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LIE ALGEBRAS WITH NILPOTENT CENTRALIZERS

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1. Introduction. We consider finite dimensional Lie algebras over an algebraically closed field F of arbitrary characteristic. Such an algebra L will be called a *centralizer nilpotent* Lie algebra (abbreviated c.n.) provided that the centralizer $\mathbf{C}(x)$ is a nilpotent subalgebra of L for all nonzero $x \in L$.

For each algebraically closed F, there is a unique simple Lie algebra of dimension 3 over F which we shall denote $S(F)$. This algebra has a basis e_{-1}, e_0, e_1 such that $[e_{-1}\, e_0] = e_{-1}$, $[e_{-1}\, e_1] = e_0$ and $[e_0\, e_1] = e_1$. (If $\mathrm{char}(F) \neq 2$, then $S(F) \cong sl_2(F)$.) It is trivial to check that $S(F)$ is a c.n. algebra for all F.

There are two other types of simple Lie algebras we consider. If $\mathrm{char}(F) = 3$, construct the octonion (Cayley) algebra over F. The subspace consisting of elements of trace zero (i.e. zero scalar part) is a Lie algebra $C(F)$ under the operation $[a\, b] = ab - ba$. Then $C(F)$ is simple of dimension 7. (In fact, $C(F) \cong sl_3(F)/F \cdot 1$.) If $\mathrm{char}(F) = p > 3$, we write $W(F)$ for the Witt algebra over F, so that $W(F)$ is simple of dimension p. One can show that $C(F)$ and $W(F)$ are c.n. Lie algebras. (A proof for $W(F)$ is sketched following Corollary 2.3.)

In this paper, we show that $S(F)$, $C(F)$ and $W(F)$ are the only simple c.n. algebras. In fact we prove more. In order to state our result, we introduce the notation $\mathcal{N}(L)$ to denote the unique largest nilpotent ideal of L, the nil radical.

THEOREM A. *Let L be a finite dimensional* c.n. *Lie algebra over an algebraically closed field F. Assume L is not nilpotent. Then $\mathcal{N}(L)$ is the unique maximal proper ideal and $L/\mathcal{N}(L)$ is either of dimension* 1 *or is isomorphic to $S(F)$, $C(F)$ or $W(F)$. Also, if L is nonsolvable, then $\mathcal{N}(L)$ is abelian.*

In proving the part of Theorem A which asserts that $\mathcal{N}(L)$ is abelian, we obtain the following related result.

THEOREM B. *Let L be a rank one nonsolvable Lie algebra over an algebraically closed field F. If* $\mathrm{char}(F) > 0$, *assume also that all eigenvalues of* ad *h lie in the prime subfield of F for some $h \in L$ such that $F \cdot h$ is a Cartan subalgebra. Then $L/\mathcal{N}(L)$ is isomorphic to one of $S(F)$, $C(F)$ or $W(F)$ and $\mathcal{N}(L)$ is abelian.*

This result generalizes theorems of Kaplansky [**4**] which consider the case where L is simple (and our proof depends on Kaplansky's work). Theorem B

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is essentially included in a result of Ermolaev [**1**], however our proof seems more elementary and direct.

In the final section of this paper we show how to construct nonsolvable c.n. Lie algebras L with $\mathcal{N}(L) \neq 0$. This can be done when $\mathrm{char}(F) \neq 2$.

Perhaps some explanation of the origin of the centralizer nilpotent hypothesis is appropriate. The study of c.n. Lie algebras was motivated by the fact that research on the analogous class of finite groups has been quite fruitful. In fact, the paper of Feit, Hall and Thompson [**2**] which proved that c.n. groups of odd order are solvable contained the germs of some of the ideas later used by Feit and Thompson to prove that all groups of odd order are solvable. There seems to be little connection, however, between the Lie algebra techniques used in this paper and the more difficult content of the Feit, Hall, Thompson paper.

2. Properties of c.n. **algebras.** Throughout this section let L be a c.n. algebra over an arbitrary algebraically closed field F. The following easy lemma from linear algebra will be useful.

LEMMA 2.1. *Let A and B be $n \times n$ matrices over F. Then there exists $\alpha, \beta \in F$, not both zero, such that $\alpha A + \beta B$ is singular.*

Proof. If not, A^{-1} exists and we let λ be an eigenvalue of $A^{-1}B$. Then $A^{-1}B - \lambda I$ is singular and hence so is $A(A^{-1}B - \lambda I) = B - \lambda A$.

THEOREM 2.2. *Let $U \subseteq L$ be a nilpotent subalgebra with* $\dim U \geqq 2$. *Then* ad *u is nilpotent on L for all $u \in U$.*

Proof. Since U is nilpotent, we may decompose L into weight spaces $L_\lambda(U)$ relative to U. We must show that $L_0(U)$ is the only nonzero component.

Suppose that $L_\lambda(U) \neq 0$. Now $L_\lambda(U)$ is a U-submodule of L and since $\dim U \geqq 2$, Lemma 2.1 yields that there exists $x \in U$ with $x \neq 0$ and $\mathrm{ad}\, x$ singular on $L_\lambda(U)$. Thus $L_\lambda(U) \cap \mathbf{C}(x) \neq 0$.

Let $z \in \mathbf{Z}(U)$, the center of U, with $z \neq 0$. Then $L_\lambda(U) \cap \mathbf{C}(x)$ is invariant under $\mathrm{ad}\, z$ and since $z \in \mathbf{C}(x)$ which is nilpotent by the c.n. hypothesis, it follows that $\mathrm{ad}\, z$ is nilpotent on $L_\lambda(U) \cap \mathbf{C}(x)$. Since $L_\lambda(U) \cap \mathbf{C}(x) \neq 0$, it follows that z centralizes some nonzero element of this space and so $L_\lambda(U) \cap \mathbf{C}(z) \neq 0$.

Now $\mathbf{C}(z) \supseteq U$ and $\mathbf{C}(z)$ is nilpotent and thus $\mathbf{C}(z) \subseteq L_0(U)$. We conclude that $L_\lambda(U) \cap L_0(U) \neq 0$ and therefore $\lambda = 0$. The result follows.

COROLLARY 2.3. *A Lie algebra over F is a* c.n. *algebra if and only if each element x is either* ad-*nilpotent or satisfies* $\mathbf{C}(x) = F \cdot x$.

Proof. In a c.n. algebra, if $\dim \mathbf{C}(x) \geqq 2$, then since $\mathbf{C}(x)$ is nilpotent, Theorem 2.2 yields that x is ad-nilpotent as desired.

For the converse, we must show that $\mathbf{C}(y)$ is nilpotent for all $y \neq 0$. We may certainly assume $\mathbf{C}(y) > F \cdot y$. Now let $x \in \mathbf{C}(y)$. If $x \in F \cdot y$, then

$\mathbf{C}(x) \supseteq \mathbf{C}(y) > F \cdot x$ and x is ad-nilpotent by hypothesis. If $x \notin F \cdot y$ then $y \in \mathbf{C}(x)$ and $y \notin F \cdot x$ and so $\mathbf{C}(x) > F \cdot x$ and x is ad-nilpotent in this case too. Thus every element of $\mathbf{C}(y)$ is ad-nilpotent and so $\mathbf{C}(y)$ is nilpotent by Engel's Theorem.

We mention that the sufficiency of the condition in Corollary 2.3 can be used to prove that the Witt algebra $W(F)$ (for $\operatorname{char}(F) > 3$) has the c.n. property. The algebra $W(F)$ has a basis $u_{-1}, u_0, u_1, \ldots, u_{p-2}$ such that

$$[u_i u_j] = \begin{cases} (j-i)u_{i+j} & \text{if} \quad -1 \leqq i+j \leqq p-2 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that if $x = \sum \alpha_i u_i \in W(F)$, then $\mathbf{C}(x) = F \cdot x$ if either $\alpha_{-1} \neq 0$ or $\alpha_0 \neq 0$ and otherwise $\operatorname{ad} x$ is nilpotent.

COROLLARY 2.4. *If L is not nilpotent, then every Cartan subalgebra has dimension* 1.

Proof. If H is a Cartan subalgebra of L, then $H = L_0(H)$. If $\dim H \geqq 2$, then every $h \in H$ is ad-nilpotent by Theorem 2.2 and $L = L_0(H)$. The result follows.

THEOREM 2.5. *For each $x \in L$, either* $\operatorname{ad} x$ *is nilpotent or $F \cdot x$ is a Cartan subalgebra.*

Note that by Corollary 2.3, the conclusion of Theorem 2.5 is also sufficient to show that a Lie algebra has the c.n. property.

Proof of Theorem 2.5. Let $x \in L$ and decompose L into generalized eigenspaces $L_\lambda(x)$ with respect to x. Assume $\operatorname{ad} x$ is not nilpotent so that $L_\lambda(x) \neq 0$ for some $\lambda \neq 0$. We must show that $L_0(x) = F \cdot x$.

Suppose $L_0(x) > F \cdot x$. Then there exists $y \in L_0(x) - F \cdot x$ with $[x\, y] \in F \cdot x$. Since $L_\lambda(x)$ is invariant under $L_0(x)$, it follows by Lemma 2.1 that for some nonzero $z \in F \cdot x + F \cdot y$ we have $\mathbf{C}(z) \cap L_\lambda(x) \neq 0$. Now $z \notin L_\lambda(x)$ because $z \in L_0(x)$ and $L_0(x) \cap L_\lambda(x) = 0$. We conclude that $\mathbf{C}(z) > F \cdot z$ and hence $\operatorname{ad} z$ is nilpotent by Corollary 2.3. However, $[z\, x] \in F \cdot [y\, x] \subseteq F \cdot x$ and thus the nilpotence of $\operatorname{ad} z$ yields that $[z\, x] = 0$. Now $\mathbf{C}(x) = F \cdot x$ by Corollary 2.3, and so we have $z \in F \cdot x$. Since $z \neq 0$, we conclude that $\mathbf{C}(z) = \mathbf{C}(x) = F \cdot x$. On the other hand, $\mathbf{C}(z) > F \cdot z$, implying that $\dim \mathbf{C}(z) \geqq 2$ and we have a contradiction.

COROLLARY 2.6. *Every proper ideal of L is nilpotent and hence is contained in $\mathcal{N}(L)$.*

Proof. Let $I < L$ be an ideal. Then I cannot contain a Cartan subalgebra of L and hence by Theorem 2.5, each $x \in I$ is ad-nilpotent. Therefore I is nilpotent by Engel's Theorem.

We have now proved the part of Theorem A which asserts that the nil radical of a c.n. algebra contains every proper ideal. Thus if L is not nilpotent, we have that either $L/\mathcal{N}(L)$ is of dimension 1 or is simple.

COROLLARY 2.7. *Let I be an ideal of L. Then L/I is a* c.n. *algebra.*

Proof. Let $x = u + I \in L/I$. By Corollary 2.3, it suffices to show that either ad x is nilpotent on L/I or that $\mathbf{C}_{L/I}(x) = F \cdot x$. If ad u is nilpotent on L, then certainly ad x is nilpotent. If ad u is not nilpotent, then $F \cdot u$ is a Cartan subalgebra of L by Theorem 2.5 and thus its image $F \cdot x$ is a Cartan subalgebra of L/I. In particular, $\mathbf{C}_{L/I}(x) = F \cdot x$ in this case.

3. Modules. We consider a certain type of module which arises naturally in the study of c.n. Lie algebras.

Definition 3.1. Let V be a finite dimensional module for a Lie algebra S. Then V is a *special* S-module if

i) every ad-nilpotent $x \in S$ acts nilpotently on V and

ii) every $x \in S$ which is not ad-nilpotent acts nonsingularly on V.

Note that if S is a Lie algebra over an algebraically closed field, then by Lemma 2.1, S cannot have a special module unless every two dimensional subspace of S contains an ad-nilpotent element.

Now let L be a c.n. algebra over F where F is algebraically closed and let $M \subseteq L$ be a nonsolvable subalgebra. Note that M is a c.n. algebra. Let $N = \mathcal{N}(M)$ so that by Corollary 2.6, N is a maximal ideal of M and $M/N = S$ is simple. Also, S is a c.n. algebra by Corollary 2.7.

View L as an M-module and let

$$L = L_0 \supseteq L_1 \supseteq \ldots \supseteq L_k = 0$$

be a composition series through M so that $M = L_m$ for some $m < k$. Then $N = L_{m+1}$ since N is the unique maximal ideal of M.

LEMMA 3.2. *Assume the above notation and let $V = L_i/L_{i+1}$ be a composition factor with $i \neq m$. Then $N \cdot V = 0$ and V may be viewed as an S-module. As such, it is special.*

Proof. Since N is an ideal of M, we see that $N \cdot V$ is an M-submodule of V and thus either $N \cdot V = 0$ or $N \cdot V = V$. However, each element $y \in N$ is ad-nilpotent on L since otherwise $F \cdot y$ is a Cartan subalgebra of L by Theorem 2.5 and this is impossible since y is in a proper ideal of M. It follows that y acts nilpotently on V for all $y \in N$ and thus by Engel's theorem, $N \cdot V < V$. Therefore $N \cdot V = 0$ as claimed and V may be viewed as an (M/N)-module.

Now suppose $x \in S = M/N$ and write $x = u + N$ for some $u \in M$. If ad x is nilpotent on S, then $F \cdot u$ cannot be a Cartan subalgebra and thus ad u is nilpotent on L. It follows in this case that u acts nilpotently on V and thus so does x. On the other hand, if ad x is not nilpotent on S, then ad u is not nilpotent on L and thus $F \cdot u$ is a Cartan subalgebra of L. It follows that the multiplicity of zero as an eigenvalue of ad u is one and thus u acts singularly on at most one of the composition factors. The exception is $L_m/L_{m+1} = M/N$ and hence u (and therefore x) is nonsingular on $V = L_i/L_{i+1}$ since $i \neq m$. We have now shown that V is a special S-module.

In the situation of Lemma 3.2, we will be especially interested in the case where $S = S(F)$ and so we wish to obtain information about special simple $S(F)$-modules.

First assume char$(F) \neq 2$. Since F is algebraically closed, we have $S(F) \cong sl_2(F)$ and the module theory for this algebra is known ([**3**] and [**5**]). For the convenience of the reader, we collect some facts below and provide their proofs. (Although we only need information in the case that both e_{-1} and e_1 act nilpotently, we consider the more general situation since it requires only very little extra work.)

LEMMA 3.3. *Let V be a simple $S(F)$-module where* char$(F) \neq 2$ *and F is algebraically closed. Let $k = \dim(V) < \infty$ and let E_j denote the transformation of V induced by e_j for $j = -1, 0, 1$. Then*

a) $k \leqq$ char(F) *if* char$(F) \neq 0$.

b) *E_0 has k distinct eigenvalues and the set of eigenvalues has the form $\{\lambda + i \mid 0 \leqq i \leqq k - 1\}$ for some $\lambda \in F$.*

c) *Let U_i be the eigenspace of E_0 corresponding to the eigenvalue $\lambda + i$. Then $V = \sum_{i=0}^{k-1} U_i$,* $\dim U_i = 1$ *and*

$$E_1(U_i) \subseteq U_{i+1} \text{ for } 0 \leqq i < k - 1$$
$$E_{-1}(U_i) \subseteq U_{i-1} \text{ for } 0 < i \leqq k - 1.$$

d) *If $k \neq$ char(F), then E_{-1} and E_1 are nilpotent and $\lambda = -(k-1)/2$ in* (b).

e) *If E_{-1} and E_1 are both nilpotent, then (after a possible change of the definition of λ if $k =$ char(F)) we have*

$$E_1(U_{k-1}) = 0 = E_{-1}(U_0)$$

and the containments in (c) *are equalities.*

Proof. Let $v \in V$ be an eigenvector for E_0 with eigenvalue λ. Write $v_i = (E_1)^i(v)$ for $i \geqq 0$ and let $W(v)$ be the span of the v_i. Write $l(v) = \dim W(v)$. Note that $E_0(v_i) = (\lambda + i)v_i$ and so nonzero v_i are eigenvectors for E_0. Since eigenvectors for distinct eigenvalues are linearly independent, we conclude for char $(F) = 0$ that $v_{l(v)} = 0$ and $v = v_0, v_1, \ldots, v_{l(v)-1}$ is a basis for $W(v)$.

If char$(F) = p > 0$, then $[E_1^p E_j] = (\mathrm{ad} E_1)^p E_j = 0$ for $j = -1, 0, 1$ since $p > 2$. Since V is simple, it follows that $E_1^p = \mu I$, a scalar matrix, and hence $v_p = \mu v$. It follows that $l(v) \leqq p$ and if $l(v) < p$, then just as in the characteristic zero case, we have $v_{l(v)} = 0$ and in all cases $v_0, \ldots, v_{l(v)-1}$ is a basis for $W(v)$.

We now impose some further conditions on the choice of v. Observe that $[E_0, E_1E_{-1}] = 0$ and so we may take v to be a common eigenvector for E_0 and

E_1E_{-1} and we write

(1) $E_1E_{-1}(v) = \alpha v.$

If E_{-1} is a singular transformation, write $X = \{x \in V | E_{-1}(x) = 0\}$. Note that $E_0(X) \subseteq X$ and so we may choose $v \in X$ and thus $\alpha = 0$ in this situation.

We consider two cases. Either one of E_{-1} or E_1 is singular or both are nonsingular. If either is singular, then by the symmetry between E_{-1} and E_1, it is no loss to assume that E_{-1} is singular and we have

(2) $E_{-1}(v) = 0$ and $\alpha = 0$.

If E_1 is nonsingular, then necessarily $\text{char}(F) = p > 0$ by the first paragraph, and $E_1^p = \mu I$ with $\mu \neq 0$. Then (1) yields

$$E_1(E_{-1}(v)) = \alpha v = E_1(\alpha\mu^{-1}v_{p-1})$$

and since E_1 is nonsingular, we have

(3) $E_{-1}(v) = \alpha\mu^{-1}v_{p-1}$ where $p = \text{char}(F)$ and $v_p = \mu v \neq 0$.

We may therefore assume that one of (2) or (3) holds.

In either situation, computation yields for $i \geqq 1$ that

(4) $E_{-1}(v_i) = (\lambda i + i(i-1)/2 + \alpha)v_{i-1}.$

Therefore $W(v)$ is invariant under E_{-1} as well as under E_0 and E_1 and by the simplicity of V we conclude that $W(v) = V$ and $l(v) = k$. Statements (a), (b) and (c) now follow and $U_i = F \cdot v_i$ in (c) for $0 \leqq i \leqq k - 1$.

If $k \neq \text{char}(F)$, then we have $v_k = 0$ and hence $E_1^k(v_i) = 0$ for all i and so $E_1^k = 0$. In this case (2) must hold since (3) does not, and (2) and (4) yield that $E_{-1}^k(v_i) = 0$ and so E_{-1} is nilpotent. Also in this case, taking $i = k$ in (4) and using $v_k = 0$ and $\alpha = 0$ but $v_{k-1} \neq 0$, we have $\lambda k + k(k-1)/2 = 0$ and thus $\lambda = -(k-1)/2$, proving (d).

For (e) we assume that E_{-1} and E_1 are nilpotent. By definition of the v_i we have $E_1(U_i) = U_{i+1}$ for $0 \leqq i < k - 1$. Since E_1 is nilpotent, we cannot have $E_1^p = \mu I$ for $\mu \neq 0$ and so we must have $v_k = 0$ and $E_1(U_{k-1}) = 0$. By (2) we have $E_{-1}(U_0) = 0$. If also $E_{-1}(U_i) = 0$ for $0 < i \leqq k - 1$, then $\sum_{j=1}^{k-1} U_j$ is invariant under E_{-1}, E_0 and E_1 and this contradicts the simplicity of V. Thus $E_{-1}(U_i) = U_{i-1}$ for $0 < i \leqq k - 1$ and the proof is complete.

If $k < \text{char}(F)$ or $\text{char}(F) = 0$, we introduce the notation

$$\Lambda(k) = \{-(k-1)/2 + i | 0 \leqq i \leqq k - 1\} \subseteq F$$

so that in the situation of Lemma 3.3, $\Lambda(k)$ is the set of eigenvalues of E_0 if $k \neq \text{char}(F)$. Note that $0 \in \Lambda(k)$ if and only if k is odd.

COROLLARY 3.4. *Let V be a special simple $S(F)$-module where* $\text{char}(F) \neq 2$ *and F is algebraically closed. Let $k = \dim V < \infty$. Then k is even and $\Lambda(k)$ is the set of eigenvalues of the transformation of V induced by e_0.*

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Proof. It suffices to show that k is even since then $k \neq \operatorname{char}(F)$ and by Lemma 3.3 (d), we have that $\Lambda(k)$ is the set of eigenvalues of the transformation induced by e_0. (Note that if we knew $k \neq \operatorname{char}(F)$, this would automatically imply that k is even since e_0 acts nonsingularly and yet $0 \in \Lambda(k)$ if k is odd as was mentioned above.)

Since V is special, both e_{-1} and e_1 act nilpotently and so by Lemma 3.3 (c, e), V has a basis $u_0, u_1, \ldots, u_{k-1}$ such that $e_1 \cdot u_i = u_{i+1}$ for $0 \leqq i < k - 1$; $e_1 \cdot u_{k-1} = 0$; $e_{-1} \cdot u_i = \alpha_i u_{i-1}$ for $0 < i \leqq k - 1$ with $0 \neq \alpha_i \in F$ and $e_{-1} \cdot u_0 = 0$.

Now $\operatorname{ad}(e_{-1} - e_1)^2(e_0) = 2e_0$ and hence $e_{-1} - e_1$ is not ad-nilpotent and the action of this element on V must be nonsingular. However, if k is odd, consider the element

$$v = \sum_{j=0}^{(k-1)/2} \left(\prod_{\nu=j+1}^{(k-1)/2} \alpha_{2\nu} \right) u_{2j}$$

$$= u_{k-1} + \alpha_{k-1} u_{k-3} + \alpha_{k-1}\alpha_{k-3} u_{k-5} + \ldots + (\alpha_{k-1} \ldots \alpha_2) u_0.$$

Then $(e_{-1} - e_1) \cdot v = 0$. This contradiction proves that k must be even.

For each even integer $k > 0$ with $k < \operatorname{char}(F)$ if $\operatorname{char}(F) > 0$, there is a unique simple $S(F)$-module of dimension k. By Corollary 3.4, these are the only simple $S(F)$-modules which can be special. In fact, all of these modules are special. This will be discussed further in Section 6.

We now consider the case where $\operatorname{char}(F) = 2$.

LEMMA 3.5. *Let* $\operatorname{char}(F) = 2$. *Then* $S(F)$ *has no nonzero special module.*

Proof. Suppose $V \neq 0$ is a special $S(F)$-module and let π be the corresponding representation. Note that $\operatorname{ad}(e_1 + e_{-1})^3 = 0$ and thus since V is special, we have that $\pi(e_1 + e_{-1})$ is nilpotent. Now computation yields

$$(*) \quad \pi(e_1 + e_{-1})^4 = \pi(e_1)^4 + \pi(e_{-1})^4 + \pi(e_0)^2 + \pi(e_0).$$

Since $(\operatorname{ad} e_1)^4 = 0 = (\operatorname{ad} e_{-1})^4$, it follows that $\pi(e_1)^4$ and $\pi(e_{-1})^4$ commute with all elements of $\pi(S(F))$ and so we can choose a common eigenvector $v \in V$ for $\pi(e_1)^4$, $\pi(e_{-1})^4$ and $\pi(e_0)$. Let $\pi(e_0)v = \lambda v$. Now $\pi(e_1)^4 v = 0 = \pi(e_{-1})^4 v$ since $\pi(e_1)$ and $\pi(e_{-1})$ are nilpotent, and (*) yields

$$\pi(e_1 + e_{-1})^4 v = \lambda(\lambda + 1)v.$$

Since $\pi(e_1 + e_{-1})$ is nilpotent, this yields $\lambda(\lambda + 1) = 0$ and so $\lambda = 0$ or $\lambda = 1$. However, since $\pi(e_0)$ is nonsingular with eigenvalue λ, we cannot have $\lambda = 0$ and so $\lambda = 1$ and $e_0 \cdot v = v$. It follows that $e_0 \cdot (e_1 \cdot v) = 0 = e_0 \cdot (e_{-1} v)$ and since $\pi(e_0)$ is nonsingular, we have $e_1 \cdot v = 0 = e_{-1} \cdot v$. Finally, this yields

$$v = e_0 \cdot v = [e_{-1}\, e_1] \cdot v = 0,$$

a contradiction.

4. Nonsolvable c.n. **algebras.** The main effort in this section will be to prove the following.

THEOREM 4.1. *Let L be a nonsolvable* c.n. *Lie algebra over an algebraically closed field F. Then there exists $h \in L$ such that $F \cdot h$ is a Cartan subalgebra and all eigenvalues of* ad *h lie in the prime subfield of F.*

When this result is combined with Theorem B (which will be proved in the next section), it will complete the proof of Theorem A. We need the simple algebra case of Theorem B in the proof of 4.1. This is the following result of Kaplansky [**4**].

THEOREM 4.2. (Kaplansky). *Let L be a simple rank one Lie algebra over an algebraically closed field F. If* $\text{char}(F) > 0$, *assume also that all eigenvalues of* ad *h lie in the prime subfield of F, where $h \in L$ is such that $F \cdot h$ is a Cartan subalgebra. Then one of the following occurs.*

a) $L \cong S(F)$

b) $L \cong C(F)$ *and* $\text{char}(F) = 3$

c) $L \cong W(F)$ *and* $\text{char}(F) > 3$.

Furthermore, if $\text{char}(F) \geqq 3$, *there exists $S \subseteq L$ with $S \cong S(F)$ such that $h \in S$ and h corresponds under the isomorphism to a scalar multiple of the standard basis vector e_0 with the scalar in the prime field.*

Proof. If $\text{char}(F) = 0$, then the classical theory yields $L \cong A_1(F) \cong S(F)$. If $\text{char}(F) = 2$, then $\dim L = 3$ by Lemma 74 of [**4**], and thus $L \cong S(F)$. If $\text{char}(F) = 3$ and $\dim L > 3$, then $L \cong C(F)$ by Theorem 9 of [**4**] and thus either (a) or (b) holds. In either case, the argument in [**4**] shows that there exist $a, z \in L$ with $[h\, a] = a$, $[h\, z] = -z$ and $[a\, z] = h$. It follows that $F \cdot a + F \cdot h + F \cdot z$ is a subalgebra isomorphic to $S(F)$ in which a, h, z corresponds to e_{-1}, e_0, e_1 respectively.

Now assume $\text{char}(F) > 3$. Then Theorem 2 of [**4**] yields that $L \cong S(F)$ or $L \cong W(F)$. In particular, the argument shows that there exist one dimensional root spaces L_λ and $L_{-\lambda}$ with $[L_\lambda\, L_{-\lambda}] = F \cdot h$. It follows that $L_{-\lambda} + F \cdot h + L_\lambda \cong S(F)$ and h corresponds to λe_0.

We mention that in the situation of Theorem 4.2 with $\text{char}(F) = 2$, one need not assume that L is simple in order to conclude that $L \cong S(F)$. In fact, it can be shown that it suffices to assume that the root space $L_1(h)$ is not an abelian ideal. One first proves that under this assumption, $\dim L = 3$. It is then possible to find a basis which demonstrates $L \cong S(F)$.

Also, when $\text{char}(F) = 2$, the last assertion of Theorem 4.2 is definitely false, since in $S(F)$ we can take $h = e_0 + \alpha e_{-1}$ with $\alpha \in F$. Then $[h\, e_{-1}] = e_{-1}$ and $[h, e_1 + \alpha e_0] = (e_1 + \alpha e_0) + \alpha^2 e_{-1}$. Then $F \cdot h$ is a Cartan subalgebra and $L_1(h) = F \cdot e_{-1} + F \cdot (e_1 + \alpha e_0)$. Thus if $\alpha \neq 0$, then ad h is not semisimple and no isomorphism can carry e_0 to a scalar multiple of h.

Note that Theorems 4.1 and 4.2 together imply that the only simple c.n. Lie algebras are $S(F)$, $C(F)$ and $W(F)$.

Proof of Theorem 4.1. Since L is nonsolvable, there exists $h \in L$ with ad h not nilpotent. By Theorem 2.5, $F \cdot h$ is a Cartan subalgebra and so $L_0(h) = F \cdot h$ where $L_\lambda(h)$ denotes the generalized eigenspace of ad h for the eigenvalue λ.

Now $[L\, L] = L$ since otherwise $[L\, L] \subseteq \mathcal{N}(L)$ by Corollary 2.6 and this would imply that L is solvable. Since $[L_\lambda(h), L_\mu(h)] \subseteq L_{\lambda+\mu}(h)$ and L is the direct sum of the various $L_\lambda(h)$, it follows from the fact that $h \in [L\, L]$ that $h \in [L_\lambda(h), L_{-\lambda}(h)]$ for some λ. Since $L_0(h) = F \cdot h$, we must have $\lambda \neq 0$ and we may replace h by $\lambda^{-1}h$ and assume $\lambda = 1$.

Now let $F_0 \subseteq F$ be the prime subfield and let

$$K = \sum_{\mu \in F_0} L_\mu(h).$$

Then $K \subseteq L$ is a subalgebra and if $K = L$, there is nothing further to prove. Suppose then that $K < L$ and note that $h \in [K\, K]$. Since $F \cdot h$ is a Cartan subalgebra of K, it follows that $[K\, K] = K$ and K is nonsolvable.

Let $N = \mathcal{N}(K)$ and $T = K/N$. By Corollary 2.6, T is a simple algebra and T satisfies the hypotheses of Theorem 4.2 with respect to the image of h in $K/N = T$. By Theorem 4.2, T is isomorphic to one of $S(F)$, $C(F)$ or $W(F)$ and in any case, T has a subalgebra $S \cong S(F)$. Write $S = M/N$.

Let e_{-1}, e_0, e_1 be the standard basis for S and let $y \in M$ be a preimage for e_0. Then ad y is not nilpotent since ad e_0 is not nilpotent on S, and thus $F \cdot y$ is a Cartan subalgebra of L. We claim that all eigenvalues of ad y lie in the prime field F_0.

Construct an M-composition series for L through M as in Lemma 3.2 and let V be a composition factor other than M/N. By 3.2, $N \cdot V = 0$ and we may view V as an S-module, and as such it is special. If $\operatorname{char}(F) = 2$, there is no such module by Lemma 3.5 and it follows that $M = L$ and $N = 0$. This is a contradiction since we are assuming $M \subseteq K < L$.

Assume then that $\operatorname{char}(F) \neq 2$. Then Corollary 3.4 applies and e_0 induces a transformation on V for which all eigenvalues lie in F_0. This transformation of V is the same as that induced by ad y, and thus all eigenvalues of ad y induced on composition factors other than M/N lie in F_0. Since the eigenvalues of ad y on $M/N = S$ are $-1, 0, 1$, the proof is complete.

COROLLARY 4.3. *Let L be a nonsolvable* c.n. *algebra over the algebraically closed field F. Then $L/\mathcal{N}(L)$ is isomorphic to one of $S(F)$, $C(F)$ or $W(F)$.*

Proof. By Corollaries 2.6 and 2.7, $L/\mathcal{N}(L)$ is a simple c.n. algebra. The result follows from Theorems 4.1 and 4.2.

5. Proof of theorem B.

LEMMA 5.1. *Let D be a nonsingular derivation of a finite dimensional Lie algebra L over an algebraically closed field F. If* $\operatorname{char}(F) > 0$, *assume in addition that all eigenvalues of D lie in the prime subfield of F. Then L is nilpotent.*

Note. This result is due to Jacobson. (See, for instance, Problem 9 on page 54 of [**3**] for the case $\mathrm{char}(F) = 0$.) The same ideas work when $\mathrm{char}(F) > 0$ as shown below.

Proof of Lemma 5.1. Let $L_\lambda(D)$ denote the generalized eigenspace of D on L with respect to the eigenvalue λ and let

$$W = \cup_\lambda L_\lambda(D).$$

Then $\{\mathrm{ad}\, w | \, w \in W\}$ is a weakly closed system of linear transformations of L (as defined on page 31 of [**3**]). Furthermore, the hypothesis on eigenvalues implies that each $\mathrm{ad}\, w$ is nilpotent. Since W spans L, it follows from the theorem on page 33 of [**3**] that L is nilpotent.

The following theorem is slightly more general than is needed here. The case $U = V$ of this result appears in [**7**].

THEOREM 5.2. *Let F be algebraically closed with* $\mathrm{char}(F) \neq 2$ *and let U, V be simple $S(F)$-modules of dimension m and n respectively with $n \equiv m$* mod 2. *If* $\mathrm{char}(F) > 0$, *assume also that $m, n <$* $\mathrm{char}(F)$. *Then $U \otimes V$ is generated as an $S(F)$-module by the eigenspace corresponding to the eigenvalue* 0 *of the action of e_0 on $U \otimes V$.*

Proof. By Lemma 3.3, the eigenvalues of e_0 on each of U and V are distinct and the hypothesis that $m, n < \mathrm{char}(F)$ if $\mathrm{char}(F) \neq 0$ allows us to conclude that the sets of eigenvalues are $\Lambda(m)$ and $\Lambda(n)$ respectively (in the notation introduced following Lemma 3.3). Assume $m \geqq n$. Since $m \equiv n \bmod 2$, it follows that $\Lambda(m) \supseteq \Lambda(n)$.

Decompose $U = \sum U_\lambda$, where U_λ is the eigenspace of e_0 corresponding to the eigenvalue $\lambda \in \Lambda(m)$ and similarly decompose $V = \sum V_\mu$ for $\mu \in \Lambda(n)$. Let $X \subseteq U \otimes V$ be the $S(F)$-submodule generated by all $U_\lambda \otimes V_\mu$ with $\lambda + \mu = 0$. To complete the proof, we show that $U_\lambda \otimes V_\mu \subseteq X$ for all $\lambda \in \Lambda(m)$ and $\mu \in \Lambda(n)$.

It is convenient to identify $\Lambda(m) \subseteq F$ with a subset of the rational numbers $\mathbf{Q}$, even when $\mathrm{char}(F) > 0$. We put

$$\Lambda(m) = \{-(m-1)/2 + i | \, 0 \leqq i \leqq m-1\} \subseteq \mathbf{Q}.$$

We can now show that $U_\lambda \otimes V_\mu \subseteq X$ by considering the three cases: $\lambda + \mu = 0$, $\lambda + \mu > 0$ and $\lambda + \mu < 0$. The first case is, of course, trivial by the definition of X.

Suppose $U_\lambda \otimes V_\mu \not\subseteq X$ with $\lambda + \mu > 0$. Choose λ and μ such that λ is as small as possible. Now $\lambda > -\mu \geqq -(n-1)/2 \geqq -(m-1)/2$ and so $\lambda - 1 \in \Lambda(m)$ and $e_1 \cdot U_{\lambda-1} = U_\lambda$ by Lemma 3.3(b). Let $u \in U_\lambda$ and $v \in V_\mu$ and write $u = e_1 \cdot y$ with $y \in U_{\lambda-1}$. Now

$$(*) \quad u \otimes v = (e_1 \cdot y) \otimes v = e_1 \cdot (y \otimes v) - y \otimes (e_1 \cdot v).$$

If $(\lambda - 1) + \mu = 0$, then $y \otimes v \in X$. Otherwise, $(\lambda - 1) + \mu > 0$ since

$\lambda + \mu \geqq 1$, and thus by the minimality of λ we have $y \otimes v \in X$. Thus in any case, $e_1 \cdot (y \otimes v) \in X$.

Since e_1 acts nilpotently on V by Lemma 3.3(c), it follows that $e_1 \cdot V_\mu = 0$ if $\mu = (n-1)/2$. Otherwise $e_1 \cdot v \in V_{\mu+1}$, and since $(\lambda - 1) + (\mu + 1) > 0$, the minimality of λ guarantees that $y \otimes (e_1 \cdot v) \in X$. Thus (*) yields that $u \otimes v \in X$. We have now proved that $U_\lambda \otimes V_\mu \subseteq X$ when $\lambda + \mu > 0$.

To show that $U_\lambda \otimes V_\mu \subseteq X$ when $\lambda + \mu < 0$, we argue similarly, choosing λ maximal such that the assertion is false and using e_{-1} in place of e_1.

COROLLARY 5.3. *Let U, V, W be $S(F)$-modules where F is algebraically closed and* $\operatorname{char}(F) \neq 2$. *Let Θ: $U \otimes V \rightarrow W$ be an $S(F)$-module homomorphism. Assume*

i) *U, V are simple*

ii) $\dim U \neq \operatorname{char}(F)$ *and* $\dim V \neq \operatorname{char}(F)$

iii) *e_0 acts nonsingularly on U, V, W.*

Then Θ is identically zero.

Proof. If $m = \dim(U)$ and $n = \dim(V)$, then by (ii) and Lemma 3.3(c), the sets of eigenvalues of e_0 on U and V are $\Lambda(m)$ and $\Lambda(n)$ respectively. By (iii), $0 \notin \Lambda(m)$ and $0 \notin \Lambda(n)$ and thus m and n are both even.

Since e_0 is nonsingular on W, we see that the zero eigenspace of e_0 on $U \otimes V$ is contained in $\ker \Theta$. The result now follows from Theorem 5.2.

Proof of Theorem B. If $\operatorname{char}(F) = 2$, then by the remark following Theorem 4.2 (which is essentially Lemma 74 of [**4**]), $L \cong S(F)$ and $\mathcal{N}(L) = 0$. We may assume then, that $\operatorname{char}(F) \neq 2$.

Let I be a maximal proper ideal of L and let $F \cdot h$ be a Cartan subalgebra of L such that all eigenvalues of ad h lie in the prime subfield of F if $\operatorname{char}(F) \neq 0$. Then $h \notin I$ and h induces a nonsingular derivation on I. By Lemma 5.1, then, I is nilpotent and so $I \subseteq \mathcal{N}(L)$.

Since L is nonsolvable, we have $I = \mathcal{N}(L)$ and L/I is simple. Also, L/I satisfies the hypotheses of Theorem 4.2 with respect to the image of h in L/I. Thus $L/\mathcal{N}(L) = L/I$ is isomorphic to $S(F)$, $C(F)$ or $W(F)$. What remains is to show that I is abelian.

If $\operatorname{char}(F) \geqq 3$, then by Theorem 4.2, L/I contains M/I, an isomorphic copy of $S(F)$. Also, replacing h by a scalar multiple if necessary, we may assume that the coset $h + I$ is the standard basis vector e_0 of $M/I \cong S(F)$. If $\operatorname{char}(F) = 0$, take $M = L$ so that in this case too, $M/I \cong S(F)$ and $h + I = e_0$. Write $M/I = S$.

Suppose I is not abelian so that the center $Z = \mathbf{Z}(I) < I$. Let U be minimal among ideals of M with $Z < U \subseteq I$ and let $C = \mathbf{C}_I(U)$, the centralizer, so that $C < I$ and C is an ideal of M. Let V be minimal among ideals of M with $C < V \subseteq I$. Then since $V \not\subseteq C$, we have $[U\ V] \neq 0$.

Since I is nilpotent and U/Z is a nonzero ideal of I/Z, we have $(U/Z) \cap \mathbf{Z}(I/Z) \neq 0$. However, U/Z is a minimal ideal of M/Z and we conclude that

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$U/Z \subseteq \mathbf{Z}(I/Z)$ and thus $[I\ U] \subseteq Z$. Similarly $[I\ V] \subseteq C$ and hence U/Z and V/C may be viewed as modules for $M/I = S$. Furthermore, by the minimality of U and V we see that $\bar{U} = U/Z$ and $\bar{V} = V/C$ are simple S-modules.

Now let $W = [U\ V] \subseteq I$ so that W is an ideal of M and $W \neq 0$. Since $[[I\ U]V] \subseteq [Z\ V] = 0$ and $[U[I\ V]] \subseteq [U\ C] = 0$, we have $[I\ W] = 0$ and W may also be viewed as an S-module. We now define

$$\Theta: \bar{U} \otimes \bar{V} \rightarrow W$$

by $\Theta(\bar{u} \otimes \bar{v}) = [uv]$ where $\bar{u} = u + Z$ and $\bar{v} = v + C$. This is a well-defined S-module homomorphism since $[Z\ V] = 0 = [C\ U]$.

We now claim that $\dim \bar{U} \neq \operatorname{char}(F)$ and $\dim \bar{V} \neq \operatorname{char}(F)$. We may assume $\operatorname{char}(F) = p > 0$. The transformation induced by e_0 on U is the same as that induced by ad h and thus has eigenvalues among the nonzero elements of the prime field of F (since ad h is nonsingular on I). By Lemma 3.3(a), the number of distinct eigenvalues is equal to $\dim \bar{U}$ and thus $\dim \bar{U} \leqq p - 1$ and similarly $\dim \bar{V} \leqq p - 1$.

Since e_0 acts nonsingularly on $\bar{U}$, $\bar{V}$ and W, we conclude from Corollary 5.3 that Θ is identically zero and thus $W = \Theta(U \otimes V) = 0$. This is a contradiction and completes the proof.

6. Construction of c.n. algebras.

We have already remarked that the simple algebras $S(F)$, $C(F)$ and $W(F)$ are c.n. algebras. In this section we show that for $\operatorname{char}(F) \neq 2$, there also exist nonsimple nonsolvable c.n. algebras. (By Lemmas 3.2 and 3.5, no such algebras can exist when $\operatorname{char}(F) = 2$.)

We begin with a partial converse to Lemma 3.2.

THEOREM 6.1. *Let L be a Lie algebra over an algebraically closed field and let $I \subseteq L$ be an abelian ideal which is special when viewed as a module for L/I. Assume that L/I is a* c.n. *algebra. Then L is a* c.n. *algebra.*

Proof. By Corollary 2.3, it suffices to show for each $x \in L$ that either $\mathbf{C}(x) = F \cdot x$ or ad x is nilpotent. Now let $u = x + I \in L/I$ which is a c.n. algebra and so either ad u is nilpotent or $\mathbf{C}_{L/I}(u) = F \cdot u$.

Suppose $(\operatorname{ad} u)^k(L/I) = 0$. Then $(\operatorname{ad} x)^k(L) \subseteq I$. However, since I is a special (L/I)-module, u acts nilpotently on I and thus $(\operatorname{ad} x)^m(I) = 0$ for some m. We conclude that $(\operatorname{ad} x)^{k+m} = 0$.

Now assume ad u is not nilpotent. Since I is a special (L/I)-module, u acts nonsingularly on I and thus $\mathbf{C}(x) \cap I = 0$. However, $\mathbf{C}_{L/I}(u) = F \cdot u = (I + F \cdot x)/I$ in this case, and so $\mathbf{C}(x) \subseteq I + F \cdot x$ and we conclude that $\mathbf{C}(x) = F \cdot x$.

It follows from Theorem 6.1 that given a c.n. algebra S and a special S-module V, we can construct a new c.n. algebra by taking $L = S \oplus V$, the "semi-direct product" with $[V\ V]$ defined to be zero.

If F is algebraically closed with $\mathrm{char}(F) \neq 2$, and $S = S(F)$, we claim that each of the simple S-modules having even dimension is special. This follows from the following five facts which we give without proof (although the first three can be deduced from the proof of Lemma 3.3).

a) If $1 \leqq k \neq \mathrm{char}(F)$, then there is a unique (up to isomorphism) simple $S(F)$-module V_k of dimension k.

b) e_1 is nilpotent on all V_k.

c) e_0 is nonsingular on V_k when k is even.

d) If $0 \neq x \in S(F)$ with $\mathrm{ad}\, x$ nilpotent, then $S(F)$ has an automorphism σ with $\sigma(x) = e_1$.

e) If $x \in S(F)$ and $\mathrm{ad}\, x$ is not nilpotent, then $S(F)$ has an automorphism σ with $\sigma(x) = e_0$.

If $\mathrm{char}(F) > 3$, we can also find special modules for $W(F)$. In fact, $W(F)$ has a unique simple module of dimension $p - 1$ (for instance see [**6**]) and using the uniqueness and facts about automorphisms of $W(F)$, one can show that this module is special.

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