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ON A QUESTION OF GROSS CONCERNING UNIQUENESS OF ENTIRE FUNCTIONS

HONG-XUN YI

In this paper, we prove that there exist two finite sets S_1 (with 1 element) and S_2 (with 3 elements) such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical. This answers a question posed by Gross. Examples are provided to show that this result is sharp.

1. INTRODUCTION

Let h be a nonconstant entire function, and let S be a subset of distinct elements in $\mathbb{C}$. Define (see [**10**])

$$E_h(S) = \bigcup_{a \in S} \{z \mid h(z) = a, \text{ counting multiplicities}\},$$

$$\overline{E}_h(S) = \bigcup_{a \in S} \{z \mid h(z) = a, \text{ ignoring multiplicities}\}.$$

Let f and g be two nonconstant entire functions, and let S be a subset of distinct elements in $\mathbb{C}$. If $E_f(S) = E_g(S)$, we say f and g share the set S CM (counting multiplicity). If $\overline{E}_f(S) = \overline{E}_g(S)$, we say f and g share the set S IM (ignoring multiplicity). As a special case, let $S = \{a\}$, where $a \in \mathbb{C}$. If $E_f(\{a\}) = E_g(\{a\})$, we say f and g share the value a CM. If $\overline{E}_f(\{a\}) = \overline{E}_g(\{a\})$, we say f and g share the value a IM (see [**2**]).

In 1976, Gross asked the following question:

QUESTION 1. (See [**1**, Question 6].) Can one find two finite sets S_j $(j = 1, 2)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

In 1994, the present author [**6**] proved the following theorem, which answered the above Question 1 in the affirmative.

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Theorem A. *Let $S_1 = \{\omega \mid \omega^n - 1 = 0\}$ and $S_2 = \{a\}$, where $n \geqslant 5$, $a \neq 0$ and $a^{2n} \neq 1$. If f and g are entire functions such that $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.*

In [**1**] Gross wrote: "If the answer to Question 6 is affirmative, it would be interesting to know how large both sets would have to be."

Now it is natural to ask the following question:

Question 2. What are the smallest cardinalities of S_1 and S_2 respectively, where S_1 and S_2 are two finite sets such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

In this paper, we prove the following theorems, which answer Question 2.

Theorem 1. *Let $S_1 = \{0\}$ and $S_2 = \left\{\omega \mid \omega^2(\omega + a) - b = 0\right\}$, where a and b are two nonzero constants such that the algebraic equation $\omega^2(\omega + a) - b = 0$ has no multiple roots. If f and g are two entire functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.*

Remark. Let $S_1 = \{0\}$ and $S_2 = \{2, -3, -6\}$. It is easy to see that $S_2 = \left\{\omega \mid \omega^2(\omega + 7) - 36 = 0\right\}$. From Theorem 1 we immediately obtain that if f and g are entire functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

Theorem 2. *If S_1 and S_2 are two finite sets such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical, then $\max\{\#(S_1), \#(S_2)\} \geqslant 3$, where $\#(S)$ denotes the cardinality of the set S.*

Remark. From Theorem 2 we immediately obtain that the smallest cardinalities of S_1 and S_2 are 1 and 3 respectively, where S_1 and S_2 are two finite sets such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical. This shows that Theorem 1 is sharp.

2. Some Lemmas

In this paper, we use the usual notations of Nevanlinna theory of meromorphic functions as explained in [**3**].

Lemma 1. (See [**9**, Lemma 5].) *Let f and g be two nonconstant meromorphic functions, and let c_1, c_2 and c_3 be three nonzero constants. If*

$$c_1 f + c_2 g = c_3,$$

then

$$T(r, f) < \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, f) + S(r, f).$$

Let h be a nonconstant meromorphic function. We denote by $N_2(r,h)$ the counting function of poles of h, where a simple pole is counted once and a multiple pole is counted two times (see [7]).

Lemma 2. (See [7, Theorem 1].) *Let F and G be two nonconstant meromorphic functions such that F and G share the value 1 CM. If*

$$\limsup_{\substack{r\to\infty \\ r\in I}} \frac{N_2(r,1/F)+N_2(r,F)+N_2(r,1/G)+N_2(r,G)}{T(r)} < 1, \tag{2.1}$$

where $T(r) = \max\{T(r,F),T(r,G)\}$, I denotes any set of infinite linear measure of $0<r<\infty$, then $F\equiv G$ or $F\cdot G\equiv 1$.

Lemma 3. *Let*

$$F = \frac{f^2(f+a)}{b}, \quad G = \frac{g^2(g+a)}{b}, \tag{2.2}$$

where f and g are two nonconstant entire functions, a and b are two nonzero constants. Then $F\cdot G\not\equiv 1$.

Proof: If $F\cdot G\equiv 1$, from (2.2) we have

$$f^2(f+a)g^2(g+a)\equiv b^2.$$

From this we know that 0 and $-a$ are Picard exceptional values of f, which is impossible. Thus $F\cdot G\not\equiv 1$. ∎

Lemma 4. *Let f and g be two nonconstant entire functions which share the value 0 IM. If $F\equiv G$, where F and G are given by (2.2), then $f\equiv g$.*

Proof: Since $F\equiv G$, we have from (2.2)

$$f^2(f+a) = g^2(g+a). \tag{2.3}$$

Noting f and g share the value 0 IM, from (2.3) we know that f and g share 0 CM. From (2.3) we have

$$f^3-g^3 = -a\left(f^2-g^2\right). \tag{2.4}$$

If $f^3\not\equiv g^3$, from (2.4) we obtain

$$g = -\frac{a(h+1)}{(h-u)(h-u^2)}, \tag{2.5}$$

where $h=f/g$ and $u=\exp\left((2\pi i)/3\right)$. From (2.5) we know that h is a nonconstant meromorphic function. Noting f and g share the value 0 CM, from $h=f/g$ we know that 0 and ∞ are Picard exceptional values of h. Since g is a nonconstant entire function, from (2.5) we know that u and u^2 are Picard exceptional values of h, which is impossible. Thus $f^3\equiv g^3$ and $f^2\equiv g^2$. From this, we get $f\equiv g$. ∎

3. Proof of Theorem 1

Let F and G be given by (2.2). Thus,

$$T(r,F)=3T(r,f)+S(r,f),\quad T(r,G)=3T(r,g)+S(r,g). \tag{3.1}$$

Set

$$H=\frac{F'}{F-1}-\frac{G'}{G-1}. \tag{3.2}$$

We discuss the following two cases.

Case 1. Suppose that $H\equiv 0$. By integration we have from (3.2)

$$F-1\equiv A(G-1), \tag{3.3}$$

where A is a nonzero constant. We discuss the following two subcases.

Case 1.1. Assume that $A=1$. From (3.3) we have $F\equiv G$. By Lemma 4 we get $f\equiv g$.

Case 1.2. Assume that $A\neq 1$. Suppose 0 is not a Picard exceptional value of f and g. Since f and g share the value 0 CM, then there exists z_0 such that $f(z_0)=g(z_0)=0$. From (2.2) we obtain $F(z_0)=G(z_0)=0$. From this and (3.3) we get $A=1$, which is a contradiction. Thus, 0 is a Picard exceptional value of f and g. From (3.3) we have

$$F-AG=1-A. \tag{3.4}$$

From this we have

$$T(r,G)=T(r,F)+O(1). \tag{3.5}$$

By Lemma 1, we obtain from (2.2), (3.1), (3.4) and (3.5)

$$\begin{aligned}3T(r,f)&\leqslant \overline{N}\left(r,\frac{1}{f+a}\right)+\overline{N}\left(r,\frac{1}{g+a}\right)+S(r,f)\\&\leqslant 2T(r,f)+S(r,f),\end{aligned}$$

which is impossible.

Case 2. Suppose that $H\not\equiv 0$. Then $F\not\equiv G$. By $E_f(S_2)=E_g(S_2)$, we know that F and G share the value 1 CM. From (3.2) we have

$$T(r,H)=m(r,H)+N(r,H)=S(r,F)+S(r,G). \tag{3.6}$$

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Since f and g share the value 0 CM, f and g have the same zeros. Let z_0 be a zero of f and g. From (2.2) and (3.2) we know that z_0 is a zero of H. From this and (3.6) we get

$$\overline{N}\left(r,\frac{1}{f}\right)=\overline{N}\left(r,\frac{1}{g}\right)\leqslant N\left(r,\frac{1}{H}\right)\leqslant S(r,F)+S(r,G).$$

From this, (2.2) and (3.1) we obtain

$$\begin{aligned}N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)&\leqslant N\left(r,\frac{1}{f+a}\right)+N\left(r,\frac{1}{g+a}\right)+S(r,F)+S(r,G)\\&\leqslant\frac{1}{3}T(r,F)+\frac{1}{3}T(r,G)+S(r,F)+S(r,G).\end{aligned}$$

Thus,

$$\limsup_{\substack{r\to\infty\\ r\in I}}\frac{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)}{T(r)}\leqslant\frac{2}{3}<1,$$

where $T(r)=\max\{T(r,F),T(r,G)\}$, I denotes any set of infinite linear measure of $0<r<\infty$. By Lemma 2, we obtain $F\cdot G\equiv 1$. Again by Lemma 3, we get a contradiction.

This completes the proof of Theorem 1. □

4. Proof of Theorem 2

4.1. Some Examples.

Example 1. Let $S_1=\{a\}$ and $S_2=\{b\}$, where a and b are any two finite distinct complex numbers. Let

$$f(z)=a+(b-a)e^{h(z)},\quad g(z)=a+(b-a)e^{-h(z)},$$

where $h(z)$ is a nonconstant entire function. It is easy to show that $E_f(S_j)=E_g(S_j)$ $(j=1,2)$, but $f\not\equiv g$.

Example 2. (See [8].) Let $S_1=\{a\}$ and $S_2=\{b_1,b_2\}$, where a, b_1 and b_2 are any three finite distinct complex numbers. Let

$$f(z)=a+(b_1-a)e^{h(z)},\quad g(z)=a+(b_2-a)e^{-h(z)},$$

where $h(z)$ is a nonconstant entire function. It is easy to show that $E_f(S_j)=E_g(S_j)$ $(j=1,2)$, but $f\not\equiv g$.

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EXAMPLE 3. (See [4].) Let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$, where a_1, a_2, b_1 and b_2 are any four finite distinct complex numbers satisfying $a_1 + a_2 \neq b_1 + b_2$. Let

$$f(z) = d + (d - a_1)e^{h(z)}, \quad g(z) = d + (d - a_2)e^{-h(z)},$$

where $h(z)$ is a nonconstant entire function, $d = (a_1a_2 - b_1b_2)/(a_1 + a_2 - b_1 - b_2)$. It is easy to show that $E_f(S_j) = E_g(S_j)$ $(j = 1, 2)$, but $f \not\equiv g$.

EXAMPLE 4. (See [5].) Let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$, where a_1, a_2, b_1 and b_2 are any four finite distinct complex numbers satisfying $a_1 + a_2 = b_1 + b_2$. Let $f(z)$ be a nonconstant entire function, $g(z) = a_1 + a_2 - f(z)$. It is easy to show that $E_f(S_j) = E_g(S_j)$ $(j = 1, 2)$, but $f \not\equiv g$.

4.2. PROOF OF THEOREM 2.

Suppose that $\max\{\#(S_1), \#(S_2)\} < 3$. We proceed to get a contradiction. If $\#(S_1) = \#(S_2) = 1$, from Example 1 we have a contradiction. If $\#(S_1) = 1$ and $\#(S_2) = 2$ or $\#(S_1) = 2$ and $\#(S_2) = 1$, from Example 2 we have again a contradiction. If $\#(S_1) = \#(S_2) = 2$, from Example 3 and Example 4 we can get a contradiction. This completes the proof of Theorem 2. $\square$

5. CONCLUDING REMARK

In fact, in Section 3 of this paper we proved the following theorem, which is an improvement of Theorem 1.

THEOREM 3. *Let $S_1 = \{0\}$ and $S_2 = \{\omega \mid \omega^2(\omega + a) - b = 0\}$, where a and b are two nonzero constants such that the algebraic equation $\omega^2(\omega + a) - b = 0$ has no multiple roots. If f and g are two entire functions satisfying $\overline{E}_f(S_1) = \overline{E}_g(S_1)$ and $E_f(S_2) = E_g(S_2)$, then $f \equiv g$.*

Proceeding as in the proof of Theorem 1, we can prove the following result, which is an extension of Theorem 3.

THEOREM 4. *Let $S_1 = \{0\}$ and $S_2 = \{\omega \mid \omega^n(\omega + a) - b = 0\}$, where $n(\geqslant 2)$ is an integer, a and b are two nonzero constants such that the algebraic equation $\omega^n(\omega + a) - b = 0$ has no multiple roots. If f and g are two entire functions satisfying $\overline{E}_f(S_1) = \overline{E}_g(S_1)$ and $E_f(S_2) = E_g(S_2)$, then $f \equiv g$.*

Let $n(\geqslant 2)$ be an integer, and let a and b be two nonzero constants. It is easy to show that if $b \neq (n^n a^{n+1})/(n+1)^{n+1}$, the algebraic equation $\omega^n(\omega + a) - b = 0$ has no multiple roots. Specially, if $b \neq 4a^3/27$, the algebraic equation $\omega^2(\omega + a) - b = 0$ has no multiple roots.

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Department of Mathematics
Shandong University
Jinan, Shandong 250100
People's Republic of China
e-mail: hxyi@sdu.edu.cn