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PSEUDO-CONFLUENT MAPPINGS AND A CLASSIFICATION OF CONTINUA

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In this paper we introduce a new class of mappings and apply it to study some local properties of continua. A solution is obtained to a problem raised in [**14**] by the first author (see 4.4 below). By a mapping we always mean a continuous function.

1. Definitions and preliminaries. Recall that, given three subsets A, B, C of a topological space, the set C is said to be *connected between A and B* provided $A, B \subset C$ and $C \neq M \cup N$, where $A \subset M$, $B \subset N$ and $\bar{M} \cap N = \emptyset = M \cap \bar{N}$. The set C is *connected* provided C is connected between $\{x\}$ and $\{y\}$ for each pair of points $x, y \in C$. Let $f: X \to Y$ be a mapping of a topological space X onto a topological space Y. The mapping f is said to be *confluent* (see [**10**, p. 223]), *pseudo-confluent*, or *weakly confluent* provided, for each connected closed non-empty set $C \subset Y$, the following conditions are satisfied, respectively:

(c) for each pair of points $x \in f^{-1}(C)$ and $y \in C$, the set $f^{-1}(C)$ is connected between $\{x\}$ and $f^{-1}(y)$;

(p) for each pair of points $y, y' \in C$, the set $f^{-1}(C)$ is connected between $f^{-1}(y)$ and $f^{-1}(y')$;

(w) there exists a point $x_0 \in f^{-1}(C)$ such that, for each point $y \in C$, the set $f^{-1}(C)$ is connected between $\{x_0\}$ and $f^{-1}(y)$.

Clearly, (c) implies both (p) and (w). It follows from Proposition 1.1 that (w) also implies (p).

1.1. *If A, B, C are subsets of a topological space, x_0 is a point of it and the set C is connected between A and $\{x_0\}$ as well as between B and $\{x_0\}$, then C is connected between A and B.*

Proof. Suppose on the contrary that $C = M \cup N$, where $A \subset M$, $B \subset N$ and $\bar{M} \cap N = \emptyset = M \cap \bar{N}$. Then $x_0 \in M$ or $x_0 \in N$, which implies that C is not connected between B and $\{x_0\}$ or C is not connected between A and $\{x_0\}$, respectively.

1.2. COROLLARY. *Each confluent mapping is weakly confluent, and each weakly confluent mapping is pseudo-confluent.*

Remark. There exist pseudo-confluent mappings that are not weakly confluent. A rather simple example of such a mapping will be described in the present paper (see 3.6).

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Given a point $x_0 \in X$, the *quasi-component* $Q(X, x_0)$ of X containing x_0 is understood to mean the set of all the points $x \in X$ such that X is connected between $\{x_0\}$ and $\{x\}$. Equivalently, $Q(X, x_0)$ is the common part of all the closed-open subsets of X which contain x_0. Thus the quasi-components of X are closed subsets of X, and two quasi-components which intersect are equal.

1.3. *If A, B are compact sets and a set C is connected between A and B, then there exists a quasi-component Q of C such that $A \cap Q \neq \emptyset \neq B \cap Q$* (see [**9**, Theorem 1, p. 168]).

In particular, if a, b are points of C, then C is connected between $\{a\}$ and $\{b\}$ if and only if a and b belong to the same quasi-component of C.

The proof given in [**9**] is valid for the weaker hypothesis of 1.3. That is, the entire space need not be compact. The second part of 1.3 is also a direct consequence of the definition of quasi-components and the fact that two intersecting quasi-components coincide.

1.4. COROLLARY. *Let $f: X \to Y$ be a mapping of a topological space X onto a topological space Y such that $f^{-1}(y)$ is compact for $y \in Y$. Then f is confluent, pseudo-confluent, or weakly confluent if and only if, for each connected closed non-empty set $C \subset Y$, the following conditions are satisfied, respectively:*

(c′) *for each quasi-component Q of $f^{-1}(C)$, we have $C = f(Q)$;*

(p′) *for each pair of points $y, y' \in C$, there exists a quasi-component Q of $f^{-1}(C)$ such that $y, y' \in f(Q)$;*

(w′) *there exists a quasi-component Q of $f^{-1}(C)$ such that $C = f(Q)$.*

Remarks. Condition (w′) is essentially stronger than (w) since it follows from (w′) that the point x_0 whose existence is claimed in (w) can be selected from each set $f^{-1}(y_0)$, where $y_0 \in C$. We shall use this observation in finding some other equivalent conditions (compare 2.3 and 3.4). Easy examples show (cf. 3.5) that (w) does not imply (w′) if not all of the sets $f^{-1}(y)$ are compact. On the other hand, if X is a compact Hausdorff space, so are the sets $f^{-1}(C)$ for closed $C \subset Y$, and then their quasi-components are connected and coincide with components [**9**, Theorem 2, p. 169]. Consequently, for compact Hausdorff spaces, the word "quasi-component" can be replaced by "component" everywhere in 1.4 (compare 5.3). This indicates that, for compact Hausdorff spaces, our notions of confluent mappings and weakly confluent mappings are the same as those previously introduced in [**1**] and [**12**], respectively. Also, for compact Hausdorff spaces, one concludes that, in particular, the composite of two pseudo-confluent mappings is itself pseudo-confluent. More precisely, we have the following proposition whose direct proof is omitted.

1.5. *Let X, Y, Z be compact Hausdorff spaces, $f: X \to Y$ be a mapping of X onto Y, and $g: Y \to Z$ be a mapping of Y onto Z. If both f and g are confluent, pseudo-confluent, or weakly confluent, then the composite $gf: X \to Z$ is confluent, pseudo-confluent, or weakly confluent, respectively* (see [**1**, III, p. 214; **12**, p. 100]).

2. Mappings of hereditarily normal spaces. The next two propositions are very much analogous and we prove them jointly.

2.1. *Let $f: X \to Y$ be a closed mapping of a hereditarily normal space X onto a topological space Y, and let $x \in X$, $y \in Y$ be points. Then the following two conditions are equivalent:*

(i) *if $U \subset Y$ is an open set connected between $\{f(x)\}$ and $\{y\}$, then the set $f^{-1}(U)$ is connected between $\{x\}$ and $f^{-1}(y)$;*

(ii) *if $Z \subset Y$ is a set connected between $\{f(x)\}$ and $\{y\}$, then the set $f^{-1}(Z)$ is connected between $\{x\}$ and $f^{-1}(y)$.*

2.2. *Let $f: X \to Y$ be a closed mapping of a hereditarily normal space X onto a topological space Y, and let $y, y' \in Y$ be points. Then the following two conditions are equivalent:*

(i) *if $U \subset Y$ is an open set connected between $\{y\}$ and $\{y'\}$, then the set $f^{-1}(U)$ is connected between $f^{-1}(y)$ and $f^{-1}(y')$;*

(ii) *if $Z \subset Y$ is a set connected between $\{y\}$ and $\{y'\}$, then the set $f^{-1}(Z)$ is connected between $f^{-1}(y)$ and $f^{-1}(y')$.*

Proof. Obviously, (ii) implies (i) in both 2.1 and 2.2. Let us suppose (ii) of 2.2 is violated, which means there exists a set $Z \subset Y$ connected between $\{y\}$ and $\{y'\}$ such that

$$f^{-1}(Z) = M \cup N, f^{-1}(y) \subset M, f^{-1}(y') \subset N \text{ and } \bar{M} \cap N = \emptyset = M \cap \bar{N}.$$

It follows [**8**, Theorem 1, p. 130] that there is an open subset $G \subset X$ such that $M \subset G$ and $\bar{G} \cap N = \emptyset$. Thus

$$f^{-1}(Z) = M \cup N \subset G \cup (X \backslash \bar{G}),$$

whence $(\bar{G} \backslash G) \cap f^{-1}(Z) = \emptyset$, and $G \cup (X \backslash \bar{G})$ is not connected between $f^{-1}(y)$ and $f^{-1}(y')$. Since f is a closed mapping, the set $U = Y \backslash f(\bar{G} \backslash G)$ is open in Y. But we have $f(\bar{G} \backslash G) \cap Z = \emptyset$, whence $Z \subset U$, which implies that U also is connected between $\{y\}$ and $\{y'\}$. Because

$$f^{-1}(U) \subset X \backslash (\bar{G} \backslash G) = G \cup (X \backslash \bar{G}),$$

we conclude that the set $f^{-1}(U)$ is not connected between $f^{-1}(y)$ and $f^{-1}(y')$, so that condition (i) of 2.2 is violated. This completes the proof of 2.2. To complete the proof of 2.1, one can use exactly the same argument, with $f^{-1}(y')$ replaced by $\{x\}$ and y' replaced by $f(x)$.

2.3. *Let $f: X \to Y$ be a closed mapping of a hereditarily normal space X onto a topological space Y, and let $y_0 \in Y$ be a point such that $f^{-1}(y_0)$ is compact. Then the following two conditions are equivalent:*

(i) *for each open set $U \subset Y$, there exists a point $x_0 \in f^{-1}(y_0)$ such that if $y \in Y$ and the set U is connected between $\{y_0\}$ and $\{y\}$, then the set $f^{-1}(U)$ is connected between $\{x_0\}$ and $f^{-1}(y)$;*

(ii) *for each set* $Z \subset Y$, *there exists a point* $x_0 \in f^{-1}(y_0)$ *such that if* $y \in Y$ *and the set* Z *is connected between* $\{y_0\}$ *and* $\{y\}$, *then the set* $f^{-1}(Z)$ *is connected between* $\{x_0\}$ *and* $f^{-1}(y)$.

Proof. It is obvious that (ii) implies (i). Suppose, on the contrary, that (i) holds and (ii) does not, which means that there exists a set $Z \subset Y$ with the following property: for each point $x \in f^{-1}(y_0)$, there is a point $y(x) \in Y$ such that Z is connected between $\{y_0\}$ and $\{y(x)\}$ but $f^{-1}(Z)$ is not connected between $\{x\}$ and $f^{-1}[y(x)]$. As in the proof of 2.2, we obtain an open subset $G(x) \subset X$ such that

$$(1) \quad f^{-1}(Z) \subset G(x) \cup [X \backslash \overline{G(x)}], \quad x \in G(x) \quad \text{and} \quad f^{-1}[y(x)] \subset X \backslash \overline{G(x)}.$$

Thus the compact set $f^{-1}(y_0)$ is covered by the open sets $G(x)$, where $x \in f^{-1}(y_0)$, and there exists a finite sequence of points $x_1, \ldots, x_n$ of $f^{-1}(y_0)$ such that

$$(2) \quad f^{-1}(y_0) \subset G(x_1) \cup \ldots \cup G(x_n).$$

Since f is a closed mapping, the set

$$(3) \quad U = Y \backslash \bigcup_{i=1}^{n} f[\overline{G(x_i)} \backslash G(x_i)]$$

is open in Y, and $Z \subset U$, by (1). Hence the set U is connected between $\{y_0\}$ and $\{y(x_i)\}$ for $i = 1, \ldots, n$. Let $x_0 \in f^{-1}(y_0)$ be a point whose existence is guaranteed by (i). Consequently, the set $f^{-1}(U)$ is connected between $\{x_0\}$ and $f^{-1}[y(x_i)]$ for $i = 1, \ldots, n$. By (2), there exists an integer $k = 1, \ldots, n$ such that $x_0 \in G(x_k)$, and it follows from (3) that

$$[\overline{G(x_k)} \backslash G(x_k)] \cap f^{-1}(U) = \emptyset,$$

whence $f^{-1}(U) \subset G(x_k) \cup [X \backslash \overline{G(x_k)}]$. By (1), we get $f^{-1}[y(x_k)] \subset X \backslash \overline{G(x_k)}$, contradicting the fact that the set $f^{-1}(U)$ is connected between $\{x_0\}$ and $f^{-1}[y(x_k)]$. The proof of 2.3 is now complete.

3. Mappings onto locally connected spaces. Confluent mappings as well as pseudo-confluent or weakly confluent ones can be characterized by means of properties of arcs and connected subsets of the range spaces provided they fulfill some additional conditions. By an *arc* we mean a topological copy of the closed unit interval [0, 1] of the real line, and by a *continuum* we mean a connected compact metric space. A mapping $f: X \to Y$ is called *perfect* provided f is closed and $f^{-1}(y)$ is compact for each point $y \in Y$.

3.1. THEOREM. *Let* $f: X \to Y$ *be a perfect mapping of a hereditarily normal space* X *onto a locally connected complete metric space* Y. *Then the following four conditions are equivalent:*

(i) *f is confluent;*

(ii) *for each arc $A \subset Y$ and each quasi-component Q of $f^{-1}(A)$, we have $A = f(Q)$;*

(iii) *for each connected set $C \subset Y$ and each quasi-component Q of $f^{-1}(C)$, we have $C = f(Q)$;*

(iv) *for each set $Z \subset Y$, each point $z \in Z$ and each point $x \in f^{-1}(z)$, we have*

$$Q(Z, z) = f[Q(f^{-1}(Z), x)]. \tag{4}$$

Proof. By 1.4, (i) implies (ii), and (iii) implies (i). Clearly, (iv) implies (iii). To prove that (ii) implies (iv), let us consider a set $Z \subset Y$ and a point $x \in X$ such that $z = f(x) \in Z$. Let $y \in Q(Z, z)$, $y \neq z$, and let $U \subset Y$ be an arbitrary open set connected between $\{z\}$ and $\{y\}$. Denote by U_0 the component of U which contains z. Since U is locally connected, U_0 is a quasi-component of U [**9**, Theorem 18, p. 235], and so $y \in U_0$, by 1.3. Moreover, U_0 is an open subset of Y, so that there exists an arc $A \subset U_0$ joining y and z [**8**, p. 408; **9**, pp. 253–254]. It follows from (ii) and from the inclusion $A \subset U$ that

$$A = f[Q(f^{-1}(A), x)] \subset f[Q(f^{-1}(U), x)],$$

whence $y \in f[Q(f^{-1}(U), x)]$, i.e., $f^{-1}(y) \cap Q(f^{-1}(U), x) \neq \emptyset$. We conclude that the set $f^{-1}(U)$ is, by 1.3, connected between $\{x\}$ and $f^{-1}(y)$. By 2.1, the set $f^{-1}(Z)$ also is connected between $\{x\}$ and $f^{-1}(y)$ because the set Z is connected between $\{z\}$ and $\{y\}$. Since $f^{-1}(y)$ is compact, there exists, by 1.3, a quasi-component of $f^{-1}(Z)$ which meets $\{x\}$ and $f^{-1}(y)$. This quasi-component is $Q(f^{-1}(Z), x)$, and we get $y \in f[Q(f^{-1}(Z), x)]$. As a result, the inclusion

$$Q(Z, z) \subset f[Q(f^{-1}(Z), x)]$$

holds. We notice that the reverse inclusion is always true for $x \in f^{-1}(z)$ since the connectedness of $f^{-1}(Z)$ between $\{x\}$ and $\{x'\}$ implies, for continuous functions f, the connectedness of Z between $\{f(x)\}$ and $\{f(x')\}$. Formula (4) is then proved, and so is 3.1.

3.2. Lemma. *Let $f: X \to A$ be a mapping of a topological space X onto an arc A with end-points a_1, a_2 such that $f^{-1}(a)$ is compact for $a \in A$, and let Q be a quasi-component of X. Then $A = f(Q)$ if and only if $a_1, a_2 \in f(Q)$.*

Proof. Assume that $a_1, a_2 \in f(Q)$, i.e., there exist points $x_1, x_2 \in Q$ such that $a_1 = f(x_1)$ and $a_2 = f(x_2)$. The space X is connected between $\{x_1\}$ and $\{x_2\}$. Let $a \in A$ be a point such that $a_1 \neq a \neq a_2$. Denote by A_1 and A_2 the subarcs of A with end-points a_1, a and a, a_2, respectively. If $f^{-1}(A_1)$ were not connected between $\{x_1\}$ and $f^{-1}(a)$, we would have

$$f^{-1}(A_1) = M \cup N,\ x_1 \in M,\ f^{-1}(a) \subset N \text{ and } \bar{M} \cap N = \emptyset = M \cap \bar{N},$$

whence $f^{-1}(A) = M \cup [N \cup f^{-1}(A_2)]$, where $x_2 \in f^{-1}(A_2)$ and

$$\bar{M} \cap f^{-1}(A_2) \subset \overline{f^{-1}(A_1)} \cap f^{-1}(A_2) = f^{-1}(A_1) \cap f^{-1}(A_2) = f^{-1}(a) \subset N,$$
$$M \cap \overline{f^{-1}(A_2)} = M \cap f^{-1}(A_2) \subset f^{-1}(A_1) \cap f^{-1}(A_2) = f^{-1}(a) \subset N.$$

Consequently, we would have

$$\bar{M} \cap [N \cup f^{-1}(A_2)] \subset \bar{M} \cap N = \emptyset \text{ and } M \cap \overline{N \cup f^{-1}(A_2)} \subset M \cap \bar{N} = \emptyset,$$

which contradicts the connectedness of $X = f^{-1}(A)$ between $\{x_1\}$ and $\{x_2\}$. Thus $f^{-1}(A_1)$ is connected between $\{x_1\}$ and $f^{-1}(a)$. By 1.3, the quasi-component $Q(f^{-1}(A_1), x_1)$ meets $f^{-1}(a)$, and therefore

$$a \in f[Q(f^{-1}(A_1), x_1)] \subset f[Q(X, x_1)] = f(Q).$$

Remark. Without the compactness of the sets $f^{-1}(a)$, the conclusion of 3.2 is not necessarily true (see 3.5 below).

3.3. THEOREM. *Let $f: X \to Y$ be a perfect mapping of a hereditarily normal space X onto a locally connected complete metric space Y. Then the following four conditions are equivalent:*

(i) *f is pseudo-confluent;*

(ii) *for each arc $A \subset Y$, there exists a quasi-component Q of $f^{-1}(A)$ such that $A = f(Q)$;*

(iii) *for each connected set $C \subset Y$ and each pair of points $y, y' \in C$, there exists a quasi-component Q of $f^{-1}(C)$ such that $y, y' \in f(Q)$;*

(iv) *for each set $Z \subset Y$ and each point $z \in Z$, we have*

$$(5) \quad Q(Z, z) = \bigcup_{x \in f^{-1}(z)} f[Q(f^{-1}(Z), x)].$$

Proof. Again, as in the proof of 3.1, (iii) implies (i), by 1.4, and (i) implies (ii), by 1.4 and 3.2. Clearly, (iv) implies (iii). To prove that (ii) implies (iv), let us consider a set $Z \subset Y$ and a point $z \in Z$. Let $y \in Q(Z, z)$, $y \neq z$, and let $U \subset Y$ be an arbitrary open set connected between $\{y\}$ and $\{z\}$. Then, as in the proof of 3.1, there exists an arc $A \subset U$ joining y and z. By (ii), the set $f^{-1}(A)$ is connected between $f^{-1}(y)$ and $f^{-1}(z)$, and hence so is the set $f^{-1}(U)$. It follows from 2.2 (for $y' = z$) that the set $f^{-1}(Z)$, too, is connected between the compact sets $f^{-1}(y)$ and $f^{-1}(z)$. By 1.3, a quasi-component of $f^{-1}(Z)$ meets both $f^{-1}(y)$ and $f^{-1}(z)$, whence $y \in f[Q(f^{-1}(Z), x)]$ for at least one point $x \in f^{-1}(z)$. Thus

$$Q(Z, z) \subset \bigcup_{x \in f^{-1}(z)} f[Q(f^{-1}(Z), x)],$$

and, as in the proof of 3.1, the reverse inclusion always holds. We get (5), which completes the proof of 3.3.

3.4. THEOREM. *Let $f: X \to Y$ be a perfect mapping of a hereditarily normal space X onto a locally connected complete metric space Y. Then the following four conditions are equivalent:*

(i) *f is weakly confluent;*

(ii) *for each continuum $C \subset Y$ which is the union of a finite collection of arcs, there exists a quasi-component Q of $f^{-1}(C)$ such that $C = f(Q)$;*

(iii) *for each connected non-empty set $C \subset Y$, there exists a quasi-component Q of $f^{-1}(C)$ such that $C = f(Q)$;*

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(iv) *for each set* $Z \subset Y$ *and each point* $z \in Z$, *there exists a point* $x_0 \in f^{-1}(z)$ *such that*

(6) $Q(Z, z) = f[Q(f^{-1}(Z), x_0)]$.

Proof. By 1.4, (i) implies (ii), and (iii) implies (i). Clearly, (iv) implies (iii). To prove that (ii) implies (iv), consider a set $Z \subset Y$ and a point $z \in Z$. We need to show that condition (i) of 2.3 is satisfied for $y_0 = z$. Suppose it is not. Then there exists an open set $U \subset Y$ with the following property: for each point $x \in f^{-1}(z)$, there is a point $y(x) \in Y$ such that U is connected between $\{z\}$ and $\{y(x)\}$ but $f^{-1}(U)$ is not connected between $\{x\}$ and $f^{-1}[y(x)]$. Consequently, there exists, for each $x \in f^{-1}(z)$, a closed-open subset $V(x)$ of $f^{-1}(U)$ such that $x \in V(x)$ and $f^{-1}[y(x)] \subset f^{-1}(U)\backslash V(x)$. Sinee $f^{-1}(z)$ is compact, there exist points $x_1, \ldots, x_n$ of $f^{-1}(z)$ such that

(7) $f^{-1}(z) \subset V(x_1) \cup \ldots \cup V(x_n)$,

and, as in the proof of 3.1, there exists, for each $i = 1, \ldots, n$, an arc $A_i \subset U$ joining $y(x_i)$ and z. But the set

$$C = A_1 \cup \ldots \cup A_n$$

is a continuum in Y which is the union of a finite collection of arcs. By (ii), we have a quasi-component Q of $f^{-1}(C)$ such that $C = f(Q)$. Since $z \in C$, there exists a point $x^* \in Q$ with $z = f(x^*)$, and thus, by (7), we also have at least one integer $k = 1, \ldots, n$ such that $x^* \in V(x_k)$. On the other hand, $C \subset U$ whence $f^{-1}(C) \subset f^{-1}(U)$, and $y(x_k) \in A_k \subset C = f(Q)$. Therefore $f^{-1}[y(x_k)] \cap Q \neq \emptyset$, which implies that the set $f^{-1}(C)$ is connected between $\{x^*\}$ and $f^{-1}[y(x_k)]$, and so is the set $f^{-1}(U)$. This is a contradiction because $V(x_k)$ is a closed-open neighborhood of x^* in $f^{-1}(U)$ which is disjoint with $f^{-1}[y(x_k)]$. Condition (i) of 2.3 then holds for $y_0 = z$, and it follows from 2.3 that there exists a point $x_0 \in f^{-1}(z)$ such that, for each point $y \in Q(Z, z)$, the set $f^{-1}(Z)$ is connected between $\{x_0\}$ and $f^{-1}(y)$. By 1.3, the quasi-component $Q(f^{-1}(Z), x_0)$ meets $f^{-1}(y)$. Thus

$$Q(Z, z) \subset f[Q(f^{-1}(Z), x_0)],$$

and the reverse inclusion is always true, proving (6). The proof of 3.4 is now complete.

Remark. That the assumption of the local connectedness of Y cannot be removed from any of Theorems 3.1, 3.3, or 3.4, can be seen on Example 3.7 at the end of this section.

3.5. *Example. There exists a weakly confluent, hence also pseudo-confluent, mapping* $f: X \rightarrow [0, 1]$ *of a metric space* X *onto* $[0, 1]$ *such that the sets* $f^{-1}(0)$ *and* $f^{-1}(1)$ *are degenerate, each quasi-component of* X *is compact and none of them is mapped by* f *onto* $[0, 1]$.

Moreover, the mapping f satisfies condition (w) *for each connected non-empty set $C \subset [0, 1]$.*

Proof. We define X to be a subset of the plane, given by the formula

$$X = \{(0,0), (1,0)\} \cup \{(2^{-n}, 0): n = 1, 2, \ldots\}$$
$$\cup \{(1 - 2^{-n}, 0): n = 1, 2, \ldots\}$$
$$\cup \bigcup_{n=1}^{\infty} \{(x, 2^{-n}): 2^{-n} \leqq x \leqq 1 - 2^{-n}\},$$

and let $f[(x, y)] = x$ for $(x, y) \in X$.

3.6. *Example. There exists a pseudo-confluent mapping f: $[0, 1] \to T$ of $[0, 1]$ onto a simple triod T such that f is not weakly confluent.*

Proof. We have $T = A_0 \cup A_1 \cup A_2$, where A_i is an arc with end-points a_i, b ($i = 0, 1, 2$) such that b is the only common point of any two of the arcs A_0, A_1 and A_2. It is enough to let f be a mapping such that $f(0) = a_0$ and f maps the intervals $[0, 1/3]$, $[1/3, 2/3]$ and $[2/3, 1]$ homeomorphically onto the arcs $A_0 \cup A_1$, $A_1 \cup A_2$ and $A_2 \cup A_0$, respectively. Then condition (ii) of 3.3 is satisfied. Condition (ii) of 3.4, however, is not satisfied for any continuum $C \subset T$ of the form $C = A_0 \cup A_1' \cup A_2'$, where A_i' is a proper subarc of A_i with the point b being one of its end-points ($i = 1, 2$).

Remarks. The existence of a mapping described in Example 3.6 is related to a more general phenomenon. Namely, each dendrite having only a finite number of end-points is the image of $[0, 1]$ under a pseudo-confluent mapping (see [**6**, Theorem 2, p. 247]; cf. 4.8 in our next section). Moreover, if the dendrite is not an arc, then no such mapping is weakly confluent [**2**, Corollary II.3].

3.7. *Example. There exists a confluent mapping f: $X \to Y$ of an arc-like continuum X onto an arc-like continuum Y and a connected open set $U \subset Y$ such that the set $f^{-1}(U)$ has only two quasi-components and neither of them is mapped by f onto U.*

Proof. Let X be the subset of the plane defined by

$$X = \{(x, 1): |x| \leqq 1\} \cup \left\{\left(\sin \frac{\pi}{y - 1}, y\right) : 1 < y \leqq 2\right\}$$
$$\cup \{(1, y): |y| \leqq 1\} \cup \left\{\left(x, \sin \frac{\pi}{x - 1}\right) : 1 < x \leqq 2\right\},$$

and let R be an equivalence relation in X, given by

$$R = \{((t, 1), (1, t)): |t| \leqq 1\} \cup \{((1, t), (t, 1)): |t| \leqq 1\}$$
$$\cup \{(p, p): p \in X\}.$$

Then the natural projection f of X onto the quotient space $Y = X/R$ is a confluent mapping [**17**, Example 3.6, p. 105]. Let $p_0 = (0, 2)$, $p_1 = (1, 1)$, $p_2 = (2, 0)$, and define $U = Y\backslash\{f(p_1)\}$. The set U is connected and the set $f^{-1}(U) = X\backslash\{p_1\}$ is composed of two quasi-components Q_0, Q_2 containing the points p_0, p_2, respectively. On the other hand, we have $f(p_0) \in U\backslash f(Q_2)$ and $f(p_2) \in U\backslash f(Q_0)$. The mapping f does not satisfy condition (iii) of 3.3, thus also neither of 3.1 nor of 3.4.

4. Mappings of hereditarily locally connected continua. We say that a continuum is of *Class A* provided each connected subset of it is arcwise connected. Clearly, all the dendrites belong to Class A. We also distinguish a larger class of continua which has been introduced in [**11**]. A continuum X is said to be *finitely Suslinian* provided each collection of pairwise disjoint subcontinua of X having diameters greater than a positive number is finite. It is known that each continuum of Class A is regular [**19**, Theorem 6, p. 323], each regular continuum is finitely Suslinian, and each finitely Suslinian continuum is hereditarily locally connected [**11**, 1.4 and 1.7, pp. 132–133]. Since, for subsets of hereditarily locally connected continua, their quasi-components coincide with components [**9**, Theorem 9 (ii), p. 272], our Propositions 4.1, 4.2 and 4.3 are consequences of Theorems 3.1, 3.4 and 3.3, respectively.

4.1. *Let f: $X \to Y$ be a confluent mapping of a hereditarily locally connected continuum X onto a continuum Y. Then, for each connected set $C \subset Y$ and each component K of $f^{-1}(C)$, we have $C = f(K)$* (see [**4**, Theorem 1.3, p. 6]).

A version of the above theorem was announced earlier in [**3**] where, however, the word "hereditarily" was omitted by mistake.

4.2. *Let f: $X \to Y$ be a weakly confluent mapping of a hereditarily locally connected continuum X onto a continuum Y. Then, for each connected set $C \subset Y$, there exists a component K of $f^{-1}(C)$ such that $C = f(K)$.*

4.3. *Let f: $X \to Y$ be a pseudo-confluent mapping of a hereditarily locally connected continuum X onto a continuum Y. Then, for each connected non-empty set $C \subset Y$ and each pair of points $y, y' \in C$, there exists a component K of $f^{-1}(C)$ such that $y, y' \in f(K)$.*

Since all the continua of Class A are hereditarily locally connected, we have the following result which generalizes a previous theorem [**4**, Theorem 1.4, p. 6] and also provides an affirmative solution to a problem [**14**, Problem II, p. 326].

4.4. COROLLARY. *The pseudo-confluent image of a continuum of Class A belongs to Class A.*

The next three theorems generalize some results of [**15**] (see also [**13**, p. 169]).

4.5. THEOREM. *The pseudo-confluent image of a regular continuum is regular.*

Proof. Let f: $X \to Y$ be a pseudo-confluent mapping of a regular continuum

X onto a continuum Y. Given two points $y, y' \in Y$, there exists an open subset $G \subset X$ such that $f^{-1}(y) \subset G$, $\bar{G} \cap f^{-1}(y') = \emptyset$ and the set $\bar{G} \backslash G$ is finite [**9**, Theorem 9, p. 287]. Then $U = Y \backslash f(\bar{G} \backslash G)$ is an open subset of Y which contains y and y'. Moreover, we have $f^{-1}(U) \subset G \cup (X \backslash \bar{G})$, so that, as in the proof of 2.2, the set $f^{-1}(U)$ is not connected between $f^{-1}(y)$ and $f^{-1}(y')$. Since Y is locally connected, the points y and y' lie in distinct components of U, by 3.3, condition (iii). Also, since Y is locally connected, the components of the open set U coincide with the quasi-components. Hence, U is not connected between $\{y\}$ and $\{y'\}$. Since $Y \backslash U = f(\bar{G} \backslash G)$ is finite, Y is regular [**20**, (4.3), p. 97].

4.6. THEOREM. *The pseudo-confluent image of a finitely Suslinian continuum is finitely Suslinian.*

Proof. Suppose $f: X \to Y$ is a pseudo-confluent mapping of a continuum X onto a continuum Y that is not finitely Suslinian, i.e., there exists a number $\epsilon_0 > 0$ and an infinite sequence of pairwise disjoint subcontinua $C_1, C_2, \ldots$ of Y such that diam $C_i > \epsilon_0$ for $i = 1, 2, \ldots$ Consequently, there are points $y_i, y_i' \in C_i$ with $\operatorname{dist}(y_i, y_i') > \epsilon_0$ for $i = 1, 2, \ldots$ By 1.4, we get continua $K_i \subset f^{-1}(C_i)$ such that $y_i, y_i' \in f(K_i)$. The sets K_i are pairwise disjoint. By the continuity of f, their diameters must all be greater than some positive number δ_0. Thus X is not finitely Suslinian.

4.7. THEOREM. *The pseudo-confluent image of a hereditarily locally connected continuum is hereditarily locally connected.*

Proof. Suppose $f: X \to Y$ is a pseudo-confluent mapping of a continuum X onto a continuum Y that is not hereditarily locally connected. Then there exists a non-degenerate continuum $C_0 \subset Y$ and an infinite sequence of subcontinua $C_1, C_2, \ldots$ of Y such that

$$C_0 = \operatorname*{Lim}_{i \to \infty} C_i \quad \text{and} \quad C_0 \cap C_i = \emptyset \ (i = 1, 2, \ldots)$$

[**9**, Theorem 2, p. 269]. Since C_0 is non-degenerate, infinitely many of the continua C_i have diameters greater than a positive number ϵ_0. As in the proof of 4.6, we get an infinite sequence of continua $K_j \subset f^{-1}(C_{i_j})$ whose diameters are all greater than a positive number δ_0, and $i_1 < i_2 < \ldots$ An infinite subsequence $K_{j_1}, K_{j_2}, \ldots$ $(j_1 < j_2 < \ldots)$ converges to a continuum K_0. The diameter of K_0 is at least δ_0, hence K_0 is non-degenerate. Each point $x \in K_0$ is the limit of a sequence of points x_k, where $x_k \in K_{j_k}$, so that $f(x_k) \in f(K_{j_k}) \subset C_{i_{j_k}}$ and

$$f(x) = \lim_{k \to \infty} f(x_k) \in \operatorname*{Lim}_{k \to \infty} C_{i_{j_k}} = C_0,$$

i.e., $f(K_0) \subset C_0$. As a result, the subcontinua $K_{j_1}, K_{j_2}, \ldots$ of X do not meet their limit K_0, and thus X is not hereditarily locally connected (ibidem). This completes the proof of 4.7.

Let $f: X \to Y$ be a mapping of a space X onto a space Y. The mapping f is said to be an *inverse-arc function* (see [**6**, p. 245]) provided, for each arc $A \subset Y$, there exists an arc $L \subset X$ such that $A = f(L)$. By 3.3, conditions (i) and (ii), we have the following easy equivalence.

4.8. *A mapping of a hereditarily arcwise connected continuum onto a locally connected continuum is pseudo-confluent if and only if it is an inverse-arc function.*

4.9. COROLLARY. *The image of a continuum of Class A under an inverse-arc function belongs to Class A.*

Since all the dendrites are continua of Class A, our Corollary 4.9 strengthens the result of [**7**] and generalizes an earlier one [**6**, Theorem 8, p. 254]. We give yet another generalization of it at the end of this paper (see 5.5).

5. Other classes of continua. It is known that each hereditarily locally connected continuum is rational [**20**, Theorem (3.3), p. 94], and each rational continuum is Suslinian [**11**, 1.3, p. 132]. We recall that, as in [**11**], a continuum X is said to be *Suslinian* provided each collection of pairwise disjoint non-degenerate subcontinua of X is countable. Several results discussed in [**15**] are generalized by the next two theorems.

5.1. THEOREM. *The locally connected pseudo-confluent image of a rational continuum is rational.*

Proof. The proof of 5.1 is a copy of the proof of 4.5 with the only difference being that the set $\bar{G}\backslash G$ is countable rather than finite. Since the range continuum Y is now assumed to be locally connected, the use of 3.3 is justified and Y turns out to be rational in the same way as it was proved to be regular in 4.5.

Remarks. It can be shown [**16**] that the pseudo-confluent mappings, even the weakly confluent mappings, do not necessarily preserve the rationality of a continuum if the image is not locally connected. On the other hand, if the image is a locally connected continuum, the pseudo-confluency is equivalent to condition (ii) of 3.3. Thus the theorem which follows also generalizes a theorem of [**7**].

5.2. THEOREM. *The pseudo-confluent image of a Suslinian continuum is Suslinian.*

Proof. The proof of 5.2 is analogous to that of 4.6.

Proposition 5.3 is a generalized version of 4.8.

5.3. *Let f: $X \to Y$ be a mapping of a compact Hausdorff space X onto a compact metric space Y. Then f is pseudo-confluent if and only if, for each irreducible continuum $C \subset Y$, there exists a component K of $f^{-1}(C)$ such that $C = f(K)$.*

Proof. The condition is obviously necessary in order that f be pseudo-confluent (cf. 1.4). To see that it is also sufficient, let us consider a continuum $C \subset Y$ and a pair of points $y, y' \in C$. There exists a continuum $C' \subset C$ which is irreducible between y and y' [**9**, Theorem 1, p. 192] and the existence of a component of $f^{-1}(C')$ which is mapped by f onto C' implies that condition (p′) of 1.4 holds. Consequently, f is pseudo-confluent.

5.4. *If X is a compact metric space and* $\dim X \geqq 2$, *then there exists a weakly confluent mapping of X onto a* 2-*cell* (see [**18**, Théorème I, p. 328]).

We say that a one-dimensional continuum X is *acyclic* provided each mapping of X into the circle is homotopic to a constant mapping.

5.5. THEOREM. *The pseudo-confluent image of a one-dimensional acyclic continuum is at most one-dimensional.*

Proof. Suppose on the contrary that $f: X \to Y$ is a pseudo-confluent mapping of a one-dimensional acyclic continuum X onto a continuum Y of dimension $\dim Y \geqq 2$. By 5.4, there exists a weakly confluent mapping $g: Y \to I^2$ of Y onto the 2-cell I^2. It follows from 1.2 and 1.5 that the composite $gf: X \to I^2$ is a pseudo-confluent mapping. We note that each subcontinuum of X is acyclic [**9**, Theorem 2, p. 354]. Thus, by 5.3, each irreducible continuum contained in I^2 is the image under gf of an acyclic continuum. However, there exists an irreducible continuum in I^2 which is not the continuous image of any acyclic continuum [**5**, Section 2, p. 542], a contradiction.

Remarks. Notice that the pseudo-confluent mappings may raise the dimension of one-dimensional continua that are not acyclic since, for instance, the 2-cell is the monotone image of a locally connected one-dimensional continuum. By 4.8, a result of [**7**] concerning the dendrites is generalized in Theorem 5.5. On the other hand, 5.5 generalizes a solution, reported in [**12**] and due to H. Cook, of a problem posed there (see [**12**, Problem 1, p. 102; **7**]). The method of the proof in the case of pseudo-confluent mappings does not differ very much from that for weakly confluent ones which were treated previously; specifically, we owe to H. Cook the idea of combining the results of [**5**] and [**18**].

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