *Bull. Austral. Math. Soc.* **14**, 325–330.

Marcel Herzog (to appear), 'On linear relations between character values', *J. Algebra*.

Zvonimir Janko (1966), 'A new finite simple group with abelian Sylow 2-subgroups and its characterization', *J. Algebra* **3**, 147–186.

Chung-Mo Kwok (1975), 'A characterization of $PSL(2, 2^m)$', *J. Algebra* **34**, 288–291.

John H. Walter (1969), 'The characterization of finite groups with abelian Sylow 2-subgroups', *Ann. of Math.* (2) **89**, 405–514.

Harold N. Ward (1966), 'On Ree's series of simple groups', *Trans. Amer. Math. Soc.* **121**, 62–89.

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