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Henning Krause

Dedicated to Ragnar-Olaf Buchweitz on the occasion of his 60th birthday

Abstract

This is a report on recent work of Chałupnik and Touzé. We explain the Koszul duality for the category of strict polynomial functors and make explicit the underlying monoidal structure which seems to be of independent interest. Then we connect this to Ringel duality for Schur algebras and describe Serre duality for strict polynomial functors.

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1. Introduction

A Koszul duality for strict polynomial functors has been introduced in recent work of Chałupnik [Cha08] and Touzé [Tou11]. In this note we give a detailed report. Our aims are:

- to make explicit the underlying monoidal structure for the category of strict polynomial functors;
- to explain the connection with Ringel duality for Schur algebras;
- to describe Serre duality for strict polynomial functors in terms of Koszul duality; and
- to remove assumptions on the ring of coefficients.

The category of strict polynomial functors was introduced by Friedlander and Suslin [FS97] in their work on the cohomology of finite group schemes. We use an equivalent description in terms of representations of divided powers, following expositions of Kuhn [Kuh98], Pirashvili [Pir03], and Bousfield [Bou67].

The construction of the Koszul duality can be summarised as follows.

Theorem 1.1. *Let k be a commutative ring and d a non-negative integer. Denote by $\mathsf{Rep}\,\Gamma_k^d$ the category of k-linear representations of the category of divided powers of degree d over k.*

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(1) *The category* $\mathsf{Rep}\,\Gamma^d_k$ *has a tensor product*

$$-\otimes_{\Gamma^d_k} -\colon\ \mathsf{Rep}\,\Gamma^d_k \times \mathsf{Rep}\,\Gamma^d_k \longrightarrow \mathsf{Rep}\,\Gamma^d_k$$

which is induced by the tensor product for the category of divided powers.

(2) *The left derived tensor functor given by the exterior power* Λ^d *of degree* d *induces for the derived category of* $\mathsf{Rep}\,\Gamma^d_k$ *an equivalence*

$$\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} -\colon \mathsf{D}(\mathsf{Rep}\,\Gamma^d_k) \xrightarrow{\sim} \mathsf{D}(\mathsf{Rep}\,\Gamma^d_k).$$

The crucial observation for proving the second part of this theorem is that classical Koszul duality provides resolutions of the exterior power Λ^d and the symmetric power S^d which give an isomorphism

$$\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} \Lambda^d \cong S^d. \tag{1.1}$$

Strict polynomial functors are closely related to modules over Schur algebras. In fact, $\mathsf{Rep}\,\Gamma^d_k$ is equivalent to the category of modules over the Schur algebra $S_k(n,d)$ for $n \geqslant d$, by a result of Friedlander and Suslin [FS97]. Schur algebras were introduced by Green [Gre80] and provide a link between representations of the general linear groups and the symmetric groups. Theorem 1.1 establishes via transport of structure a tensor product for the category of modules over the Schur algebra $S_k(n,d)$; it seems that this tensor product has not been noticed before.

A Koszul type duality for modules over Schur algebras seems to appear first in work of Akin and Buchsbaum [AB88]. For instance, they construct resolutions of Schur and Weyl modules, and it is Koszul duality which maps one to the other; see Proposition 4.16. A few years later, Ringel [Rin91] introduced characteristic tilting modules for quasi-hereditary algebras. These tilting modules give rise to derived equivalences, relating a quasi-hereditary algebra and its Ringel dual. Donkin [Don93] described those tilting modules for Schur algebras, and it turns out that Koszul duality as introduced by Chałupnik [Cha08] and Touzé [Tou11] is equivalent to Ringel duality for modules over Schur algebras; this is the content of Theorem 5.1.

The Koszul duality $\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} -$ is an endofunctor and it would be interesting to have a description of its powers. It is somewhat surprising that its square is a Serre duality functor in the sense of [RVdB02]; this is another consequence of the isomorphism (1.1) and the content of Corollary 5.5.

Convention

Throughout this work we fix a commutative ring k.

2. Divided powers and strict polynomial functors

The category of strict polynomial functors was introduced by Friedlander and Suslin [FS97]. We use an equivalent description in terms of representations of divided powers, following expositions of Kuhn [Kuh98], Pirashvili [Pir03], and Bousfield [Bou67]. Note that divided powers have their origin in the computation of the homology of Eilenberg–MacLane spaces [Car54/55, EM54]. Some classical references on polynomial functors and polynomial representations are [ABW82, Gre80, Mac95].

Finitely generated projective modules

Let P_k denote the category of finitely generated projective k-modules. Given V, W in P_k, we write $V \otimes W$ for their tensor product over k and $\operatorname{Hom}(V,W)$ for the group of k-linear maps $V \to W$.

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This provides two bifunctors

$$- \otimes - \colon \mathsf{P}_k \times \mathsf{P}_k \longrightarrow \mathsf{P}_k,$$
$$\operatorname{Hom}(-,-) \colon (\mathsf{P}_k)^{\mathrm{op}} \times \mathsf{P}_k \longrightarrow \mathsf{P}_k$$

with a natural isomorphism

$$\operatorname{Hom}_{\mathsf{P}_k}(U \otimes V, W) \cong \operatorname{Hom}_{\mathsf{P}_k}(U, \operatorname{Hom}(V, W)).$$

The functor sending V to $V^* = \operatorname{Hom}(V, k)$ yields a duality

$$(\mathsf{P}_k)^{\mathrm{op}} \xrightarrow{\sim} \mathsf{P}_k.$$

Note that for U, V, W, V', W' in P_k there are natural isomorphisms

$$V^* \otimes W \cong \operatorname{Hom}(V, W), \tag{2.1}$$
$$\operatorname{Hom}(U, V) \otimes W \cong \operatorname{Hom}(U, V \otimes W), \tag{2.2}$$
$$\operatorname{Hom}(V, W) \otimes \operatorname{Hom}(V', W') \cong \operatorname{Hom}(V \otimes V', W \otimes W'). \tag{2.3}$$

Divided and symmetric powers

Fix a positive integer d and denote by $\mathfrak{S}_d$ the symmetric group permuting d elements. For each $V \in \mathsf{P}_k$, the group $\mathfrak{S}_d$ acts on $V^{\otimes d}$ by permuting the factors of the tensor product. Denote by $\Gamma^d V$ the submodule $(V^{\otimes d})^{\mathfrak{S}_d}$ of $V^{\otimes d}$ consisting of the elements which are invariant under the action of $\mathfrak{S}_d$; it is called the module of *divided powers of degree d*.[1] The maximal quotient of $V^{\otimes d}$ on which $\mathfrak{S}_d$ acts trivially is denoted by $S^d V$ and is called the module of *symmetric powers of degree d*. Set $\Gamma^0 V = k$ and $S^0 V = k$.

From the definition, it follows that $(\Gamma^d V)^* \cong S^d(V^*)$. Note that $S^d V$ is a free k-module provided that V is free; see [Bou70, III.6.6]. Thus $\Gamma^d V$ and $S^d V$ belong to P_k for all $V \in \mathsf{P}_k$, and we obtain functors $\Gamma^d, S^d \colon \mathsf{P}_k \to \mathsf{P}_k$.

For further material on divided and symmetric powers, see [ABW82, Bou81, Rob63].

The category of divided powers

We consider the category $\Gamma^d \mathsf{P}_k$ which is defined as follows. The objects are the finitely generated projective k-modules and for two objects V, W set

$$\operatorname{Hom}_{\Gamma^d \mathsf{P}_k}(V, W) = \Gamma^d \operatorname{Hom}(V, W).$$

Note that this identifies with $\operatorname{Hom}(V^{\otimes d}, W^{\otimes d})^{\mathfrak{S}_d}$ via the isomorphism (2.3), where $\mathfrak{S}_d$ acts on $\operatorname{Hom}(V^{\otimes d}, W^{\otimes d})$ via

$$(f\sigma)(v_1 \otimes \cdots \otimes v_d) = \sigma^{-1}(f(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)})).$$

Using this identification one defines the composition of morphisms in $\Gamma^d \mathsf{P}_k$.

Example 2.1. Let n be a positive integer and set $V = k^n$. Then $\operatorname{End}_{\Gamma^d \mathsf{P}_k}(V)$ is isomorphic to the *Schur algebra* $S_k(n, d)$ as defined by Green [Gre80, Theorem 2.6c]. We view this isomorphism as an identification.

The bifunctors $- \otimes -$ and $\operatorname{Hom}(-,-)$ for P_k induce bifunctors

$$- \otimes - \colon \Gamma^d \mathsf{P}_k \times \Gamma^d \mathsf{P}_k \longrightarrow \Gamma^d \mathsf{P}_k,$$
$$\operatorname{Hom}(-,-) \colon (\Gamma^d \mathsf{P}_k)^{\mathrm{op}} \times \Gamma^d \mathsf{P}_k \longrightarrow \Gamma^d \mathsf{P}_k$$

[1] The original definition of the module $\Gamma^d V$ of divided powers is different; it is, however, isomorphic to the module of symmetric tensors which is used here; see [Bou81, IV.5, Exercise 8].

with a natural isomorphism

$$\mathrm{Hom}_{\Gamma^d \mathsf{P}_k}(U \otimes V, W) \cong \mathrm{Hom}_{\Gamma^d \mathsf{P}_k}(U, \mathrm{Hom}(V, W)).$$

More precisely, the tensor product for $\Gamma^d \mathsf{P}_k$ coincides on objects with the one for P_k, while on morphisms it is for pairs V, V' and W, W' of objects the composite

$$\begin{aligned}\Gamma^d \mathrm{Hom}(V, V') \times \Gamma^d \mathrm{Hom}(W, W') &\to \Gamma^d(\mathrm{Hom}(V, V') \otimes \mathrm{Hom}(W, W')) \\ &\xrightarrow{\sim} \Gamma^d \mathrm{Hom}(V \otimes W, V' \otimes W')\end{aligned}$$

where the second map is induced by (2.3).

The duality for P_k induces a duality

$$(\Gamma^d \mathsf{P}_k)^{\mathrm{op}} \xrightarrow{\sim} \Gamma^d \mathsf{P}_k.$$

Strict polynomial functors

Let M_k denote the category of k-modules. We study the category of k-linear *representations* of $\Gamma^d \mathsf{P}_k$. This is by definition the category of k-linear functors $\Gamma^d \mathsf{P}_k \to \mathsf{M}_k$, and we write, by slight abuse of notation,

$$\mathsf{Rep}\, \Gamma_k^d = \mathrm{Fun}_k(\Gamma^d \mathsf{P}_k, \mathsf{M}_k).$$

The set of morphisms between two representations X, Y in $\mathsf{Rep}\, \Gamma_k^d$ is denoted by $\mathrm{Hom}_{\Gamma_k^d}(X, Y)$.

The representations of $\Gamma^d \mathsf{P}_k$ form an abelian category, where (co)kernels and (co)products are computed pointwise over k.

A representation X is called *finite* when $X(V)$ is finitely generated projective for each $V \in \Gamma^d \mathsf{P}_k$. It is sometimes convenient to restrict to the full subcategory

$$\mathsf{rep}\, \Gamma_k^d = \mathrm{Fun}_k(\Gamma^d \mathsf{P}_k, \mathsf{P}_k)$$

of finite representations; it is an extension closed subcategory of $\mathsf{Rep}\, \Gamma_k^d$ and therefore a Quillen exact category [Qui73].

Remark 2.2. A representation $X \in \mathsf{Rep}\, \Gamma_k^d$ is by definition a pair of functions, the first of which assigns to each $V \in \mathsf{P}_k$ a k-module $X(V)$ and the second assigns to each pair $V, W \in \mathsf{P}_k$ a k-linear map

$$X_{V,W} \colon \Gamma^d \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(X(V), X(W)).$$

The map $X_{V,W}$ admits an interpretation which is based on classical properties of symmetric tensors and divided powers, to be found in [Bou81, Rob63]. Given a pair of k-modules M, N with $M \in \mathsf{P}_k$, there is a canonical k-linear map

$$\mathrm{Hom}(\Gamma^d M, N) \longrightarrow \mathrm{Pol}^d(M, N)$$

where $\mathrm{Pol}^d(M, N)$ denotes the k-module consisting of *homogeneous polynomial maps of degree* d from M to N; it is an isomorphism when k is an infinite integral domain and N is torsion-free [Bou81, IV.5.9]. On the other hand, there is a canonical bijection

$$\mathrm{Hom}(\Gamma^d M, N) \longrightarrow P^d(M, N)$$

where $P^d(M, N)$ denotes the set of *homogeneous polynomial laws of degree* d of M in N, by [Bou81, IV.5, Exercise 10]. These observations explain the term *strict polynomial functor* used by Friedlander and Suslin in [FS97, § 2].

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The Yoneda embedding

The Yoneda embedding

$$(\Gamma^d \mathsf{P}_k)^{\mathrm{op}} \longrightarrow \mathsf{Rep}\,\Gamma^d_k, \quad V \mapsto \operatorname{Hom}_{\Gamma^d \mathsf{P}_k}(V, -)$$

identifies $\Gamma^d \mathsf{P}_k$ with the full subcategory consisting of the representable functors. For $V \in \Gamma^d \mathsf{P}_k$ we write

$$\Gamma^{d,V} = \operatorname{Hom}_{\Gamma^d \mathsf{P}_k}(V, -).$$

For $X \in \mathsf{Rep}\,\Gamma^d_k$ there is the Yoneda isomorphism

$$\operatorname{Hom}_{\Gamma^d_k}(\Gamma^{d,V}, X) \xrightarrow{\sim} X(V),$$

and it follows that $\Gamma^{d,V}$ is a projective object in $\mathsf{Rep}\,\Gamma^d_k$. For $W \in \Gamma^d \mathsf{P}_k$ this yields

$$\operatorname{Hom}_{\Gamma^d_k}(\Gamma^{d,V}, \Gamma^{d,W}) \cong \operatorname{Hom}_{\Gamma^d \mathsf{P}_k}(W, V) = \Gamma^d \operatorname{Hom}(W, V).$$

A representation X is *finitely generated* if there are objects $V_1, \dots, V_r \in \Gamma^d \mathsf{P}_k$ and an epimorphism

$$\Gamma^{d,V_1} \oplus \cdots \oplus \Gamma^{d,V_r} \longrightarrow X.$$

Note that each X in $\mathsf{Rep}\,\Gamma^d_k$ can be written canonically as a colimit of representable functors

$$\operatorname*{colim}_{\Gamma^{d,V} \to X} \Gamma^{d,V} \xrightarrow{\sim} X$$

where the colimit is taken over the category of morphisms $\Gamma^{d,V} \to X$ and V runs through the objects of $\Gamma^d \mathsf{P}_k$.

Duality

Given a representation $X \in \mathsf{Rep}\,\Gamma^d_k$, its *dual* X° is defined by

$$X^\circ(V) = X(V^*)^*.$$

This is also known as *Kuhn dual* [Kuh94]; see [Gre80, § 2.7] for its use in representation theory.

For each pair of k-modules V, W there is a natural isomorphism

$$\operatorname{Hom}(V, W^*) \cong \operatorname{Hom}(W, V^*).$$

This induces for all $X, Y \in \mathsf{Rep}\,\Gamma^d_k$ a natural isomorphism

$$\operatorname{Hom}_{\Gamma^d_k}(X, Y^\circ) \cong \operatorname{Hom}_{\Gamma^d_k}(Y, X^\circ) \tag{2.4}$$

and the evaluation morphism $X \to X^{\circ\circ}$, which is an isomorphism when X is finite.

Example 2.3. The divided power functor Γ^d and the symmetric power functor S^d belong to $\mathsf{Rep}\,\Gamma^d_k$. In fact

$$\Gamma^d = \operatorname{Hom}_{\Gamma^d \mathsf{P}_k}(k, -) \quad \text{and} \quad S^d = (\Gamma^d)^\circ.$$

The internal tensor product

The category of representations of $\Gamma^d \mathsf{P}_k$ inherits a tensor product from the tensor product for $\Gamma^d \mathsf{P}_k$. We provide a construction in terms of Kan extensions which is also known as Day convolution [Day70, IK86]. The internal Hom functor appears in work of Touzé [Tou10, § 2].

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Proposition 2.4. *The bifunctors* $-\otimes-$ *and* $\operatorname{Hom}(-,-)$ *for* $\Gamma^d\mathsf{P}_k$ *induce, via the Yoneda embedding, bifunctors*

$$-\otimes_{\Gamma^d_k}-\colon \operatorname{\mathsf{Rep}}\Gamma^d_k\times\operatorname{\mathsf{Rep}}\Gamma^d_k\longrightarrow\operatorname{\mathsf{Rep}}\Gamma^d_k,$$
$$\mathcal{H}om_{\Gamma^d_k}(-,-)\colon (\operatorname{\mathsf{Rep}}\Gamma^d_k)^{\mathrm{op}}\times\operatorname{\mathsf{Rep}}\Gamma^d_k\longrightarrow\operatorname{\mathsf{Rep}}\Gamma^d_k$$

with a natural isomorphism

$$\operatorname{Hom}_{\Gamma^d_k}(X\otimes_{\Gamma^d_k}Y,Z)\cong\operatorname{Hom}_{\Gamma^d_k}(X,\mathcal{H}om_{\Gamma^d_k}(Y,Z)).$$

To be precise, one requires for $V,W\in\Gamma^d\mathsf{P}_k$ that

$$\Gamma^{d,V}\otimes_{\Gamma^d_k}\Gamma^{d,W}=\Gamma^{d,V\otimes W},\tag{2.5}$$
$$\mathcal{H}om_{\Gamma^d_k}(\Gamma^{d,V},\Gamma^{d,W})=\Gamma^{d,\operatorname{Hom}(V,W)},\tag{2.6}$$

and this determines both bifunctors.

Proof. Given $X,Y\in\operatorname{\mathsf{Rep}}\Gamma^d_k$ and $V\in\Gamma^d\mathsf{P}_k$, one defines

$$\Gamma^{d,V}\otimes_{\Gamma^d_k}Y=\operatorname*{colim}_{\Gamma^{d,W}\to Y}\Gamma^{d,V\otimes W},$$
$$\mathcal{H}om_{\Gamma^d_k}(\Gamma^{d,V},Y)=\operatorname*{colim}_{\Gamma^{d,W}\to Y}\Gamma^{d,\operatorname{Hom}(V,W)}$$

and then

$$X\otimes_{\Gamma^d_k}Y=\operatorname*{colim}_{\Gamma^{d,V}\to X}\Gamma^{d,V}\otimes_{\Gamma^d_k}Y,$$
$$\mathcal{H}om_{\Gamma^d_k}(X,Y)=\lim_{\Gamma^{d,V}\to X}\mathcal{H}om_{\Gamma^d_k}(\Gamma^{d,V},Y).$$

For the adjunction isomorphism, write $X=\operatorname{colim}_\alpha X_\alpha$, $Y=\operatorname{colim}_\beta Y_\beta$, and $Z=\operatorname{colim}_\gamma Z_\gamma$ as colimits of representable functors as before. Then we have

$$\begin{aligned}\operatorname{Hom}_{\Gamma^d_k}(X\otimes_{\Gamma^d_k}Y,Z)&\cong\operatorname{Hom}_{\Gamma^d_k}(\operatorname*{colim}_\alpha\operatorname*{colim}_\beta X_\alpha\otimes_{\Gamma^d_k}Y_\beta,\operatorname*{colim}_\gamma Z_\gamma)\\&\cong\lim_\alpha\lim_\beta\operatorname*{colim}_\gamma\operatorname{Hom}_{\Gamma^d_k}(X_\alpha\otimes_{\Gamma^d_k}Y_\beta,Z_\gamma)\\&\cong\lim_\alpha\lim_\beta\operatorname*{colim}_\gamma\operatorname{Hom}_{\Gamma^d_k}(X_\alpha,\mathcal{H}om_{\Gamma^d_k}(Y_\beta,Z_\gamma))\\&\cong\operatorname{Hom}_{\Gamma^d_k}(\operatorname*{colim}_\alpha X_\alpha,\lim_\beta\operatorname*{colim}_\gamma\mathcal{H}om_{\Gamma^d_k}(Y_\beta,Z_\gamma))\\&\cong\operatorname{Hom}_{\Gamma^d_k}(X,\mathcal{H}om_{\Gamma^d_k}(Y,Z)).\end{aligned}$$ □

The tensor product and the internal Hom functor enjoy the usual categorical properties. For instance, the tensor product is right exact and can be computed using projective presentations.

Lemma 2.5. *Given* $X\in\operatorname{\mathsf{Rep}}\Gamma^d_k$ *and* $V\in\Gamma^d\mathsf{P}_k$, *there are natural isomorphisms*

$$\Gamma^{d,V}\otimes_{\Gamma^d_k}X\cong X\circ\operatorname{Hom}(V,-),$$
$$\mathcal{H}om_{\Gamma^d_k}(\Gamma^{d,V},X)\cong X\circ V\otimes-.$$

Proof. The isomorphisms are clear from (2.5) and (2.6) when X is representable. The general case follows since $\Gamma^{d,V}\otimes_{\Gamma^d_k}-$ and $\mathcal{H}om_{\Gamma^d_k}(\Gamma^{d,V},-)$ preserve colimits. □

Hom-tensor identities

We collect some basic identities for the internal products $-\otimes_{\Gamma^d_k}-$ and $\mathcal{H}om_{\Gamma^d_k}(-,-)$.

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Lemma 2.6. *Given $X, Y, Z \in \mathsf{Rep}\,\Gamma^d_k$, the natural morphism*

$$\mathcal{H}om_{\Gamma^d_k}(X, Y) \otimes_{\Gamma^d_k} Z \longrightarrow \mathcal{H}om_{\Gamma^d_k}(X, Y \otimes_{\Gamma^d_k} Z)$$

is an isomorphism provided that X or Z is finitely generated projective.[2]

Proof. The isomorphism is clear from (2.2) when all objects are representable functors. The general case follows by writing functors as colimits of representable functors, keeping in mind that $\Gamma^{d,V} \otimes_{\Gamma^d_k} -$ and $\mathcal{H}om_{\Gamma^d_k}(\Gamma^{d,V}, -)$ preserve colimits. □

Lemma 2.7. *Let X, Y be in $\mathsf{Rep}\,\Gamma^d_k$ and suppose that X is finitely presented. Then there are natural isomorphisms*

$$X \otimes_{\Gamma^d_k} Y^\circ \cong \mathcal{H}om_{\Gamma^d_k}(X, Y)^\circ,$$
$$(X \otimes_{\Gamma^d_k} Y)^\circ \cong \mathcal{H}om_{\Gamma^d_k}(X, Y^\circ).$$

Proof. Suppose first that X is representable. Then the isomorphism follows from Lemma 2.5, using (2.1). For a representation X that admits a presentation

$$\Gamma^{d,V_1} \longrightarrow \Gamma^{d,V_0} \longrightarrow X \longrightarrow 0$$

one uses exactness. □

Lemma 2.8. *Given $X, Y \in \mathsf{Rep}\,\Gamma^d_k$, there is a natural isomorphism*

$$\mathcal{H}om_{\Gamma^d_k}(X, Y^\circ) \cong \mathcal{H}om_{\Gamma^d_k}(Y, X^\circ).$$

Proof. Choose a representable functor P. Using (2.4) and Lemma 2.7, we have

$$\begin{aligned}
\operatorname{Hom}_{\Gamma^d_k}(P, \mathcal{H}om_{\Gamma^d_k}(X, Y^\circ)) &\cong \operatorname{Hom}_{\Gamma^d_k}(P \otimes_{\Gamma^d_k} X, Y^\circ) \\
&\cong \operatorname{Hom}_{\Gamma^d_k}(Y, (P \otimes_{\Gamma^d_k} X)^\circ) \\
&\cong \operatorname{Hom}_{\Gamma^d_k}(Y, \mathcal{H}om_{\Gamma^d_k}(P, X^\circ)) \\
&\cong \operatorname{Hom}_{\Gamma^d_k}(P, \mathcal{H}om_{\Gamma^d_k}(Y, X^\circ)).
\end{aligned}$$

The assertion now follows from Yoneda's lemma. □

The external tensor product

Let d, e be non-negative integers and denote by $(\Gamma^d \otimes \Gamma^e)\mathsf{P}_k$ the category having as objects the finitely generated projective k-modules and as morphisms $V \to W$

$$\Gamma^d \operatorname{Hom}(V, W) \otimes \Gamma^e \operatorname{Hom}(V, W).$$

The inclusion $\mathfrak{S}_d \times \mathfrak{S}_e \subseteq \mathfrak{S}_{d+e}$ induces for each $V \in \mathsf{P}_k$ an inclusion

$$\Gamma^{d+e}V = (V^{\otimes d+e})^{\mathfrak{S}_{d+e}} \subseteq (V^{\otimes d+e})^{\mathfrak{S}_d \times \mathfrak{S}_e} \cong (V^{\otimes d})^{\mathfrak{S}_d} \otimes (V^{\otimes e})^{\mathfrak{S}_e} = \Gamma^d V \otimes \Gamma^e V.$$

This yields a k-linear functor

$$C^{d,e} \colon \Gamma^{d+e}\mathsf{P}_k \longrightarrow (\Gamma^d \otimes \Gamma^e)\mathsf{P}_k$$

which is the identity on objects. It induces a tensor product

$$- \otimes - : \mathsf{Rep}\,\Gamma^d_k \times \mathsf{Rep}\,\Gamma^e_k \longrightarrow \mathsf{Rep}\,\Gamma^{d+e}_k.$$

[2] This means that finitely generated projective objects are *strongly dualisable* in the sense of [LMS86, III.1]. We refer to these notes for a discussion of further tensor categorical properties.

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To be precise, one defines for $X \in \mathsf{Rep}\,\Gamma_k^d$, $Y \in \mathsf{Rep}\,\Gamma_k^e$, and $V \in \mathsf{P}_k$

$$(X \otimes Y)(V) = X(V) \otimes Y(V),$$

and $X \otimes Y$ acts on morphisms via $C^{d,e}$.

Note that the external tensor product is exact when restricted to finite representations and that

$$(X \otimes Y)^\circ \cong X^\circ \otimes Y^\circ$$

for finite representations X, Y.

For integers $d \geqslant 0$ and $n > 0$, we denote by $\Lambda(n, d)$ the set of sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of non-negative integers such that $\sum \lambda_i = d$. Given $\lambda \in \Lambda(n, d)$, we write

$$\Gamma^\lambda = \Gamma^{\lambda_1} \otimes \cdots \otimes \Gamma^{\lambda_n} \quad \text{and} \quad S^\lambda = S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_n}.$$

Note that

$$\Gamma^{(1,\ldots,1)} \cong \otimes^n \cong S^{(1,\ldots,1)}. \tag{2.7}$$

Graded representations

It is sometimes convenient to consider the category

$$\prod_{d \geqslant 0} \mathsf{Rep}\,\Gamma_k^d$$

consisting of graded representations $X = (X^0, X^1, X^2, \ldots)$. An example is for each $V \in \mathsf{P}_k$ the representation

$$\Gamma^V = (\Gamma^{0,V}, \Gamma^{1,V}, \Gamma^{2,V}, \ldots).$$

There are two tensor products for graded representations. Given representations X, Y, they are defined in degree d by

$$(X \otimes Y)^d = \bigoplus_{i+j=d} X^i \otimes Y^j \quad \text{and} \quad (X \otimes_{\Gamma_k} Y)^d = X^d \otimes_{\Gamma_k^d} Y^d.$$

Decomposing divided powers

Given $V \in \mathsf{P}_k$, let $SV = \bigoplus_{d \geqslant 0} S^d V$ denote the *symmetric algebra*. This gives a functor from P_k to the category of commutative k-algebras which preserves coproducts. Thus

$$SV \otimes SW \cong S(V \oplus W)$$

and therefore by duality

$$\Gamma V \otimes \Gamma W \cong \Gamma(V \oplus W).$$

This yields an isomorphism of graded representations

$$\Gamma^V \otimes \Gamma^W \cong \Gamma^{V \oplus W}$$

since for each $d \geqslant 0$

$$(\Gamma^V \otimes \Gamma^W)^d \cong \bigoplus_{i+j=d} (\Gamma^{i,V} \otimes \Gamma^{j,W}) \cong \Gamma^{d, V \oplus W}.$$

Thus, for each positive integer n, one obtains a decomposition

$$\Gamma^{d,k^n} = \bigoplus_{i=0}^{d} (\Gamma^{d-i,k^{n-1}} \otimes \Gamma^i)$$

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and, using induction a canonical decomposition,

$$\Gamma^{d,k^n} = \bigoplus_{\lambda \in \Lambda(n,d)} \Gamma^{\lambda}. \tag{2.8}$$

This decomposition of divided powers has the following immediate consequence.

Proposition 2.9. *The category of finitely generated projective objects of $\mathsf{Rep}\,\Gamma_k^d$ is equivalent to the category of direct summands of finite direct sums of representations of the form Γ^λ, where $\lambda = (\lambda_1, \dots, \lambda_n)$ is a sequence of non-negative integers satisfying $\sum \lambda_i = d$ and n is a positive integer.*

Proof. This follows from Yoneda's lemma and the fact that each representable functor admits a decomposition (2.8) into representations of the form Γ^λ. □

Representations of Schur algebras

Strict polynomial functors and modules over Schur algebras are closely related by a result due to Friedlander and Suslin [FS97, Theorem 3.2]; it is an immediate consequence of Proposition 2.9. Given any ring A, we denote by $\mathsf{Mod}\,A$ the category of A-modules.

Theorem 2.10. *Let d, n be positive integers. Evaluation at k^n induces a functor $\mathsf{Rep}\,\Gamma_k^d \to \mathsf{Mod}\,S_k(n,d)$ which is an equivalence if $n \geqslant d$.*

Proof. Let $P \in \mathsf{Rep}\,\Gamma_k^d$ be a *small projective generator*, that is, P is a projective object and the functor $\mathrm{Hom}_{\Gamma_k^d}(P, -)$ is faithful and preserves set-indexed direct sums. Then

$$\mathrm{Hom}_{\Gamma_k^d}(P, -)\colon\ \mathsf{Rep}\,\Gamma_k^d \longrightarrow \mathsf{Mod}\,\mathrm{End}_{\Gamma_k^d}(P)$$

is an equivalence.

From Yoneda's lemma it follows that the representable functors $\Gamma^{d,V}$ with $V \in \mathsf{P}_k$ form a family of small projective generators of $\mathsf{Rep}\,\Gamma_k^d$. Now assume $n \geqslant d$. The decomposition (2.8) then implies that the functors Γ^λ with $\lambda \in \Lambda(n,d)$ form a family of small projective generators. Thus $P = \Gamma^{d,k^n}$ is a small projective generator since each Γ^λ occurs as a direct summand. It remains to observe that $\mathrm{End}_{\Gamma_k^d}(P) = S_k(n,d)$ and that $\mathrm{Hom}_{\Gamma_k^d}(P,-)$ equals evaluation at k^n, by Yoneda's lemma. □

Remark 2.11. Let $n \geqslant d$. The evaluation functor $\mathsf{Rep}\,\Gamma_k^d \xrightarrow{\sim} \mathsf{Mod}\,S_k(n,d)$ restricts to an equivalence $\mathsf{rep}\,\Gamma_k^d \xrightarrow{\sim} \mathsf{mod}\,S_k(n,d)$, where $\mathsf{mod}\,S_k(n,d)$ denotes the full subcategory of modules that are finitely generated projective over k.

Remark 2.12. The tensor product for the category $\mathsf{Rep}\,\Gamma_k^d$ from Proposition 2.4 induces via transport of structure a tensor product for the category $\mathsf{Mod}\,S_k(n,d)$ when $n \geqslant d$. It seems that this tensor product has not been noticed before, despite the fact that polynomial representations of the general linear groups have been studied for more than a hundred years.

Representations of symmetric groups

Schur–Weyl duality yields a relation between representations of the general linear groups and representations of the symmetric groups. In our context this takes the following form. Let $\omega = (1, \dots, 1)$ be a sequence of length d. Then Γ^ω is the functor taking V to $V^{\otimes d}$ and

$$\mathrm{End}_{\Gamma_k^d}(\Gamma^\omega) \cong k\mathfrak{S}_d,$$

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where $k\mathfrak{S}_d$ denotes the group algebra of the symmetric group $\mathfrak{S}_d$. This observation gives rise to the functor $\operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, -)\colon \operatorname{\mathsf{Rep}} \Gamma^d_k \to \operatorname{\mathsf{Mod}} k\mathfrak{S}_d$, which appears in Schur's thesis [Sch01, § III, IV]; see also [Gre80, § 6]. The functor is not relevant for the rest of this work, but we mention that it has a fully faithful left adjoint $\operatorname{\mathsf{Mod}} k\mathfrak{S}_d \to \operatorname{\mathsf{Rep}} \Gamma^d_k$ taking $X \in \operatorname{\mathsf{Mod}} k\mathfrak{S}_d$ to the functor

$$V \mapsto X \otimes_{k\mathfrak{S}_d} \operatorname{Hom}_{\Gamma^d_k}(\Gamma^{d,V}, \Gamma^\omega)$$

and a fully faithful right adjoint $\operatorname{\mathsf{Mod}} k\mathfrak{S}_d \to \operatorname{\mathsf{Rep}} \Gamma^d_k$ taking $X \in \operatorname{\mathsf{Mod}} k\mathfrak{S}_d$ to the functor

$$V \mapsto \operatorname{Hom}_{k\mathfrak{S}_d}(\operatorname{Hom}_{\Gamma^d_k}(\Gamma^\omega, \Gamma^{d,V}), X).$$

3. Exterior powers and Koszul duality

Koszul duality expresses the intimate homological relation between symmetric and exterior powers. For strict polynomial functors, Koszul duality has been introduced by Chałupnik [Cha08, § 2] and Touzé [Tou11, § 3] as the derived functor of $\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, -)$. In this section we treat the non-derived version. A crucial ingredient is the compatibility of the internal and external tensor products; for this we follow closely Touzé [Tou10, § 2]. For some earlier work relating symmetric and exterior powers for Schur algebras, we refer to [AB88, ABW82, Aki89].

Exterior powers

Given $V \in \mathsf{P}_k$, let $\Lambda V = \bigoplus_{d \geqslant 0} \Lambda^d V$ denote the *exterior algebra*, which is obtained from the tensor algebra $TV = \bigoplus_{d \geqslant 0} V^{\otimes d}$ by taking the quotient with respect to the ideal generated by the elements $v \otimes v$, $v \in V$.

For each $d \geqslant 0$, the k-module $\Lambda^d V$ is free provided that V is free; see [Bou70, III.7.8]. Thus $\Lambda^d V$ belongs to P_k for all $V \in \mathsf{P}_k$, and this gives a functor $\Gamma^d \mathsf{P}_k \to \mathsf{P}_k$, since the ideal generated by the elements $v \otimes v$ is invariant under the action of $\mathfrak{S}_d$ on $V^{\otimes d}$. There is a natural isomorphism

$$\Lambda^d(V^*) \cong (\Lambda^d V)^*$$

induced by $(f_1 \wedge \cdots \wedge f_d)(v_1 \wedge \cdots \wedge v_d) = \det(f_i(v_j))$, and therefore $(\Lambda^d)^\circ \cong \Lambda^d$.

For each $V \in \Gamma^d \mathsf{P}_k$, we use the notation

$$\Lambda^{d,V} = \Lambda^d \circ \operatorname{Hom}(V, -) \cong \Lambda^d \otimes_{\Gamma^d_k} \Gamma^{d,V}$$

and this gives a graded representation

$$\Lambda^V = (\Lambda^{0,V}, \Lambda^{1,V}, \Lambda^{2,V}, \ldots).$$

Note that multiplication in ΛV is graded commutative and that

$$\Lambda V \otimes \Lambda W \cong \Lambda(V \oplus W).$$

This yields an isomorphism of graded representation

$$\Lambda^V \otimes \Lambda^W \cong \Lambda^{V \oplus W}. \tag{3.1}$$

Internal versus external tensor product

We investigate the compatibility of internal and external tensor product.

Lemma 3.1. *Let $X_1, X_2 \in \operatorname{\mathsf{Rep}} \Gamma^d_k$ and $Y_1, Y_2 \in \operatorname{\mathsf{Rep}} \Gamma^e_k$. Then there is a natural morphism*

$$\tau : (X_1 \otimes_{\Gamma^d_k} X_2) \otimes (Y_1 \otimes_{\Gamma^e_k} Y_2) \longrightarrow (X_1 \otimes Y_1) \otimes_{\Gamma^{d+e}_k} (X_2 \otimes Y_2).$$

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Proof. Consider first the case that $X_2 = \Gamma^{d,V}$ and $Y_2 = \Gamma^{e,W}$ for some $V, W \in \mathsf{P}_k$. From the decomposition (2.8) it follows that $\Gamma^{d,V} \otimes \Gamma^{e,W}$ is canonically a direct summand of $\Gamma^{d+e,V\oplus W}$. Also, we use the description of $\Gamma^{d,V} \otimes_{\Gamma_k^d} -$ from Lemma 2.5. The projections $V \oplus W \to V$ and $V \oplus W \to W$ induce a morphism

$$X_1 \circ \operatorname{Hom}(V, -) \otimes Y_1 \circ \operatorname{Hom}(W, -) \longrightarrow (X_1 \otimes Y_1) \circ \operatorname{Hom}(V \oplus W, -),$$

and composition with the canonical projection

$$(X_1 \otimes Y_1) \circ \operatorname{Hom}(V \oplus W, -) \longrightarrow (X_1 \otimes Y_1) \otimes_{\Gamma_k^{d+e}} (\Gamma^{d,V} \otimes \Gamma^{e,W})$$

gives τ which is natural in $V \in \Gamma^d \mathsf{P}_k$ and $W \in \Gamma^e \mathsf{P}_k$. This last observation yields τ for general X_2 and Y_2. □

Lemma 3.2. *Let $X \in \operatorname{\mathsf{Rep}} \Gamma_k^d$ and $Y \in \operatorname{\mathsf{Rep}} \Gamma_k^e$. Then the composite*

$$(\Lambda^d \otimes_{\Gamma_k^d} X) \otimes (\Lambda^e \otimes_{\Gamma_k^e} Y) \xrightarrow{\tau} (\Lambda^d \otimes \Lambda^e) \otimes_{\Gamma_k^{d+e}} (X \otimes Y) \to \Lambda^{d+e} \otimes_{\Gamma_k^{d+e}} (X \otimes Y)$$

(where the second map is induced by the multiplication $\Lambda^d \otimes \Lambda^e \to \Lambda^{d+e}$) is a natural isomorphism.

Proof. Consider first the case that $X = \Gamma^{d,V}$ and $Y = \Gamma^{e,W}$ for some $V, W \in \mathsf{P}_k$. From (3.1) we have an isomorphism of graded representations

$$\begin{aligned}(\Lambda \otimes_{\Gamma_k} \Gamma^V) \otimes (\Lambda \otimes_{\Gamma_k} \Gamma^W) &\cong \Lambda^V \otimes \Lambda^W \\ &\cong \Lambda^{V \oplus W} \\ &\cong \Lambda \otimes_{\Gamma_k} \Gamma^{V\oplus W} \\ &\cong \Lambda \otimes_{\Gamma_k} (\Gamma^V \otimes \Gamma^W)\end{aligned}$$

which restricts in degree $d + e$ to an isomorphism

$$\bigoplus_{i+j=d+e} \Lambda^{i,V} \otimes \Lambda^{j,W} \xrightarrow{\sim} \Lambda^{d+e} \otimes_{\Gamma_k^{d+e}} \Bigg(\bigoplus_{i'+j'=d+e} \Gamma^{i',V} \otimes \Gamma^{j',W} \Bigg).$$

The $(i, j) = (d, e)$ summand then maps onto the $(i', j') = (d, e)$ summand (requires checking). This establishes the isomorphism for the case $X = \Gamma^{d,V}$ and $Y = \Gamma^{e,W}$. Naturality in V, W yields the general case. □

Remark 3.3. The multiplication maps $\Lambda^d \otimes \Lambda^e \to \Lambda^{d+e}$ and $\Lambda^e \otimes \Lambda^d \to \Lambda^{d+e}$ are equal up to $(-1)^{de}\sigma$, where $\sigma\colon \Lambda^d \otimes \Lambda^e \xrightarrow{\sim} \Lambda^e \otimes \Lambda^d$ permutes the factors of the tensor product. This sign comes from the graded commutativity of the exterior algebra and appears as well in the isomorphism of Lemma 3.2 when the factors X and Y are permuted.

For a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of non-negative integers we write

$$\Lambda^\lambda = \Lambda^{\lambda_1} \otimes \cdots \otimes \Lambda^{\lambda_n}.$$

Proposition 3.4. *For each sequence $\lambda \in \Lambda(n, d)$ there is an isomorphism*

$$\Lambda^\lambda \cong \Lambda^d \otimes_{\Gamma_k^d} \Gamma^\lambda.$$

Proof. Apply Lemma 3.2. □

Koszul duality

We compute $\Lambda^d \otimes_{\Gamma_k^d} -$ and $\mathcal{H}om_{\Gamma_k^d}(\Lambda^d, -)$ using the relation between exterior and symmetric powers.

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Lemma 3.5. *Given $d \geqslant 1$ and $V \in \mathsf{P}_k$, there are exact sequences*

$$\bigoplus_{i=1}^{d-1} V^{\otimes i-1} \otimes \Gamma^2 V \otimes V^{\otimes d-i-1} \xrightarrow{1\otimes\Delta\otimes 1} V^{\otimes d} \longrightarrow \Lambda^d V \longrightarrow 0,$$
$$\bigoplus_{i=1}^{d-1} V^{\otimes i-1} \otimes \Lambda^2 V \otimes V^{\otimes d-i-1} \xrightarrow{1\otimes\Delta\otimes 1} V^{\otimes d} \longrightarrow S^d V \longrightarrow 0$$

where $\Delta\colon \Gamma^2 V \to V \otimes V$ is the component of the comultiplication which is dual to the multiplication $V^* \otimes V^* \to S^2(V^*)$, and $\Delta\colon \Lambda^2 V \to V \otimes V$ is the component of the comultiplication given by $\Delta(v \wedge w) = v \otimes w - w \otimes v$.

Proof. See [ABW82, pp. 214–216] or [Tot97, p. 6]. □

Proposition 3.6. *Let $d \geqslant 0$. Then*

$$\Lambda^d \otimes_{\Gamma_k^d} \Lambda^d \cong S^d.$$

Proof. Tensoring the projective presentation of Λ^d in Lemma 3.5 with Λ^d yields an exact sequence which identifies with the presentation of S^d via the isomorphism of Proposition 3.4. Here, we use the isomorphism (2.7). □

Corollary 3.7. *For each sequence $\lambda \in \Lambda(n, d)$ there is an isomorphism*

$$\Lambda^d \otimes_{\Gamma_k^d} \Lambda^\lambda \cong S^\lambda.$$

Proof. Combine Lemma 3.2 and Proposition 3.6. To be explicit, we have

$$\Lambda^d \otimes_{\Gamma_k^d} (\Lambda^{\lambda_1} \otimes \cdots \otimes \Lambda^{\lambda_n}) \cong (\Lambda^{\lambda_1} \otimes_{\Gamma_k^{\lambda_1}} \Lambda^{\lambda_1}) \otimes \cdots \otimes (\Lambda^{\lambda_n} \otimes_{\Gamma_k^{\lambda_n}} \Lambda^{\lambda_n}) \cong S^\lambda. \qquad \square$$

For a set C of objects of an additive category, we denote by $\operatorname{\mathsf{add}} \mathsf{C}$ the full subcategory consisting of all finite direct sums of objects in C and their direct summands.

Corollary 3.8. *Let $d, n \geqslant 1$. The functor $\Lambda^d \otimes_{\Gamma_k^d} -$ induces equivalences*

$$\operatorname{\mathsf{add}}\{\Gamma^\lambda \mid \lambda \in \Lambda(n, d)\} \xrightarrow{\sim} \operatorname{\mathsf{add}}\{\Lambda^\lambda \mid \lambda \in \Lambda(n, d)\},$$
$$\operatorname{\mathsf{add}}\{\Lambda^\lambda \mid \lambda \in \Lambda(n, d)\} \xrightarrow{\sim} \operatorname{\mathsf{add}}\{S^\lambda \mid \lambda \in \Lambda(n, d)\}$$

with quasi-inverses induced by $\mathcal{H}om_{\Gamma_k^d}(\Lambda^d, -)$.

Proof. From Lemma 2.7, we have, for $X \in \operatorname{\mathsf{Rep}} \Gamma_k^d$,

$$\mathcal{H}om_{\Gamma_k^d}(\Lambda^d, X)^\circ \cong \Lambda^d \otimes_{\Gamma_k^d} X^\circ.$$

The assertion then follows from Proposition 3.4 and Corollary 3.7, using that $(S^\lambda)^\circ \cong \Gamma^\lambda$ and $(\Lambda^\lambda)^\circ \cong \Lambda^\lambda$ for each sequence λ. □

4. Derived Koszul duality

In this section we establish the Koszul duality for strict polynomial functors, working in the unbounded derived category and over an arbitrary ring of coefficients. We refer to [Cha08, Corollary 2.3] and [Tou11, Theorem 3.4] for the corresponding results of Chałupnik and Touzé.

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The derived category of strict polynomial functors

Let $\mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$ denote the unbounded derived category of the abelian category $\mathsf{Rep}\,\Gamma_k^d$. The objects are $\mathbb{Z}$-graded complexes with differential of degree $+1$ and the morphisms are chain maps with all quasi-isomorphisms inverted. This is a triangulated category which admits set-indexed products and coproducts. The set of morphisms between two complexes X, Y in $\mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$ is denoted by $\operatorname{Hom}_{\mathsf{D}(\Gamma_k^d)}(X, Y)$. An object in $\mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$ is called *perfect* if it is isomorphic to a bounded complex of finitely generated projective objects.

Derived functors

We construct the derived functors of the internal tensor product and the internal Hom functor.

Proposition 4.1. *The bifunctors* $-\otimes_{\Gamma_k^d} -$ *and* $\mathcal{H}om_{\Gamma_k^d}(-,-)$ *for* $\mathsf{Rep}\,\Gamma_k^d$ *have derived functors*

$$- \otimes^{\mathbf{L}}_{\Gamma_k^d} - \colon \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d) \times \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d) \longrightarrow \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d),$$
$$\mathbf{R}\mathcal{H}om_{\Gamma_k^d}(-,-) \colon \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)^{\mathrm{op}} \times \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d) \longrightarrow \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$$

with a natural isomorphism

$$\operatorname{Hom}_{\mathsf{D}(\Gamma_k^d)}(X \otimes^{\mathbf{L}}_{\Gamma_k^d} Y, Z) \cong \operatorname{Hom}_{\mathsf{D}(\Gamma_k^d)}(X, \mathbf{R}\mathcal{H}om_{\Gamma_k^d}(Y, Z)).$$

To compute these functors, one chooses for complexes X, Y in $\mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$ a K-projective resolution $\mathbf{p}Y \to Y$ and a K-injective resolution $Y \to \mathbf{i}Y$ (in the sense of [Spa88]). Then one gets

$$X \otimes^{\mathbf{L}}_{\Gamma_k^d} Y = X \otimes_{\Gamma_k^d} \mathbf{p}Y,$$
$$\mathbf{R}\mathcal{H}om_{\Gamma_k^d}(X, Y) = \mathcal{H}om_{\Gamma_k^d}(X, \mathbf{i}Y).$$

This involves total complexes which are defined in degree n by

$$(X \otimes_{\Gamma_k^d} Y)^n = \bigoplus_{p+q=n} X^p \otimes_{\Gamma_k^d} Y^q$$
$$\mathcal{H}om_{\Gamma_k^d}(X, Y)^n = \prod_{p+q=n} \mathcal{H}om_{\Gamma_k^d}(X^{-p}, Y^q).$$

Note that $\mathbf{p}Y = Y$ when Y is a bounded above complex of projective objects. Also, we have in $\mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$

$$\mathcal{H}om_{\Gamma_k^d}(\mathbf{p}X, Y) \xrightarrow{\sim} \mathcal{H}om_{\Gamma_k^d}(\mathbf{p}X, \mathbf{i}Y) \xleftarrow{\sim} \mathcal{H}om_{\Gamma_k^d}(X, \mathbf{i}Y).$$

Proof. Let $\mathsf{K}(\mathsf{Rep}\,\Gamma_k^d)$ denote the homotopy category of $\mathsf{Rep}\,\Gamma_k^d$ and consider the quotient functor $Q \colon \mathsf{K}(\mathsf{Rep}\,\Gamma_k^d) \to \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$ that inverts all quasi-isomorphisms. This functor has a left adjoint $\mathbf{p}$ (taking a complex to its K-projective resolution) and a right adjoint $\mathbf{i}$ (taking a complex to its K-injective resolution).

Given X in $\mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$, one obtains the following pairs of adjoint functors:

$$\mathsf{D}(\mathsf{Rep}\,\Gamma_k^d) \underset{Q}{\overset{\mathbf{p}}{\rightleftarrows}} \mathsf{K}(\mathsf{Rep}\,\Gamma_k^d) \underset{\mathcal{H}om_{\Gamma_k^d}(X,-)}{\overset{X \otimes_{\Gamma_k^d} -}{\rightleftarrows}} \mathsf{K}(\mathsf{Rep}\,\Gamma_k^d) \underset{\mathbf{i}}{\overset{Q}{\rightleftarrows}} \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d).$$

The composite from left to right gives the derived functor $X \otimes^{\mathbf{L}}_{\Gamma_k^d} -$, while the composite from right to left gives the derived functor $\mathbf{R}\mathcal{H}om_{\Gamma_k^d}(X, -)$. □

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Lemma 4.2. *Let $P \in \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$ be perfect. Then $P \otimes^{\mathbf{L}}_{\Gamma_k^d} -$ and $\mathbf{R}\mathcal{H}om_{\Gamma_k^d}(P,-)$ preserve set-indexed products and coproducts.*

Proof. The objects $X \in \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$ such that $X \otimes^{\mathbf{L}}_{\Gamma_k^d} -$ and $\mathbf{R}\mathcal{H}om_{\Gamma_k^d}(X,-)$ preserve set-indexed products and coproducts form a triangulated subcategory of $\mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$ which contains all finitely generated projective objects when viewed as complexes concentrated in degree zero. It follows that each perfect object belongs to this subcategory. □

Duality

The duality taking $X \in \mathsf{Rep}\,\Gamma_k^d$ to X° has a derived functor. For a complex $X = (X^n, d_X^n)$ we define its *dual* X° by

$$(X^\circ)^n = (X^{-n})^\circ \quad \text{and} \quad d^n_{X^\circ} = (-1)^{n+1}(d_X^{-n-1})^\circ$$

and its *derived dual* by

$$X^\diamond = (\mathbf{p}X)^\circ.$$

Note that

$$\mathcal{H}om_{\Gamma_k^d}(X, S^d) \cong X^\diamond.$$

Lemma 4.3. *Given $X, Y \in \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathsf{D}(\Gamma_k^d)}(X, Y^\diamond) \cong \mathrm{Hom}_{\mathsf{D}(\Gamma_k^d)}(Y, X^\diamond).$$

Proof. The assertion means that $(-)^\diamond$ as a functor $\mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)^{\mathrm{op}} \to \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$ is self-adjoint. From (2.4) we know that $(-)^\circ$ as a functor $(\mathsf{Rep}\,\Gamma_k^d)^{\mathrm{op}} \to \mathsf{Rep}\,\Gamma_k^d$ is self-adjoint. Passing to complexes, one obtains the following pairs of adjoint functors.

$$\mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)^{\mathrm{op}} \underset{\mathbf{p}^{\mathrm{op}}}{\overset{Q^{\mathrm{op}}}{\rightleftarrows}} \mathsf{K}(\mathsf{Rep}\,\Gamma_k^d)^{\mathrm{op}} \underset{(-)^\circ}{\overset{(-)^\circ}{\rightleftarrows}} \mathsf{K}(\mathsf{Rep}\,\Gamma_k^d) \underset{Q}{\overset{\mathbf{p}}{\rightleftarrows}} \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$$

The assertion now follows. □

Hom-tensor identities

We collect some basic identities for the internal products $- \otimes^{\mathbf{L}}_{\Gamma_k^d} -$ and $\mathbf{R}\mathcal{H}om_{\Gamma_k^d}(-,-)$.

Lemma 4.4. *Given $X, Y, Z \in \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$, the natural morphism*

$$\mathbf{R}\mathcal{H}om_{\Gamma_k^d}(X,Y) \otimes^{\mathbf{L}}_{\Gamma_k^d} Z \longrightarrow \mathbf{R}\mathcal{H}om_{\Gamma_k^d}(X, Y \otimes^{\mathbf{L}}_{\Gamma_k^d} Z)$$

is an isomorphism provided that X or Z is perfect.

Proof. Adapt the proof of Lemma 2.6. □

Lemma 4.5. *Let X, Y be in $\mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$ and suppose that X is perfect. Then there are natural isomorphisms*

$$X \otimes^{\mathbf{L}}_{\Gamma_k^d} Y^\diamond \cong \mathbf{R}\mathcal{H}om_{\Gamma_k^d}(X,Y)^\diamond,$$
$$(X \otimes^{\mathbf{L}}_{\Gamma_k^d} Y)^\diamond \cong \mathbf{R}\mathcal{H}om_{\Gamma_k^d}(X,Y^\diamond).$$

Proof. Adapt the proof of Lemma 2.7. □

Lemma 4.6. *Given $X, Y \in \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d)$, there is a natural isomorphism*

$$\mathbf{R}\mathcal{H}om_{\Gamma_k^d}(X, Y^\diamond) \cong \mathbf{R}\mathcal{H}om_{\Gamma_k^d}(Y, X^\diamond).$$

Proof. Adapt the proof of Lemma 2.8, using Lemma 4.3. □

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Koszul duality

We compute $\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} -$ and $\mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, -)$ using the relation between exterior and symmetric powers.

LEMMA 4.7. *Given $d \geqslant 1$ and $V \in \mathsf{P}_k$, the normalised bar resolutions computing $\operatorname{Ext}_{S(V^*)}(k,k) \cong \Lambda V$ and $\operatorname{Ext}_{\Lambda(V^*)}(k,k) \cong SV$ yield exact sequences*

$$0 \to \Gamma^d V \to \bigoplus_{i_1+i_2=d} \Gamma^{i_1}V \otimes \Gamma^{i_2}V \to \cdots \\ \to \bigoplus_{i_1+\cdots+i_{d-1}=d} \Gamma^{i_1}V \otimes \cdots \otimes \Gamma^{i_{d-1}}V \to V^{\otimes d} \to \Lambda^d V \to 0,$$

$$0 \to \Lambda^d V \to \bigoplus_{i_1+i_2=d} \Lambda^{i_1}V \otimes \Lambda^{i_2}V \to \cdots \\ \to \bigoplus_{i_1+\cdots+i_{d-1}=d} \Lambda^{i_1}V \otimes \cdots \otimes \Lambda^{i_{d-1}}V \to V^{\otimes d} \to S^d V \to 0$$

which are natural in V and where $i_1, i_2, \ldots$ run through all positive integers.

Proof. We refer to [Aki89, p. 359] or [Tot97, p. 6] for the proof and sketch the argument for the convenience of the reader. Take the normalised bar resolution [Mac95, ch. X] of k over $S(V^*)$:

$$\cdots \to S(V^*) \otimes S^{>0}(V^*)^{\otimes 2} \to S(V^*) \otimes S^{>0}(V^*) \to S(V^*) \to k \to 0.$$

Then apply $\operatorname{Hom}_{S(V^*)}(-,k)$:

$$0 \to k \longrightarrow S^{>0}(V^*)^* \to (S^{>0}(V^*)^*)^{\otimes 2} \to \cdots .$$

The cohomology of this complex gives $\operatorname{Ext}_{S(V^*)}(k,k)$ and identifies with ΛV via classical Koszul duality. Taking the degree d part (with $S^i(V^*)^*$ replaced by $\Gamma^i V$) yields the resolution of $\Lambda^d V$. The construction of the resolution of $S^d V$ is analogous. □

The resolution of Λ^d shows that it is a perfect object in $\mathsf{D}(\mathsf{Rep}\,\Gamma^d_k)$ when viewed as a complex concentrated in degree zero.

PROPOSITION 4.8. *Let $d \geqslant 0$. Then*

$$\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} \Lambda^d \cong S^d.$$

Proof. Tensoring the projective resolution of Λ^d in Lemma 4.7 with Λ^d yields an exact sequence which identifies with the resolution of S^d via the isomorphism of Proposition 3.4. □

THEOREM 4.9. *The functors $\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} -$ and $\mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, -)$ provide mutually quasi-inverse equivalences*

$$\mathsf{D}(\mathsf{Rep}\,\Gamma^d_k) \xrightarrow{\sim} \mathsf{D}(\mathsf{Rep}\,\Gamma^d_k).$$

Proof. Using Hom-tensor identities, the isomorphism $(\Lambda^d)^\diamond \cong \Lambda^d$, and Proposition 4.8, one computes

$$\mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, \Lambda^d) \cong (\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} \Lambda^d)^\diamond \cong (S^d)^\diamond \cong \Gamma^d$$

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and this gives for a complex X

$$\begin{aligned}\mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, \Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} X) &\cong \mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, \Lambda^d) \otimes^{\mathbf{L}}_{\Gamma^d_k} X \\ &\cong \Gamma^d \otimes^{\mathbf{L}}_{\Gamma^d_k} X \\ &\cong X\end{aligned}$$

and

$$\begin{aligned}\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} \mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, X) &\cong \mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, \Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d} X) \\ &\cong \mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, \Lambda^d) \otimes^{\mathbf{L}}_{\Gamma^d_k} X \\ &\cong \Gamma^d \otimes^{\mathbf{L}}_{\Gamma^d_k} X \\ &\cong X.\end{aligned}$$ □

The following consequence is the contravariant version of Koszul duality.

COROLLARY 4.10. *Given $X \in \mathsf{D}(\mathsf{Rep}\,\Gamma^d_k)$, there is a natural isomorphism*

$$\mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\mathbf{R}\mathcal{H}om_{\Gamma^d_k}(X, \Lambda^d), \Lambda^d) \cong X^{\diamond\diamond}.$$

Proof. We combine Hom-tensor identities, the isomorphism $(\Lambda^d)^\diamond \cong \Lambda^d$, and Theorem 4.9. This gives

$$\begin{aligned}\mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\mathbf{R}\mathcal{H}om_{\Gamma^d_k}(X, \Lambda^d), \Lambda^d) &\cong \mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, X^\diamond), \Lambda^d) \\ &\cong \mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, \mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, X^\diamond)^\diamond) \\ &\cong \mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, \Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} X^{\diamond\diamond}) \\ &\cong X^{\diamond\diamond}.\end{aligned}$$ □

The bounded derived category of finite representations

Let $\mathsf{D}^b(\mathsf{rep}\,\Gamma^d_k)$ denote the bounded derived category of the exact category $\mathsf{rep}\,\Gamma^d_k$.

LEMMA 4.11. *The inclusion $\mathsf{rep}\,\Gamma^d_k \to \mathsf{Rep}\,\Gamma^d_k$ induces a fully faithful and exact functor*

$$\mathsf{D}^b(\mathsf{rep}\,\Gamma^d_k) \longrightarrow \mathsf{D}(\mathsf{Rep}\,\Gamma^d_k).$$

Proof. The proof of Theorem 2.10 shows that Γ^{d,k^d} is a projective generator of $\mathsf{Rep}\,\Gamma^d_k$ which belongs to $\mathsf{rep}\,\Gamma^d_k$. Thus each object X in $\mathsf{rep}\,\Gamma^d_k$ admits an epimorphism $P \to X$ such that P is a projective object and belongs to $\mathsf{rep}\,\Gamma^d_k$. It follows that the inclusion $\mathsf{rep}\,\Gamma^d_k \to \mathsf{Rep}\,\Gamma^d_k$ induces a fully faithful functor $\mathsf{D}^b(\mathsf{rep}\,\Gamma^d_k) \to \mathsf{D}(\mathsf{Rep}\,\Gamma^d_k)$; see for instance [Ver96, Proposition III.2.4.1]. □

The duality $(\mathsf{rep}\,\Gamma^d_k)^{\mathrm{op}} \xrightarrow{\sim} \mathsf{rep}\,\Gamma^d_k$ taking X to X° is an exact functor and induces therefore a duality

$$\mathsf{D}^b(\mathsf{rep}\,\Gamma^d_k)^{\mathrm{op}} \xrightarrow{\sim} \mathsf{D}^b(\mathsf{rep}\,\Gamma^d_k).$$

Note that $X^\diamond \cong X^\circ$ for all $X \in \mathsf{D}^b(\mathsf{rep}\,\Gamma^d_k)$.

Given $X, Y \in \mathsf{D}^b(\mathsf{rep}\,\Gamma^d_k)$, the objects $X \otimes^{\mathbf{L}}_{\Gamma^d_k} Y$ and $\mathbf{R}\mathcal{H}om_{\Gamma^d_k}(X, Y)$ belong to $\mathsf{D}^b(\mathsf{rep}\,\Gamma^d_k)$ provided that X is perfect. This observation yields the following immediate consequence of Theorem 4.9.

COROLLARY 4.12. *The functors $\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} -$ and $\mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, -)$ induce mutually quasi-inverse equivalences*

$$\mathsf{D}^b(\mathsf{rep}\,\Gamma^d_k) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{rep}\,\Gamma^d_k).$$

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A consequence of Corollary 4.10 is the following.

Corollary 4.13. *The functor* $\mathbf{R}\mathcal{H}om_{\Gamma_k^d}(-,\Lambda^d)$ *induces an equivalence*

$$D\colon \mathsf{D}^b(\mathsf{rep}\,\Gamma_k^d)^{\mathrm{op}} \xrightarrow{\sim} \mathsf{D}^b(\mathsf{rep}\,\Gamma_k^d)$$

satisfying $D^2 \cong \mathrm{Id}$.

Finite global dimension

The Schur algebra $S_k(n,d)$ has finite global dimension provided that k is a field or $k=\mathbb{Z}$. This is a classical fact [AB88, Don86] and follows also from the more general fact that Schur algebras are quasi-hereditary; precise dimensions were computed by Totaro [Tot97]. We extend this result as follows.

Proposition 4.14. *Suppose that* k *is noetherian and let* $X,Y\in\mathsf{rep}\,\Gamma_k^d$. *Then* $\mathrm{Ext}^i_{\Gamma_k^d}(X,Y)=0$ *for all* $i>2d$.

The proof is based on the following elementary lemma which is taken from [CPS90]; it also explains the noetherianess hypothesis. For a prime ideal $\mathfrak{p}\subseteq k$, let $k(\mathfrak{p})$ denote the residue field $k_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$.

Lemma 4.15. *Let* A *be a noetherian* k*-algebra and* M *a finitely generated* A*-module. Suppose that* A *and* M *are* k*-projective. Then* M *is projective over* A *if and only if* $M\otimes_k k(\mathfrak{p})$ *is projective over* $A\otimes_k k(\mathfrak{p})$ *for all prime ideals* $\mathfrak{p}\subseteq k$.

Proof. One direction is clear. So suppose that $M\otimes_k k(\mathfrak{p})$ is projective over $A\otimes_k k(\mathfrak{p})$ for all $\mathfrak{p}$. It suffices to prove the assertion when k is local with maximal ideal $\mathfrak{m}$, and we may assume that k is complete since k is noetherian. Thus A is semi-perfect, and a projective cover $P\to M\otimes_k k(\mathfrak{m})$ lifts to a projective cover $P\to M$, which is an isomorphism since $P\otimes_k k(\mathfrak{m})\to M\otimes_k k(\mathfrak{m})$ is one. It follows that M is projective over A. □

Proof of Proposition 4.14. We work in the category of modules over $S_k(d,d)$ which is equivalent to $\mathsf{Rep}\,\Gamma_k^d$ by Theorem 2.10. Note that for all $n\geqslant 1$ and each ring homomorphism $k\to k'$

$$S_{k'}(n,d)\cong S_k(n,d)\otimes_k k'.$$

This is a consequence of the base change formula

$$\Gamma_{k'}^d(V\otimes_k k')\cong(\Gamma_k^d V)\otimes_k k'$$

which holds for each $V\in\mathsf{P}_k$. The global dimension of $S_{k(\mathfrak{p})}(d,d)$ is bounded by $2d$ for all prime ideals $\mathfrak{p}\subseteq k$, by [Tot97, Theorem 2]. Thus the assertion follows from Lemma 4.15. □

Schur and Weyl functors

Fix a positive integer d. A *partition* of *weight* d is a sequence $\lambda=(\lambda_1,\lambda_2,\ldots)$ of non-negative integers satisfying $\lambda_1\geqslant\lambda_2\geqslant\cdots$ and $\sum\lambda_i=d$. Its *conjugate* λ' is the partition where λ'_i equals the number of terms of λ that are greater or equal than i.

Fix a partition λ of weight d. Each integer $r\in\{1,\ldots,d\}$ can be written uniquely as sum $r=\lambda_1+\cdots+\lambda_{i-1}+j$ with $1\leqslant j\leqslant\lambda_i$. The pair (i,j) describes the position (ith row and jth column) of r in the *Young diagram* corresponding to λ. The partition λ determines a permutation $\sigma_\lambda\in\mathfrak{S}_d$ by $\sigma_\lambda(r)=\lambda'_1+\cdots+\lambda'_{j-1}+i$, where $1\leqslant i\leqslant\lambda_j$. Note that $\sigma_{\lambda'}=\sigma_\lambda^{-1}$.

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Fix a partition λ of weight d, and assume that $\lambda_1 + \cdots + \lambda_n = d = \lambda'_1 + \cdots + \lambda'_m$. Following [ABW82, II.1], there is defined for $V \in \mathsf{P}_k$ the *Schur module* $S_\lambda V$ as image of the map

$$\Lambda^{\lambda'_1} V \otimes \cdots \otimes \Lambda^{\lambda'_m} V \xrightarrow{\Delta \otimes \cdots \otimes \Delta} V^{\otimes d} \xrightarrow{s_{\lambda'}} V^{\otimes d} \xrightarrow{\nabla \otimes \cdots \otimes \nabla} S^{\lambda_1} V \otimes \cdots \otimes S^{\lambda_n} V.$$

Here, we denote for an integer r by $\Delta \colon \Lambda^r V \to V^{\otimes r}$ the component of the comultiplication given by

$$\Delta(v_1 \wedge \cdots \wedge v_r) = \sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)},$$

$\nabla \colon V^{\otimes r} \to S^r V$ is the multiplication, and $s_\lambda \colon V^{\otimes d} \to V^{\otimes d}$ is given by

$$s_\lambda(v_1 \otimes \cdots \otimes v_d) = v_{\sigma_\lambda(1)} \otimes \cdots \otimes v_{\sigma_\lambda(d)}.$$

The corresponding *Weyl module* W_λ is by definition the image of the analogous map

$$\Gamma^{\lambda_1} V \otimes \cdots \otimes \Gamma^{\lambda_n} V \xrightarrow{\Delta \otimes \cdots \otimes \Delta} V^{\otimes d} \xrightarrow{s_\lambda} V^{\otimes d} \xrightarrow{\nabla \otimes \cdots \otimes \nabla} \Lambda^{\lambda'_1} V \otimes \cdots \otimes \Lambda^{\lambda'_m} V.$$

Note that $(W_\lambda V)^* \cong S_\lambda(V^*)$. Moreover, $S_\lambda V$ is free when V is free, by [ABW82, Theorem II.2.16]. Thus $S_\lambda V$ and $W_\lambda V$ belong to P_k for all $V \in \mathsf{P}_k$.

The definition of Schur and Weyl modules gives rise to functors S_λ and W_λ in $\mathsf{Rep}\, \Gamma^d_k$ for each partition λ of weight d. In [AB88, § 4], a finite resolution $\mathbf{\Gamma}(W_\lambda)$ of W_λ in terms of divided powers is constructed, and in [AB88, Theorem 6.1] it is shown that $\Lambda^d \otimes_{\Gamma^d_k} \mathbf{\Gamma}(W_\lambda)$ is a resolution of $S_{\lambda'}$, basically using an explicit description of the functor

$$\mathsf{add}\{\Gamma^\mu \mid \mu \in \Lambda(n,d)\} \xrightarrow{\sim} \mathsf{add}\{\Lambda^\mu \mid \mu \in \Lambda(n,d)\}$$

from Corollary 3.8. Summarising this discussion, we have the following result.

Proposition 4.16. *For each partition λ of weight d, there is an isomorphism*

$$\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} W_\lambda \cong S_{\lambda'}.$$

The functor $\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} -$ gives an equivalence $\mathsf{D}(\mathsf{Rep}\, \Gamma^d_k) \xrightarrow{\sim} \mathsf{D}(\mathsf{Rep}\, \Gamma^d_k)$, by Theorem 4.9. Thus the classical formula

$$\operatorname{Ext}^*_{\Gamma^d_k}(W_\lambda, W_\mu) \cong \operatorname{Ext}^*_{\Gamma^d_k}(S_{\lambda'}, S_{\mu'})$$

due to Akin and Buchsbaum [AB88, Theorem 7.7] and Donkin [Don93, Corollary 3.9] follows.

An explicit example

Set $k = \mathbb{F}_2$. We compute $\mathsf{rep}\, \Gamma^2_k$. This is an abelian *length category*; that is, each object has finite composition length. The simple objects are indexed by the partitions $\alpha = (2)$ and $\omega = (1,1)$. We describe the indecomposable objects by specifying the factors of a composition series; see also [Erd93] for a description of the corresponding Schur algebra which is of finite representation type. The indecomposable projective objects are

$$\Gamma^2 = \Gamma^\alpha = \begin{bmatrix} \alpha \\ \omega \end{bmatrix} \quad \text{and} \quad \Gamma^\omega = \begin{bmatrix} \omega \\ \alpha \\ \omega \end{bmatrix},$$

and the indecomposable injective objects are

$$S^2 = S^\alpha = \begin{bmatrix} \omega \\ \alpha \end{bmatrix} \quad \text{and} \quad S^\omega = \begin{bmatrix} \omega \\ \alpha \\ \omega \end{bmatrix}.$$

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The resolutions from Lemma 4.7 have the form

$$0 \longrightarrow \Gamma^\alpha \longrightarrow \Gamma^\omega \longrightarrow \Lambda^\alpha \longrightarrow 0,$$
$$0 \longrightarrow \Lambda^\alpha \longrightarrow \Lambda^\omega \longrightarrow S^\alpha \longrightarrow 0$$

and therefore

$$\Lambda^2 = \Lambda^\alpha = [\omega] \quad \text{and} \quad \Lambda^\omega = \begin{bmatrix} \omega \\ \alpha \\ \omega \end{bmatrix}.$$

The Schur and Weyl functors are

$$S_\alpha = \begin{bmatrix} \omega \\ \alpha \end{bmatrix}, \quad S_\omega = [\omega], \quad W_\alpha = \begin{bmatrix} \alpha \\ \omega \end{bmatrix}, \quad W_\omega = [\omega].$$

The decomposition (2.8) of Γ_k^{2,k^2} has the form

$$\Gamma_k^{2,k^2} = \Gamma^\alpha \oplus \Gamma^\omega \oplus \Gamma^\alpha$$

and the endomorphism algebra of Γ_k^{2,k^2} is the Schur algebra $S_k(2,2)$. The functor $\operatorname{Hom}_{\Gamma_k^2}(\Gamma_k^{2,k^2}, -)$ induces an equivalence $\mathsf{rep}\,\Gamma_k^2 \xrightarrow{\sim} \mathsf{mod}\, S_k(2,2)$. The following table gives the tensor product for each pair of indecomposable representations in $\mathsf{rep}\,\Gamma_k^2$.

$\otimes_{\Gamma_k^2}$	$[\alpha]$	$[\omega]$	$\begin{bmatrix} \alpha \\ \omega \end{bmatrix}$	$\begin{bmatrix} \omega \\ \alpha \end{bmatrix}$	$\begin{bmatrix} \omega \\ \alpha \\ \omega \end{bmatrix}$
$[\alpha]$	$[\alpha]$	0	$[\alpha]$	0	0
$[\omega]$	0	$\begin{bmatrix} \omega \\ \alpha \end{bmatrix}$	$[\omega]$	$\begin{bmatrix} \omega \\ \alpha \end{bmatrix}$	$\begin{bmatrix} \omega \\ \alpha \\ \omega \end{bmatrix}$
$\begin{bmatrix} \alpha \\ \omega \end{bmatrix}$	$[\alpha]$	$[\omega]$	$\begin{bmatrix} \alpha \\ \omega \end{bmatrix}$	$\begin{bmatrix} \omega \\ \alpha \end{bmatrix}$	$\begin{bmatrix} \omega \\ \alpha \\ \omega \end{bmatrix}$
$\begin{bmatrix} \omega \\ \alpha \end{bmatrix}$	0	$\begin{bmatrix} \omega \\ \alpha \end{bmatrix}$	$\begin{bmatrix} \omega \\ \alpha \end{bmatrix}$	$\begin{bmatrix} \omega \\ \alpha \end{bmatrix}$	$\begin{bmatrix} \omega \\ \alpha \\ \omega \end{bmatrix}$
$\begin{bmatrix} \omega \\ \alpha \\ \omega \end{bmatrix}$	0	$\begin{bmatrix} \omega \\ \alpha \\ \omega \end{bmatrix}$	$\begin{bmatrix} \omega \\ \alpha \\ \omega \end{bmatrix}$	$\begin{bmatrix} \omega \\ \alpha \\ \omega \end{bmatrix}$	$\begin{bmatrix} \omega \\ \alpha \\ \omega \end{bmatrix} \oplus \begin{bmatrix} \omega \\ \alpha \\ \omega \end{bmatrix}$

I am grateful to Dieter Vossieck for pointing out the following connection.

Remark 4.17 (Vossieck). Let $k[\varepsilon]$ denote the k-algebra of dual numbers ($\varepsilon^2 = 0$). The category $\mathsf{rep}\,\Gamma_k^2$ is equivalent to the category of finitely generated modules over the Auslander algebra [Aus71] of $k[\varepsilon]$. The category of $k[\varepsilon]$-modules admits (at least) three different tensor products which induce tensor products on the category of modules over the Auslander algebra (via Day convolution): we have $- \otimes_{k[\varepsilon]} -$ and $- \otimes_k -$, where in the second case the action of $k[\varepsilon]$ is either given by $\varepsilon \mapsto \varepsilon \otimes 1 + 1 \otimes \varepsilon + \varepsilon \otimes \varepsilon$ (the group scheme μ_2) or by $\varepsilon \mapsto \varepsilon \otimes 1 + 1 \otimes \varepsilon$ (the group scheme α_2). The last case yields a tensor product which is equivalent to $- \otimes_{\Gamma_k^2} -$.

5. Koszul versus Ringel and Serre duality

In this section we explain the connection between the Koszul duality from Theorem 4.9 and Ringel duality for Schur algebras, as developed in work of Ringel [Rin91] and Donkin [Don93]. Then we show that Koszul duality applied twice yields a Serre functor in the sense of Reiten and Van den Bergh [RVdB02].

Ringel duality

Fix integers $d, n \geqslant 1$ with $n \geqslant d$ and recall from Theorem 2.10 that evaluation at k^n gives an equivalence

$$\operatorname{Hom}_{\Gamma_k^d}(\Gamma^{d,k^n}, -) \colon \mathsf{Rep}\,\Gamma_k^d \xrightarrow{\sim} \mathsf{Mod}\, S_k(n, d).$$

The module

$$T = \operatorname{Hom}_{\Gamma_k^d}(\Gamma^{d,k^n}, \Lambda^{d,k^n})$$

is the characteristic tilting module[3] for the Schur algebra $S_k(n, d)$; see [Don93, Rin91]. In particular, it induces an equivalence

$$\mathbf{R}\operatorname{Hom}_{S_k(n,d)}(T, -)\colon \mathsf{D}(\mathsf{Mod}\, S_k(n, d)) \xrightarrow{\sim} \mathsf{D}(\mathsf{Mod}\, \operatorname{End}_{S_k(n,d)}(T)).$$

The connection between this equivalence and the equivalence of Theorem 4.9 is explained as follows. First observe that

$$\Lambda^{d,k^n} = \Lambda^d \otimes_{\Gamma_k^d} \Gamma^{d,k^n}.$$

It follows that the composite of the equivalences $\Lambda^d \otimes^{\mathbf{L}}_{\Gamma_k^d} -$ and $\operatorname{Hom}_{\Gamma_k^d}(\Gamma^{d,k^n}, -)$ induces an isomorphism

$$\phi\colon S_k(n, d) = \operatorname{End}_{\Gamma_k^d}(\Gamma^{d,k^n}) \xrightarrow{\sim} \operatorname{End}_{\Gamma_k^d}(\Lambda^{d,k^n}) \xrightarrow{\sim} \operatorname{End}_{S_k(n,d)}(T).$$

Theorem 5.1. *The following diagram commutes up to a natural isomorphism.*

$$\begin{array}{ccccc}
\mathsf{D}(\mathsf{Rep}\,\Gamma_k^d) & & \xrightarrow[\sim]{\mathbf{R}\mathcal{H}om(\Lambda^d,-)} & & \mathsf{D}(\mathsf{Rep}\,\Gamma_k^d) \\
\wr\downarrow {\scriptstyle \mathbf{R}\mathrm{Hom}(\Gamma^{d,k^n},-)} & & & & {\scriptstyle \mathbf{R}\mathrm{Hom}(\Gamma^{d,k^n},-)}\downarrow\wr \\
\mathsf{D}(\mathsf{Mod}\, S_k(n,d)) & \xrightarrow[\sim]{\mathbf{R}\mathrm{Hom}(T,-)} & \mathsf{D}(\mathsf{Mod}\,\operatorname{End}(T)) & \xrightarrow[\sim]{\phi_*} & \mathsf{D}(\mathsf{Mod}\, S_k(n,d))
\end{array}$$

Proof. We have

$$\begin{aligned}
\mathbf{R}\operatorname{Hom}_{\Gamma_k^d}(\Gamma^{d,k^n}, \mathbf{R}\mathcal{H}om_{\Gamma_k^d}(\Lambda^d, -)) &\cong \mathbf{R}\operatorname{Hom}_{\Gamma_k^d}(\Lambda^{d,k^n}, -) \\
&\cong \phi_*\, \mathbf{R}\operatorname{Hom}_{S_k(n,d)}(T, \mathbf{R}\operatorname{Hom}_{\Gamma_k^d}(\Gamma^{d,k^n}, -))
\end{aligned}$$

where the first isomorphism is adjunction and the second is induced by evaluation at k^n. □

Remark 5.2. It follows from Theorems 4.9 and 5.1 that T is a tilting module over the Schur algebra $S_k(n, d)$.

Serre duality

Fix an integer $d \geqslant 0$ and suppose that k is a field. Our aim is to describe a Serre duality functor for $\mathsf{D}^b(\mathsf{rep}\,\Gamma_k^d)$.

Recall from work of Reiten and Van den Bergh [RVdB02] the following notion of Serre duality. Let C be a k-linear triangulated category and suppose that $\operatorname{Hom}_{\mathsf{C}}(X, Y)$ is finite dimensional over k for all objects X, Y in C. A *Serre functor* is by definition an equivalence $F\colon \mathsf{C} \xrightarrow{\sim} \mathsf{C}$ together

[3] A module T over any ring A is a *tilting module* if and only if $\mathbf{R}\operatorname{Hom}_A(T, -)$ induces an equivalence $\mathsf{D}(\mathsf{Mod}\, A) \xrightarrow{\sim} \mathsf{D}(\mathsf{Mod}\, \operatorname{End}_A(T))$. If A is quasi-hereditary, then this extra structure determines a *characteristic tilting module*. Its endomorphism ring A' is called *Ringel dual* of A; it is again quasi-hereditary and A'' is Morita equivalent to A; see [Rin91]. Note that Schur algebras are quasi-hereditary.

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with a natural isomorphism

$$\operatorname{Hom}_{\mathsf{C}}(X,-)^* \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{C}}(-,FX)$$

for each $X \in \mathsf{C}$. When a Serre functor exists, it is unique up to isomorphism.

Our description of a Serre functor is based on the following lemma.

Lemma 5.3. *Let X, Y be in $\mathsf{Rep}\,\Gamma_k^d$ and suppose that X is finitely generated projective. Then there is a natural isomorphism*

$$\operatorname{Hom}_{\Gamma_k^d}(X,Y)^* \cong \operatorname{Hom}_{\Gamma_k^d}(Y, S^d \otimes_{\Gamma_k^d} X).$$

Proof. Using Hom-tensor identities, we compute for $X = \Gamma^{d,V}$

$$\begin{aligned}\operatorname{Hom}_{\Gamma_k^d}(\Gamma^{d,V},Y)^* &\cong Y(V)^* \\ &\cong \operatorname{Hom}_{\Gamma_k^d}(\Gamma^{d,V^*}, Y^\circ) \\ &\cong \operatorname{Hom}_{\Gamma_k^d}((S^d \otimes_{\Gamma_k^d} \Gamma^{d,V})^\circ, Y^\circ) \\ &\cong \operatorname{Hom}_{\Gamma_k^d}(Y, S^d \otimes_{\Gamma_k^d} \Gamma^{d,V}).\end{aligned}$$ □

Proposition 5.4. *Let k be a field. Given $X, Y \in \mathsf{D}^b(\mathsf{rep}\,\Gamma_k^d)$, there is a natural isomorphism*

$$\operatorname{Hom}_{\mathsf{D}(\Gamma_k^d)}(X,Y)^* \cong \operatorname{Hom}_{\mathsf{D}(\Gamma_k^d)}(Y, S^d \otimes^{\mathbf{L}}_{\Gamma_k^d} X).$$

Proof. One can assume that X and Y are bounded complexes of finitely generated projective objects, by Proposition 4.14. Form the total complex $\operatorname{Hom}_{\Gamma_k^d}(X,Y)$ defined in degree n by

$$\operatorname{Hom}_{\Gamma_k^d}(X,Y)^n = \prod_{p+q=n} \operatorname{Hom}_{\Gamma_k^d}(X^{-p}, Y^q).$$

Note that this involves in each degree a product having only finitely many non-zero factors. Taking cohomology in degree zero gives

$$H^0 \operatorname{Hom}_{\Gamma_k^d}(X,Y) \cong \operatorname{Hom}_{\mathsf{K}(\Gamma_k^d)}(X,Y) \cong \operatorname{Hom}_{\mathsf{D}(\Gamma_k^d)}(X,Y),$$

where the first isomorphism is a general fact while the second uses that X is perfect. Now the assertion follows from the isomorphism in Lemma 5.3, using that the duality with respect to k is exact. □

In Corollary 4.12 it has been shown that $\Lambda^d \otimes^{\mathbf{L}}_{\Gamma_k^d} -$ induces an equivalence

$$\mathsf{D}^b(\mathsf{rep}\,\Gamma_k^d) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{rep}\,\Gamma_k^d)$$

and it would be interesting to have a description of its powers $(\Lambda^d \otimes^{\mathbf{L}}_{\Gamma_k^d} -)^n$. The first step is the following case $n = 2$.

Corollary 5.5. *Let k be a field. The functor $(\Lambda^d \otimes^{\mathbf{L}}_{\Gamma_k^d} -)^2 \cong S^d \otimes^{\mathbf{L}}_{\Gamma_k^d} -$ induces a Serre functor $\mathsf{D}^b(\mathsf{rep}\,\Gamma_k^d) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{rep}\,\Gamma_k^d)$.*

Proof. Combine Propositions 4.8 and 5.4. □

I am grateful to Bernhard Keller and Claus Michael Ringel for pointing out some different perspectives.

Remark 5.6 (Keller). A perhaps related phenomenon was observed in [Kel94, § 10.3]: if A is an augmented dg algebra and A^* its Koszul dual, there is a canonical triangle functor F_A

from the derived category of A^* to that of A and a canonical triangle functor can_A from the derived category of A to that of A^{**}. With these notations, there is a canonical morphism $F_A F_{A^*} \mathrm{can}_A \to S$, where S is the 'Serre functor' of A, i.e. the derived tensor product by the k-dual bimodule of A.

Remark 5.7 (Ringel). Let A be a quasi-hereditary algebra, with indecomposable projective modules $P(i)$, and indecomposable injective modules $Q(i)$. Let T be the characteristic tilting module with indecomposable summands $T(i)$. Let $B = \mathrm{End}_A(T)$. Then $\mathrm{Hom}_A(T, -)$ sends the ∇-filtered A-modules to the Δ-filtered B-modules, in particular it sends $Q(i)$ to $\mathrm{Hom}_A(T, Q(i))$, which is a direct summand of the characteristic tilting module for B. Now assume that $B = A$ (as quasi-hereditary algebras). Thus $F = \mathbf{R}\mathrm{Hom}_A(T, -)$ is an auto-equivalence of $\mathsf{D}^b(\mathsf{mod}\, A)$. As we just have seen, F sends $Q(i)$ to $T(i)$. Of course, however, F sends $T(i)$ to $P(i)$. Thus F^2 sends $Q(i)$ to $P(i)$ which is just the (inverse of the) Serre functor.

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