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Free Group Actions on Spaces Homotopy Equivalent to a Sphere

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Abstract. For infinite-dimensional homotopy space forms X/G with G nontrivial finite cyclic groups, we study the homotopy type of X/G. We show that the Euler class of X/G is not zero if and only if X/G is finitely dominated.

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1. Introduction

Let Γ be a discrete group. A *free* Γ*-complex* is a connected CW-complex X together with an action of Γ which permutes freely the cells of X. If X is contractible, then X/Γ is a $K(\Gamma, 1)$-complex and if, in addition, Γ is a finitely presented group, then X/Γ is finitely dominated if and only if Γ is of type *FP*, and X/Γ is homotopy equivalent to a finite complex if and only if Γ is of type *FL* ([B]). However, in general, if X is homotopy equivalent to a finite complex, it is not known when X/Γ is finitely dominated or homotopy equivalent to a finite complex. If X is a simply connected free Γ-complex such that $H_*(X) \cong H_*(S^m)$ for some m, then we shall say that the quotient space X/Γ is a *homotopy space form*. Since X is simply connected and $H_*(X) \cong H_*(S^m)$, X is homotopy equivalent to S^m and $m \geqslant 2$. The purpose of this paper is to discuss the homotopy type of homotopy space forms. The authors would like to thank Prof. Hyunkoo Lee for suggesting this problem and thank the referee for suggesting the apt title.

2. C. T. C. Wall's Finiteness Characterization for CW-Complexes

A space Y is *finitely dominated* if there is a finite complex K such that Y is a retract of K in the homotopy category, i.e., there exist maps $i\colon Y \to K$ and $r\colon K \to Y$ such that $r \circ i$ is homotopic to the identity id_Y. Let us recall a characterization given by C. T. C. Wall for connected complexes to be finitely dominated ([W]). A connected complex

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Y is dominated by a finite complex of dimension n if and only if Y satisfies the following conditions $\mathbf{D}_n$ and $\mathbf{F}_n$:

$\mathbf{D}_n$: $H_i(\widetilde{Y}) = 0$ for $i > n$, and $H^{n+1}(Y; \mathfrak{B}) = 0$ for all coefficient bundles $\mathfrak{B}$ (possibly non-Abelian (cf. [D, LR]) if $n = 1$).

$\mathbf{F}_1$: The group $\pi_1(Y)$ is finitely generated.

$\mathbf{F}_2$: The group $\pi_1(Y)$ is finitely presented, and for all finite 2-complex L and map $\phi\colon L \to Y$ inducing an isomorphism $\phi_*\colon \pi_1(L) \to \pi_1(Y)$, $\pi_2(\phi)$ is a finitely generated $\mathbb{Z}[\pi_1(Y)]$-module.

$\mathbf{F}_n$ ($n \geqslant 3$): Condition $\mathbf{F}_{n-1}$ holds, and for all finite $(n-1)$-complex L and $(n-1)$-connected map $\phi\colon L \to Y$, $\pi_n(\phi)$ is a finitely generated $\mathbb{Z}[\pi_1(Y)]$-module.

For a $\mathbb{Z}[\pi_1(Y)]$-module B, the cohomology groups of Y with coefficients in the coefficient bundle $\mathfrak{B}$ induced by the module B is ([CE, S])

$$H^*(Y; \mathfrak{B}) = H^*(\mathrm{Hom}_{\mathbb{Z}[\pi_1(Y)]}(C_*(\widetilde{Y}), B)).$$

Let $\phi\colon L \to Y$, and $M = Y \cup_\phi (L \times I)$ its mapping cylinder. We define $\pi_n(\phi) = \pi_n(M, L \times I)$, and call ϕ *n-connected* if L and Y are connected and $\pi_i(\phi) = 0$ for $1 \leqslant i \leqslant n$.

3. Conditions $\mathbf{F}_n$ for Homotopy Space Forms

A homotopy space form X/Γ with Γ a finitely presented group satisfies all of the $\mathbf{F}_n$. I.e.,

THEOREM 1. *Let X/Γ be a homotopy space form with $H_*(X) \cong H_*(S^m)$. If Γ is finitely presented, then X/Γ satisfies all of the $\mathbf{F}_n$. Thus X/Γ is homotopy equivalent to a complex with finitely many cells in each dimension.*

Proof. Since $m \geqslant 2$, $\pi_1(X/\Gamma) = \Gamma$ and $\pi_i(X/\Gamma) \cong \pi_i(X) \cong \pi_i(S^m)$ is finitely generated Abelian for each $i \geqslant 2$. For any $\phi\colon L \to X/\Gamma$ which induces an isomorphism on π_1, since X/Γ is a deformation retract of the mapping cylinder M of ϕ, $\pi_*(M) \cong \pi_*(X/\Gamma)$ and hence the $\pi_i(M)$ are finitely generated Abelian for all $i \geqslant 2$. Consider the following long exact sequence of homotopy groups induced by the pair $(L \times I, M)$ and commutative diagrams:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \pi_i(L \times I) & \longrightarrow & \pi_i(M) & \longrightarrow & \pi_i(\phi) & \longrightarrow & \pi_{i-1}(L \times I) & \longrightarrow \\
 & \uparrow & & \uparrow i_* & & & & & \\
 & \pi_i(L) & \xrightarrow{\phi_*} & \pi_i(X/\Gamma) & & & & &
\end{array}$$

This is an exact sequence of $\mathbb{Z}\Gamma$-modules. Suppose L is a finite 2- or $(n-1)$-complex and ϕ induces an isomorphism on π_1 or $(n-1)$-connected, respectively. Since the $\pi_i(M)$ and the $\pi_{i-1}(L \times I)$ are finitely generated for all $i \geqslant 2$, it follows from the above commutative diagram that $\pi_2(\phi)$ or $\pi_n(\phi)$, respectively, is finitely generated as an Abelian group and hence as a $\mathbb{Z}\Gamma$-module. Hence X/Γ satisfies all of the $\mathbf{F}_n$. By [W], X/Γ is homotopy equivalent to a complex of finite type. □

Remark 1. Let X/Γ be a homotopy space form where X is finite dimensional and Γ is finitely presented. Then X/Γ is also finite dimensional and so it satisfies some $\mathbf{D}_n$ by [W] and all of the $\mathbf{F}_n$ by the above theorem. Thus X/Γ is finitely dominated.

4. Induced Action

Consider a finite group G and a finite dimensional free G-complex X with $H^*(X) \cong H^*(S^m)$. Then we see that the induced G-action on $H^*(X)$ is trivial or nontrivial according as m is odd or even. For, suppose G is a cyclic group. Then we have the transfer isomorphism

$$\mathrm{tr}^* : H^*(X/G, \mathbb{R}) \to H^*(X, \mathbb{R})^G.$$

Note that $\chi(X) = \chi(X/G) \cdot |G|$. If m is odd, then $0 = \chi(X) = \chi(X/G) = 1 - \dim H^m(X, \mathbb{R})^G$ or $\dim H^m(X, \mathbb{R})^G = 1$, which can only happen if G acts trivially on $H^m(X)$. Suppose m is even. Then $2 = \chi(X) = \chi(X/G) \cdot |G|$, so G is a trivial group or isomorphic to $\mathbb{Z}/2$. If $G \cong \mathbb{Z}/2$, then $1 = \chi(X/G) = 1 + \dim H^m(X, \mathbb{R})^G$, or $\dim H^m(X, \mathbb{R})^G = 0$, which can only happen if G acts nontrivially on $H^m(X)$. Moreover, it is well known that such a group G has periodic Tate cohomology groups ([L]).

When X/Γ is an infinite-dimensional homotopy space form, we would like to investigate the relation between the nontriviality of the induced Γ-action on $H^*(X)$ and the homotopy type of X/Γ in Section 5. The following examples show that the trivial induced action on $H^*(X)$ has nothing to do with the homotopy type of X/Γ.

EXAMPLE 1. (1) *Let $X = E \times S^{2k-1}$ where E is a contractible space and let G be a finite group acting trivially on E and freely on S^{2k-1}. Then the diagonal G-action on X is free and by the above argument, it induces a trivial action on $H^*(X)$. This homotopy space form is homotopy equivalent to a compact space; $X/G = E \times_G S^{2k-1} = E \times (S^{2k-1}/G) \simeq S^{2k-1}/G$.*

(2) *Consider a nontrivial finite group G acting on the product space $E_G \times S^m$ where E_G is a contractible space with free G-action and S^m has the trivial G-action. Then the diagonal G-action on $E_G \times S^m$ is free and the induced G-action on $H^*(E_G \times S^m)$ is trivial, and the quotient space $E_G \times_G S^m = B_G \times S^m$ is not dominated by any finite complex.*

(3) *Example of Section 5 provides infinite-dimensional homotopy space forms X/G which are not finitely dominated and the induced G-action on $H^*(X)$ is trivial.*

5. Infinite Dimensional Homotopy Space Forms

Let X/G be an infinite dimensional homotopy space form where G is a finite group. Then the triviality of the induced G-action on $H^*(X) \cong H^*(S^m)$ does not depend on whether m is odd or even. We will assume that the induced G-action on $H^*(X) \cong H^*(S^m)$ is nontrivial and then we will study the homotopy type of X/G. The action homomorphism $G \to \mathrm{Aut}(H^m(S^m)) \cong \mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2$ is then surjective and so it gives rise to a short exact sequence $1 \to G_1 \to G \to \mathbb{Z}/2 \to 1$ where the G_1-action on $H^*(X)$ is trivial. In particular, the order of G is even.

EXAMPLE 2. *Let $G = G_1 \times \mathbb{Z}/2$ be a finite group with $G_1 \neq 1$. Let $\mathbb{Z}/2$ act freely on S^{2k}. Then G acts componentwisely and, hence, freely on $E_{G_1} \times S^{2k}$. Since $\mathbb{Z}/2$ acts nontrivially on $H^*(S^{2k})$, G acts nontrivially on $H^*(E_{G_1} \times S^{2k})$. The quotient space $E_{G_1} \times_G S^{2k} = B_{G_1} \times (S^{2k}/(\mathbb{Z}/2))$ is not dominated by any finite complex.*

Since X/G satisfies all of the $\mathbf{F}_n$, and since $X \simeq S^m$ and $m \geqslant 2$, the following are equivalent:

(1) X/G is finitely dominated.
(2) X/G satisfies $\mathbf{D}_n$ for some n.
(3) There is $n \geqslant m$ such that $H^{n+1}(X/G; \mathfrak{B}) = 0$ for all Abelian coefficient bundle $\mathfrak{B}$.

Consider the covering spectral sequence ([CE], Theorem 8.4, p. 354)

$$H^p(G, H^q(X, B)) \Longrightarrow H^{p+q}(X/G; \mathfrak{B}).$$

Let $H^m(X, \mathbb{Z}) = \mathbb{Z}^*$ as a $\mathbb{Z}G$-module and $B^* = \mathbb{Z}^* \otimes B$ with diagonal G-action. Then $H^m(X, B) = H^m(X, \mathbb{Z}) \otimes B = B^*$ as a $\mathbb{Z}G$-module. Since $H^q(X, B) = B$ for $q = 0, m$ and the remaining groups are zero, we obtain a long exact sequence

$$\begin{aligned} \cdots \to H^i(G, B) \to H^i(X/G; \mathfrak{B}) \to H^{i-m}(G, \mathfrak{B}^*) \\ \to H^{i+1}(G, B) \to H^{i+1}(X/G; \mathfrak{B}) \to H^{i+1-m}(G, B^*) \to \cdots. \end{aligned}$$

In this sequence with $i = m$ and $B = \mathbb{Z}^*$, the map $H^{i-m}(G, B^*) \to H^{i+1}(G, B)$ becomes the map $H^0(G, \mathbb{Z}) = \mathbb{Z} \to H^{m+1}(G, \mathbb{Z}^*)$. Let Ω be the image of $1 \in H^0(G, \mathbb{Z})$. We will call Ω the *Euler class* of X/G. Then we obtain a long exact sequence ([L])

$$\begin{aligned} \cdots \xrightarrow{-\cup\Omega} H^i(G, B) \to H^i(X/G; \mathfrak{B}) \to H^{i-m}(G, B^*) \\ \xrightarrow{-\cup\Omega} H^{i+1}(G, B) \to H^{i+1}(X/G; \mathfrak{B}) \to H^{i+1-m}(G, B^*) \xrightarrow{-\cup\Omega} \cdots. \end{aligned}$$

Thus $H^i(G, B) \cong H^i(X/G; \mathfrak{B})$ for all $i < m$ and, hence, we obtain the following theorem:

THEOREM 2. *Let X/G be an infinite-dimensional homotopy space form where G is a finite group and the induced G-action on $H^*(X) \cong H^*(S^m)$ is nontrivial. Then the following are equivalent:*

(1) *X/G is finitely dominated.*
(2) *There is $n \geqslant m$ such that $- \cup \Omega\colon H^{n-m}(G, B^*) \to H^{n+1}(G, B)$ is an epimorphism and $- \cup \Omega\colon H^{n-m+1}(G, B^*) \to H^{n+2}(G, B)$ is a monomorphism for all $\mathbb{Z}G$-module B, where $H^m(X, \mathbb{Z}) = \mathbb{Z}^*$ as a $\mathbb{Z}G$-module and $B^* = B \otimes \mathbb{Z}^*$ with diagonal G-action.*

LEMMA 3. *Let G be a nontrivial finite cyclic group. Then*

$$H^i(G, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{if } i \text{ is odd}, \\ \mathbb{Z}/|G| & \text{if } i \text{ is even}, \end{cases}$$

and

$$H^i(G, \mathbb{Z}^*) = \begin{cases} 0 & \text{if } i \text{ is even}, \\ \mathbb{Z}/2 & \text{if } i \text{ is odd}. \end{cases}$$

LEMMA 4. *Let X/G be an infinite-dimensional homotopy space form where G is a finite cyclic group and the induced G-action on $H^*(X) \cong H^*(S^m)$ is nontrivial. If m is odd, then the Euler class Ω of X/G is zero.*

Proof. Since the map $H^0(G, \mathbb{Z}) \to H^{m+1}(G, \mathbb{Z}^*) = H^{2i}(G, \mathbb{Z}^*) = 0$ is trivial, the Euler class Ω is zero. □

THEOREM 5. *Let X/G be an infinite-dimensional homotopy space form where G is a finite cyclic group and the induced G-action on $H^*(X) \cong H^*(S^m)$ is nontrivial. If the Euler class Ω of X/G is zero, then X/G is not finitely dominated. In particular, if m is odd, by Lemma 4, X/G is not finitely dominated.*

Proof. Since the Euler class Ω is zero, the maps $- \cup \Omega$ are trivial. From the above long exact sequence with $B = \mathbb{Z}$ together with Lemma 3, we see that the cohomology groups $H^*(X/G, \mathbb{Z})$ are not eventually zero. Hence X/G is not finitely dominated. □

The following example provides infinite-dimensional homotopy space forms Y/G with trivial induced G-action on $H^*(Y)$ which are not finitely dominated.

EXAMPLE 3. *Let X/G be an infinite dimensional homotopy space form where G is a finite cyclic group and the induced G-action on $H^*(X) \cong H^*(S^{2k-1})$ is nontrivial. Then by Lemma 4, the Euler class $\Omega \in H^{2k+1}(G, \mathbb{Z}^*)$ of X/G is zero. Consider the join $Y = X * X$ of X with itself and consider the canonical G-action on Y. Then Y/G is an infinite-dimensional homotopy space form and the induced G-action on $H^*(Y) \cong H^*(S^{4k+1})$ is trivial, and the Euler class of Y/G is $\Omega^2 = 0 \in H^{4k+2}(G, \mathbb{Z})$.*

The covering spectral sequence applied to Y/G gives rise to a long exact sequence

$$\cdots \xrightarrow{-\cup\Omega^2} H^i(G,B) \to H^i(Y/G;\mathfrak{B}) \to H^{i-4k-1}(G,B)$$
$$\xrightarrow{-\cup\Omega^2} H^{i+1}(G,B) \to H^{i+1}(Y/G;\mathfrak{B}) \to H^{i-4k}(G,B) \xrightarrow{-\cup\Omega^2} \cdots.$$

Since $\Omega^2 = 0$, the cohomology groups $H^*(Y/G,\mathbb{Z})$ are frequently non-zero, which implies that Y/G is not finitely dominated.

THEOREM 6. *Let X/G be an infinite-dimensional homotopy space form where G is a finite cyclic group and the induced G-action on $H^*(X) \cong H^*(S^{2k})$ is nontrivial. If the Euler class Ω of X/G is nonzero, then X/G is finitely dominated.*

Proof. Let t be a generator of G, ℓ the order of G, $F = F_i = \mathbb{Z}G$ for all $i \geqslant 0$, and let B be a $\mathbb{Z}G$-module. Then $\cdots \to F_2 \xrightarrow{N} F_1 \xrightarrow{t-1} F_0 \twoheadrightarrow \mathbb{Z}$ is a free resolution of $\mathbb{Z}G$-modules of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, where $N = 1 + t + t^2 + \cdots + t^{\ell-1}$ is the norm element of the group ring $\mathbb{Z}G$. Then it is obvious that $\widetilde{t-1}\colon \mathrm{Hom}_{\mathbb{Z}G}(F_{2i},B) \to \mathrm{Hom}_{\mathbb{Z}G}(F_{2i+1},B)$ is $(t-1)\colon B \to B$, and $\widetilde{N}\colon \mathrm{Hom}_{\mathbb{Z}G}(F_{2i+1},B) \to \mathrm{Hom}_{\mathbb{Z}G}(F_{2i+2},B)$ is $(1 + t + \cdots + t^{\ell-1})\colon B \to B$ for all $i \geqslant 0$. Note also that $\widetilde{t-1}\colon \mathrm{Hom}_{\mathbb{Z}G}(F_{2i},B^*) \to \mathrm{Hom}_{\mathbb{Z}G}(F_{2i+1},B^*)$ is $(-1-t)\colon B \to B$, and $\widetilde{N}\colon \mathrm{Hom}_{\mathbb{Z}G}(F_{2i+1},B^*) \to \mathrm{Hom}_{\mathbb{Z}G}(F_{2i+2},B^*)$ is $(1 - t + \cdots - t^{\ell-1})\colon B \to B$ for all $i \geqslant 0$.

By assumption, the Euler class $\Omega \in H^{2k+1}(G,\mathbb{Z}^*) = \mathbb{Z}/2$ is nonzero. For $\{x\} \in H^j(G,B^*)$, let us calculate $\{x\} \cup \Omega = \{x \cup \omega\}$: Let $\omega = 1 \in \mathrm{Hom}_{\mathbb{Z}G}(F_{2k+1},\mathbb{Z}^*) = \mathbb{Z}^*$ represent Ω. The cochain cup product is $x \cup \omega = x \times \omega \circ \Delta_{j,2k+1} \in \mathrm{Hom}_{\mathbb{Z}G}(F_{j+2k+1}, B^* \otimes \mathbb{Z}^*) = B^* \otimes \mathbb{Z}^* = B$, where $\Delta : F \to F \otimes F$ is a diagonal approximation and its $(j, 2k+1)$-component $\Delta_{j,k+1} : F_{j+2k+1} \to F_j \otimes F_{2k+1}$ is given by $\Delta_{j,k+1}(1) = 1 \otimes 1$ if j is even and $\Delta_{j,k+1}(1) = \sum_{0 \leqslant h<i<\ell} t^h \otimes t^i$ if j is odd ([B]). Then as an element of the $\mathbb{Z}G$-module B,

$$x \cup \omega = \begin{cases} -x & \text{if } j \text{ is even,} \\ x + t^2\cdot x + \cdots + t^{\ell-2}\cdot x & \text{if } j \text{ is odd.} \end{cases}$$

Since, for j even, the map $-\cup\Omega\colon H^j(G,B^*) \to H^{j+2k+1}(G,B)$ maps $\{x\}$ to $\{-x\}$, it is a monomorphism. Let j be odd and $\{y\} \in H^{j+2k+1}(G,B)$. Then $y \in \mathrm{Hom}_{\mathbb{Z}G}(F_{j+2k+1},B) = B$ and $(t-1)\cdot y = 0$ or $t \cdot y = y$. Consider y as an element of $B = \mathrm{Hom}_{\mathbb{Z}G}(F_j,B)$. Since $\widetilde{N}\cdot y = 0$, we can consider $\{y\} \in H^j(G,B^*)$. Since $y \cup \omega - y = (\ell-2)/(2)y = (1 + t + \cdots + t^{\ell-1})\cdot((\ell-2)/(2\ell)y)$, we have $\{y\} = \{y\} \cup \Omega$. This shows that for j odd, the map $-\cup\Omega : H^j(G,B^*) \to H^{j+2k+1}(G,B)$ is an epimorphism. By Theorem 2, X/G is finitely dominated. □

THEOREM 7. *Suppose an infinite-dimensional homotopy space form X/G is finitely dominated. If $G \cong \mathbb{Z}/p$ where $p = 2, 3, 5, 7, 11, 13, 17,$ or 19, then X/G is homotopy equivalent to a finite complex.*

Proof. By [W], for the finitely dominated complex X/G there is an obstruction $\sigma(X/G) \in \widetilde{K}^0(\mathbb{Z}G)$, the reduced projective class group, such that if $\sigma(X/G) = 0$ then X/G is homotopy equivalent to a finite complex. Since $G \cong \mathbb{Z}/p$ and $p = 2, 3, 5, 7, 11, 13, 17$, or 19, by [H], $\widetilde{K}^0(\mathbb{Z}G) = \{0\}$. This proves the theorem. □

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