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# ON CHUNG'S STRONG LAW OF LARGE NUMBERS IN GENERAL BANACH SPACES

BONG DAE CHOI AND SOO HAK SUNG

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent Banach valued random variables and $\{a_n, n \geqslant 1\}$ a sequence of real numbers such that $0 < a_n \uparrow \infty$. It is shown that, under the assumption $\sum_{n=1}^{\infty} E\phi(\|X_n\|)/\phi(a_n) < \infty$ with some restrictions on $\phi$, $S_n/a_n \to 0$ a.s. if and only if $S_n/a_n \to 0$ in probability if and only if $S_n/a_n \to 0$ in $L^1$. From this result several known strong laws of large numbers in Banach spaces are easily derived.

## 1. INTRODUCTION

Let $(B, \| \ \|)$ be a real separable Banach space. The laws of large numbers for Banach-valued random variables have been studied by many authors ([**1**], [**3**], [**4**], [**5**], [**8**]). Hoffmann-Jorgensen and Pisier [**3**] and Korzeniowski [**4**] have investigated the geometric structure on the Banach space for which an analogue of the strong laws of large numbers (SLLN) holds true. de Acosta [**1**] and Kuelbs and Zinn [**5**] have shown that many classical SLLN hold for random variables taking values in a general Banach space under the assumption that the weak law of large numbers (WLLN) holds.

In this paper, we apply several inequalities (maximal inequality [**2**] and de Acosta inequality [**1**]) to obtain Chung's SLLN in a general Banach space under the assumption that WLLN holds. From this result several known SLLN in Banach spaces are easily obtained.

## 2. MAIN RESULT

To prove the main theorem we will need the following several lemmas. The following lemma is a generalisation of a classical result (Stout [**7**], P. 127-128), but its proof is standard and is omitted.

LEMMA 1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables stochastically dominated by $X$ with $E|X|^r < \infty$ for $0 < r < \infty$; that is, $P(|X_n| \geqslant t) \leqslant P(|X| \geqslant t)$, for $t \geqslant 0$. Then*

(i) $\sum_{n=1}^{\infty} \frac{1}{n^{\beta/r}} E|X_n|^{\beta} I(|X_n| \leqslant n^{1/r}) < \infty$ *for* $0 < r < \beta$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha/r}} E|X_n|^{\alpha} I(|X_n| > n^{1/r}) < \infty$ *for* $0 < \alpha < r$.

Recently Etemadi [**2**] proved the following maximal inequality which holds for $B$-valued random variables.

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LEMMA 2. *Let $X_1, \dots, X_n$ be independent random variables. Let $S_i = \sum_{j=1}^{i} X_j$ for $i = 1, \dots, n$, and $t > 0$. Then*

$$P(\max_{1\leqslant i\leqslant n} |S_i| > t) \leqslant 4 \max_{1\leqslant i\leqslant n} P(|S_i| > t/4).$$

The following lemma plays an essential role in our main Theorem.

LEMMA 3. *(de Acosta [1]). Let $X_1, \dots, X_n$ be independent $B$-valued random variables with $E\|X_i\|^r < \infty$ for $i = 1, \dots, n$ and $1 \leqslant r \leqslant 2$. Then*

$$E\,|\|S_n\| - E\|S_n\||^r \leqslant C_r \sum_{i=1}^{n} E\|X_i\|^r,$$

*where $C_r$ is a positive constant depending only on $r$; if $r = 2$ then it is possible to take $C_2 = 4$.*

Let $\phi$ be a positive, even and continuous function on R such that as $|x|$ increases,

$$\frac{\phi(x)}{x}\uparrow \quad \text{and} \quad \frac{\phi(x)}{x^2}\downarrow \tag{1}$$

THEOREM 4. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent $B$-valued random variables and $\{a_n, n \geqslant 1\}$ constants such that $0 < a_n \uparrow \infty$. Assume*

$$\sum_{n=1}^{\infty} \frac{E\phi(\|X_n\|)}{\phi(a_n)} < \infty. \tag{2}$$

*Then the following are equivalent:*

(i) $E\|S_n\|/a_n \to 0$;
(ii) $S_n/a_n \to 0$ *a.s.*;
(iii) $S_n/a_n \to 0$ *in probability*.

PROOF: (i) $\Longrightarrow$ (ii). Let $X_n' = X_n I(\|X_n\| \leqslant a_n), X_n'' = X_n I(\|X_n\| > a_n), S_n' = \sum_{i=1}^{n} X_i'$ and $S_n'' = \sum_{i=1}^{n} X_i''$. Since $\phi$ is increasing, we have

$$\sum_{i=1}^{\infty} P(\|X_i\| > a_i) \leqslant \sum_{i=1}^{\infty} P(\phi(\|X_i\|) > \phi(a_i)) \leqslant \sum_{i=1}^{\infty} \frac{E\phi(\|X_i\|)}{\phi(a_i)} < \infty.$$

Thus it follows by the Borel-Cantelli lemma that $S_n''/a_n \to 0$ a.s. The proof will be completed by showing that

$$S_n'/a_n \to 0 \quad \text{a.s.} \tag{3}$$

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From the first hypothesis in (1), we have

$$\frac{|x|}{a_i} \leqslant \frac{\phi(|x|)}{\phi(a_i)} \quad \text{for } |x| > a_i.$$

It follows that

$$\sum_{i=1}^{\infty} \frac{E\|X_i''\|}{a_i} \leqslant \sum_{i=1}^{\infty} \frac{E\phi(\|X_i''\|)}{\phi(a_i)} < \infty.$$

Thus $E\|S_n''\|/a_n \leqslant \sum_{i=1}^{n} E\|X_i''\|/a_n \to 0$ by the Kronecker lemma. From this result and (i), we obtain

$$E\|S_n'\|/a_n \to 0. \tag{4}$$

To prove (3), for $k \geqslant 0$ we define $m_k = \inf\{n,\, a_n \geqslant 2^k\}$. First we show that

$$\frac{S'_{m_k}}{a_{m_k}} \to 0 \quad \text{a.s.} \tag{5}$$

By (4), it is enough to show that $\left(\left\|S'_{m_k}\right\| - E\left\|S'_{m_k}\right\|\right)/a_{m_k} \to 0$ a.s. From Lemma 3,

$$\begin{aligned}
&\sum_{\substack{k=0,\\ m_k \neq m_{k+1}}}^{\infty} P\left(\left|\frac{\left\|S'_{m_k}\right\| - E\left\|S'_{m_k}\right\|}{a_{m_k}}\right| > \varepsilon\right) \\
&\quad \leqslant \frac{1}{\varepsilon^2} \sum_{\substack{k=0,\\ m_k \neq m_{k+1}}}^{\infty} \frac{1}{\left(a_{m_k}\right)^2} E\left|\left\|S'_{m_k}\right\| - E\left\|S'_{m_k}\right\|\right|^2 \\
&\quad \leqslant \frac{4}{\varepsilon^2} \sum_{\substack{k=0,\\ m_k \neq m_{k+1}}}^{\infty} \frac{1}{\left(a_{m_k}\right)^2} \sum_{i=1}^{m_k} E\|X_i'\|^2 \\
&\quad = \frac{4}{\varepsilon^2} \sum_{i=1}^{\infty} E\|X_i'\|^2 \left( \sum_{\substack{\{k: m_k \geqslant i,\\ m_k \neq m_{k+1}\}}} 1/\left(a_{m_k}\right)^2 \right),
\end{aligned}$$

where $\sum_{k=0,\, m_k \neq m_{k+1}}^{\infty}$ means that the summation is taken over all k such that $m_k \neq m_{k+1}$. Now we estimate $\sum_{\{k: m_k \geqslant i,\, m_k \neq m_{k+1}\}} 1/\left(a_{m_k}\right)^2$. Let $k_0 = \min\{k : m_k \geqslant$

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$i,\ m_k \neq m_{k+1}\}$. Then $m_{k_0} \geqslant i, m_{k_0+1} > m_{k_0}$ and $a_{m_{k_0}} < 2^{k_0+1}$. Hence we have

$$\begin{aligned}\sum_{\substack{\{k: m_k \geqslant i,\\ m_k \neq m_{k+1}\}}} 1/\left(a_{m_k}\right)^2 &\leqslant \sum_{k=k_0}^{\infty} \frac{1}{\left(a_{m_k}\right)^2} \\ &\leqslant \sum_{k=k_0}^{\infty} \frac{1}{\left(2^k\right)^2} = \frac{1}{1-(1/2)^2} \frac{1}{\left(2^{k_0}\right)^2} = \frac{16}{3} \frac{1}{\left(2^{k_0+1}\right)^2} \\ &< \frac{16}{3} \frac{1}{\left(a_{m_{k_0}}\right)^2} \leqslant \frac{16}{3} \frac{1}{a_i^2}.\end{aligned}$$

Thus we have

$$\begin{aligned}\sum_{\substack{k=0,\\ m_k \neq m_{k+1}}}^{\infty} & P\left(\left|\frac{\left\|S'_{m_k}\right\| - E\left\|S'_{m_k}\right\|}{a_{m_k}}\right| > \varepsilon\right) \\ &\leqslant \frac{4}{\varepsilon^2} \frac{16}{3} \sum_{i=1}^{\infty} \frac{E\left\|X'_i\right\|^2}{a_i^2} \\ &\leqslant \frac{4}{\varepsilon^2} \frac{16}{3} \sum_{i=1}^{\infty} \frac{E\phi(\|X_i\|)}{\phi(a_i)} < \infty.\end{aligned}$$

It follows that $\left(\left\|S'_{m_k}\right\| - E\left\|S'_{m_k}\right\|\right)/a_{m_k} \to 0$ a.s. By observing that

$$\max_{m_k \leqslant n < m_{k+1}} \frac{\|S'_n\|}{a_n} \leqslant \frac{\left\|S'_{m_k}\right\|}{a_{m_k}} + \max_{m_k \leqslant n < m_{k+1}} \frac{\left\|S'_n - S'_{m_k}\right\|}{a_{m_k}},$$

we will obtain $S'_n/a_n \to 0$ a.s. if we show that

(6) $$\max_{m_k \leqslant n < m_{k+1}} \frac{\left\|S'_n - S'_{m_k}\right\|}{a_{m_k}} \to 0 \text{ a.s.}$$

First we observe that $\max_{m_k \leqslant n < m_{k+1}} E\left\|S'_n - S'_{m_k}\right\|/a_{m_k} \to 0$ and hence we have that

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$\max_{m_k \leqslant n < m_{k+1}} E\left\|S'_n - S'_{m_k}\right\| / a_{m_k} \leqslant \varepsilon/8$ for $k \geqslant k_1$, because we have by (4)

$$\begin{aligned}
\max_{m_k \leqslant n < m_{k+1}} \frac{E\left\|S'_n - S'_{m_k}\right\|}{a_{m_k}} &\leqslant \frac{E\left\|S'_{m_k}\right\|}{a_{m_k}} + \max_{m_k \leqslant n < m_{k+1}} \frac{E\left\|S'_n\right\|}{a_{m_k}} \\
&\leqslant \frac{E\left\|S'_{m_k}\right\|}{a_{m_k}} + \frac{1}{2^k} \max_{m_k \leqslant n < m_{k+1}} \frac{a_n E\left\|S'_n\right\|}{a_n} \\
&\leqslant \frac{E\left\|S'_{m_k}\right\|}{a_{m_k}} + \frac{a_{m_{k+1}-1}}{2^k} \max_{m_k \leqslant n < m_{k+1}} \frac{E\left\|S'_n\right\|}{a_n} \\
&\leqslant \frac{E\left\|S'_{m_k}\right\|}{a_{m_k}} + 2 \max_{m_k \leqslant n < m_{k+1}} \frac{E\left\|S'_n\right\|}{a_n} \to 0
\end{aligned}$$

as $k \to \infty$. By Lemma 2 and Lemma 3, we obtain

$$\begin{aligned}
&\sum_{\substack{k=k_1, \\ m_k \neq m_{k+1}}}^{\infty} P\left( \max_{m_k \leqslant n < m_{k+1}} \frac{\left\|S'_n - S'_{m_k}\right\|}{a_{m_k}} > \varepsilon \right) \\
&\leqslant 4 \sum_{\substack{k=k_1, \\ m_k \neq m_{k+1}}}^{\infty} \max_{m_k \leqslant n < m_{k+1}} P\left( \frac{\left\|S'_n - S'_{m_k}\right\|}{a_{m_k}} > \frac{\varepsilon}{4} \right) \\
&\leqslant 4 \sum_{\substack{k=k_1, \\ m_k \neq m_{k+1}}}^{\infty} \max_{m_k \leqslant n < m_{k+1}} P\left( \frac{\left| \left\|S'_n - S'_{m_k}\right\| - E\left\|S'_n - S'_{m_k}\right\| \right|}{a_{m_k}} > \frac{\varepsilon}{8} \right) \\
&\leqslant 4 \frac{8^2}{\varepsilon^2} \sum_{\substack{k=k_1, \\ m_k \neq m_{k+1}}}^{\infty} \frac{1}{\left(a_{m_k}\right)^2} \max_{m_k \leqslant n < m_{k+1}} E\left| \left\|S'_n - S'_{m_k}\right\| - E\left\|S'_n - S'_{m_k}\right\| \right|^2 \\
&\leqslant \frac{4^2 8^2}{\varepsilon^2} \sum_{\substack{k=0, \\ m_k \neq m_{k+1}}}^{\infty} \frac{1}{\left(a_{m_k}\right)^2} \sum_{i=1}^{m_{k+1}-1} E\left\|X'_i\right\|^2 \\
&= \frac{4^2 8^2}{\varepsilon^2} \sum_{i=1}^{\infty} E\left\|X'_i\right\|^2 \left( \sum_{\substack{\{k: m_{k+1}-1 \geqslant i, \\ m_k \neq m_{k+1}\}}} 1/\left(a_{m_k}\right)^2 \right) \\
&< \frac{4^2 8^2}{\varepsilon^2} \frac{16}{3} \sum_{i=1}^{\infty} \frac{E\left\|X'_i\right\|^2}{a_i^2} \\
&\leqslant \frac{4^2 8^2}{\varepsilon^2} \frac{16}{3} \sum_{i=1}^{\infty} \frac{E\phi(\|X_i\|)}{\phi(a_i)} < \infty.
\end{aligned}$$

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Hence the desired result (6) follows by the Borel-Cantelli Lemma. The implication (ii)$\Longrightarrow$(iii) is obvious. Now we show that (iii)$\Longrightarrow$(i). Assume $S_n/a_n \to 0$ in probability. From the proof of (i)$\Longrightarrow$(ii), we have $E\|S_n''\|/a_n \to 0$ and $S_n''/a_n \to 0$ a.s. Hence we obtain $S_n'/a_n \to 0$ in probability. Thus it is enough to show that $E\|S_n'\|/a_n \to 0$. By Lemma 3,

$$E\left(\frac{\|S_n'\| - E\|S_n'\|}{a_n}\right)^2 \leqslant \frac{4}{a_n^2}\sum_{i=1}^{n} E\|X_i'\|^2 \to 0,$$

since

$$\sum_{i=1}^{\infty}\frac{E\|X_i'\|^2}{a_i^2} \leqslant \sum_{i=1}^{\infty}\frac{E\phi(\|X_i\|)}{\phi(a_i)} < \infty.$$

Hence $(\|S_n'\| - E\|S_n'\|)/a_n \to 0$ in probability. Recalling that, $S_n'/a_n \to 0$ in probability we have $E\|S_n'\|/a_n \to 0$. ∎

Corollary 5. *([1], [5]). Let $\{X_n, n \geqslant 1\}$ be a sequence of independent $B$-valued random variables such that $\sum_{n=1}^{\infty} E\|X_n\|^{\alpha}/n^{\alpha} < \infty$ for some $1 \leqslant \alpha \leqslant 2$. Then*

*$S_n/n \to 0$ in probability if and only if $S_n/n \to 0$ a.s. if and only if $S_n/n \to 0$ in $L^1$.*

Proof: It is clear that $\phi(x) = x^{\alpha}$ satisfies the condition (1). ∎

Corollary 6. *(Marcinkiewicz SLLN). ([4]). Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. $B$-valued random variables with $E\|X_1\|^r < \infty$ for $1 \leqslant r < 2$. Then the following are equivalent:*

(i) $S_n/n^{1/r} \to 0$ *a.s.;*
(ii) $S_n/n^{1/r} \to 0$ *in probability;*
(iii) $S_n/n^{1/r} \to 0$ *in* $L^1$*;*
(iv) $S_n/n^{1/r} \to 0$ *in* $L^r$*.*

Proof: Let $X_n' = X_n I(\|X_n\| \leqslant n^{1/r})$, $X_n'' = X_n I(\|X_n\| > n^{1/r})$, and $S_n' = \sum_{i=1}^{n} X_i'$ and $S_n'' = \sum_{i=1}^{n} X_i''$. First we show that (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii). Since $E\|X_1\|^r < \infty$, we have $S_n''/n^{1/r} \to 0$ a.s. and $S_n''/n^{1/r} \to 0$ in $L^1$. Hence it is enough to show that $S_n'/n^{1/r} \to 0$ a.s. if and only if $S_n'/n^{1/r} \to 0$ in probability if and only if $S_n'/n^{1/r} \to 0$ in $L^1$. Since $\sum_{n=1}^{\infty} E\|X_n'\|^2/n^{2/r} < \infty$ by Lemma 1, these equivalences are seen by applying Theorem 4 to $(X_n')$ with $\phi(x) = x^2$. Now we show that (iii)$\Longleftrightarrow$(iv). Since the implication (iv)$\Longrightarrow$(iii) is obvious, it remains to show that(iii)$\Longrightarrow$(iv). Assume $S_n/n^{1/r} \to 0$ in $L^1$.

$$E\|S_n\|^r \leqslant 2^{r-1}E|\,\|S_n\| - E\|S_n\|\,|^r + 2^{r-1}(E\|S_n\|)^r.$$

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Thus it is enough to show that

(7) $$\frac{1}{n} E\,|\,\|S_n\| - E\,\|S_n\|\,|^r \to 0.$$

From Lemma 3,

$$\begin{aligned}
E\,|\,\|S_n\| - E\,\|S_n\|\,|^r &= E\,|\,\|S'_n + S''_n\| - E\,\|S'_n + S''_n\|\,|^r \\
&\leqslant E(|\,\|S'_n\| - E\,\|S'_n\|\,| + |\,\|S''_n\| - E\,\|S''_n\|\,| + 2E\,\|S''_n\|)^r \\
&\leqslant 2^{2r-2} E\,|\,\|S'_n\| - E\,\|S'_n\|\,|^r + 2^{2r-2} E\,|\,\|S''_n\| - E\,\|S''_n\|\,|^r + 2^{2r-1}(E\,\|S''_n\|)^r \\
&\leqslant 2^{2r-2}\left(E\,|\,\|S'_n\| - E\,\|S'_n\|\,|^2\right)^{r/2} + 2^{2r-2} E\,|\,\|S''_n\| - E\,\|S''_n\|\,|^r + 2^{2r-1}(E\,\|S''_n\|)^r \\
&\leqslant 2^{3r-2}\left(\sum_{i=1}^n E\,\|X'_i\|^2\right)^{r/2} + 2^{2r-2} C_r \sum_{i=1}^n E\,\|X''_i\|^r + 2^{2r-1}\left(\sum_{i=1}^n E\,\|X''_i\|\right)^r.
\end{aligned}$$

By a standard calculation, we have

$$\sum_{i=1}^n E\,\|X'_i\|^2 / n^{2/r} \to 0,$$

$$\sum_{i=1}^n E\,\|X''_i\|^r / n \to 0 \quad \text{and}$$

$$\sum_{i=1}^n E\,\|X''_i\| / n^{1/r} \to 0.$$

Thus the proof is completed. ∎

REMARK: For i.i.d. real valued random variables, Pyke and Root [6] have shown that

$$E\,|X_1|^r < \infty \iff S_n/n^{1/r} \to 0 \text{ a.s.} \iff S_n/n^{1/r} \to 0 \text{ in } L^r.$$

Recall that the Banach space $(B, \|\;\|)$ is of type $p$ if there exists $C > 0$ such that

$$E\left\|\sum_{k=1}^n X_k\right\|^p \leqslant C \sum_{k=1}^n E\,\|X_k\|^p$$

for all independent $B$-valued random variables $X_1, \ldots, X_n$ with mean zero and finite $p$th moments; ([3], [8]).

COROLLARY 7. ([10]). *If the Banach space $B$ is of type $p$ for $1 < p \leqslant 2$ and $\{X_n, n \geqslant 1\}$ are independent $B$-valued random variables with $EX_n = 0$ for $n = 1, 2, \ldots$ and $E\,\|X_n\|^p \leqslant \Gamma$ for some constant $\Gamma$, then*

$$S_n/n^{1/r} \to 0 \text{ a.s.} \qquad \text{for } 1 < r < p.$$

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PROOF: To apply Theorem 4, we will show that

$$\sum_{n=1}^{\infty} E\|X_n\|^p / \left(n^{1/r}\right)^p < \infty \text{ and } S_n/n^{1/r} \to 0 \text{ in } L^1.$$

The first one is obvious. The second one is true by the following fact:

$$\begin{aligned} E\left\|\frac{S_n}{n^{1/r}}\right\| &\leqslant \frac{1}{n^{1/r}}(E\|S_n\|^p)^{1/p} \\ &\leqslant \frac{C^{1/p}}{n^{1/r}}\left(\sum_{i=1}^{n} E\|X_i\|^p\right)^{1/p} \\ &\leqslant \frac{C^{1/p}\Gamma^{1/p}n^{1/p}}{n^{1/r}} \to 0. \end{aligned}$$

∎

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Department of Applied Mathematics
Korea Advanced Institute of Science and Technology
P.O.Box 150
Cheongryang
Seoul
Korea


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