THE JOURNAL OF SYMBOLIC LOGIC
Volume 11, Number 1, March 1946

EIGHTH MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

The meeting of the Association for Symbolic Logic scheduled for April 27 and 28, 1945, and previously announced in this JOURNAL was canceled in response to such a request from the Office of Defense Transportation.

A meeting of the Association was held on the morning of February 23, 1946, at Columbia University in New York City, in conjunction with the American Mathematical Society. The Program Committee consisted of Professors Carl G. Hempel, Saunders MacLane, and Ernest Nagel (chairman). Because a sufficient number of volunteered papers were not submitted in time, the meeting was devoted entirely to two invited addresses of one hour's length each, by Professor Karl Menger on *Analysis without variables*, and by Professor Alonzo Church on *A formulation of the logic of sense and denotation*. Professor C. A. Baylis presided at the meeting. Abstracts of the addresses follow immediately. ERNEST NAGEL

KARL MENGER. *Analysis without variables*.

If in a ring (with addition denoted by $+$, and multiplication denoted by $\cdot$) a third binary operation (a substitution denoted by juxtaposition) is defined and connected with the ring operations by the distributive laws $(f + g)h = fh + gh$, $(f\cdot g)h = fh\cdot gh$, then we speak about a *tri-operational algebra* ("T.O.A."). E.g., the classical functions of one variable with a common domain, each with a range contained in this domain, form a T.O.A. A general formula in T.O.A. is valid for *all* functions of one variable provided that all terms in the formula are meaningful. For $n > 1$, in absence of a binary substitution, functions of n variables do not form a T.O.A. However, the n-tuples of functions of n variables do. More generally, for every abstract set Σ, the mappings of Σ into a ring R form a ring R'. If with each element f of R and each element G of R' an element fG of R is associated in such a way that

$$(*)\qquad (fG)H = f(GH), \qquad (f + g)H = fH + gH, \qquad (f\cdot g)H = fH\cdot gH,$$

then R' is a T.O.A. If Σ consists of the numbers $1, 2, \ldots, n$, then we obtain the functions of n variables, and each mapping of Σ into R is usually expressed as an ordered n-tuple $(f_1, f_2, \ldots, f_n)$. A synthesis of the theories of functions of any set Σ of variables is contained in the following concept of an *Algebra of Analysis* ("A.o.A."): a ring and a T.O.A., called R and T, respectively, about which we assume that (1) with each f of R and each G of T an element fG of R is associated in such a way that the formulae (*) hold, (2) there is a tri-operational isomorphism between R and a subset of T associating with each f of R an element H_f of T in such a way that

$$H_{f+g} = H_f + H_g, \qquad H_{f\cdot g} = H_f\cdot H_g, \qquad H_{fG} = H_fG.$$

For functions of n variables, examples of such isomorphisms are the mappings of each f on the n-tuple $H_f = (f, f, \ldots, f)$, or on $(f, 0, \ldots, 0)$, or on $(0, f, 0, \ldots, 0)$, etc. In case that $T = R$ and $H_f = f$, the A.o.A. is a T.O.A. One can formulate intrinsic characterizations of those A.o.A. which correspond to algebras of functions of Σ variables. Each algebra reflects an analysis without variables.

Our theory, which has obvious connections with operator theory and with combinatory logic, has the same relation to the classical theory of functions as our Algebra of Geometry has to classical geometry. The classical theories start with a class of undefined entities (points in geometry; numbers in analysis) in terms of which the other entities and the operations are set-theoretically defined (lines and planes as certain sets of points, a new concept for each higher dimension; functions of one and two variables as certain sets of pairs and triples of numbers, a new concept for each additional variable). In our algebras we start out with one comprehensive undefined concept ("flats" in geometry; "functions" in analysis) and undefined operations (joining and intersecting in geometry; adding, multiplying, and substituting in analysis). In terms of these operations we define specific entities (a point is that which has no parts; a function c is constant, and thus plays the role of a number, if $c0 = c$). On the basis of algebraic assumptions about the operations and dispensing with a great deal of set theory, we develop algebras synthesizing geometries of different dimensions and theories of functions of different numbers of variables. Cf. the author's paper, *General algebra of analysis*, ***Reports of a mathematical colloquium***, ser. 2 no. 7 (1946), pp. 46–60.

ALONZO CHURCH. *A formulation of the logic of sense and denotation.*

The distinction of *Sinn und Bedeutung* (see abstract of a paper by the author in this JOURNAL, vol. 7, p. 47) is incorporated into a logistic system. Types: ι_0, names of individuals; ι_{i+1}, names of senses of names of type ι_i; o_0, names of truth-values; o_{i+1}, names senses of names of type o_i; for any types α, β, a type $(\alpha\beta)$ of names of functions, such that if $\boldsymbol{F}_{(\alpha\beta)}$ and $\boldsymbol{A}_\beta$ are names, of types indicated by the subscripts, then $(\boldsymbol{F}_{(\alpha\beta)}\boldsymbol{A}_\beta)$ is a name of type α. Subscripts upon constants and variables of the system, and upon syntactical variables (bold letters) indicate the type. To represent variable or undetermined type symbols are used Greek letters, α, β, etc.; and subscript i is used upon such Greek letters to indicate the result of increasing all subscripts in the type symbol by i. As an abbreviation, parentheses are omitted under the convention of association to the left. Primitive symbols: constants f_{o_i}, $C_{o_io_io_i}$, $\Pi_{o_i(o_i\alpha_i)}$, $\iota_{\alpha_i(o_i\alpha_i)}$, $\Delta_{o_i\iota_{j+i+1}\iota_{j+i}}$, $\Delta_{o_io_{j+i+1}o_{j+i}}$; an infinite list of variables of each type; the abstraction operator λ; parentheses. Definitions: $[\boldsymbol{A}_{o_0} \supset \boldsymbol{B}_{o_0}] \rightarrow C_{o_0o_0o_0}\boldsymbol{B}_{o_0}\boldsymbol{A}_{o_0}$, $[(\boldsymbol{x}_{\alpha_i})\boldsymbol{A}_{o_i}] \rightarrow \Pi_{o_i(o_i\alpha_i)}(\lambda \boldsymbol{x}_{\alpha_i}\boldsymbol{A}_{o_i})$, $Q_{o_i\alpha\alpha} \rightarrow \lambda x_\alpha \lambda y_\alpha (f_{o_i\alpha}) . f_{o_i\alpha}y_\alpha \supset f_{o_i\alpha}x_\alpha$, $[\boldsymbol{A}_\alpha = \boldsymbol{B}_\alpha] \rightarrow Q_{o_0\alpha\alpha}$ $\boldsymbol{B}_\alpha\boldsymbol{A}_\alpha$, $\Delta_{o_i(\alpha_{i+1}\beta_{i+1})(\alpha_i\beta_i)} \rightarrow \lambda f_{\alpha_i\beta_i}\lambda f_{\alpha_{i+1}\beta_{i+1}}(x_{\beta_i})(x_{\beta_{i+1}}) . \Delta_{o_i\beta_{i+1}\beta_i}x_{\beta_i}x_{\beta_{i+1}} \supset \Delta_{o_i\alpha_{i+1}\alpha_i}$ $(f_{\alpha_i\beta_i}x_{\beta_i})(f_{\alpha_{i+1}\beta_{i+1}}x_{\beta_{i+1}})$. Conventions for omission of brackets and punctuation by dots are the same as in the author's *Formulation of the simple theory of types* (this JOURNAL, vol. 5, pp. 56–68). Notation is such that a (constant) name becomes a name of its sense by increasing subscripts in all type symbols by 1; and $\Delta_{o_0\alpha_1\alpha}$ is interpreted as the relation between the sense and the denotation of a name of type α. Rules of inference and axioms are based on those of *Formulation of the simple theory of types*, but with modifications such that no asserted formula contains free variables, with the Wajsberg-Quine axioms for the propositional calculus (this JOURNAL, vol. 3, p. 37), and with the added axiom $(p)(q) . p \supset q \supset . q \supset p \supset . p = q$ (where the omitted subscripts are o_0). To these are added the axioms $\Delta_{o_0o_{i+1}o_i}\mathrm{f}_{o_i}\mathrm{f}_{o_{i+1}}$, $\Delta_{o_0(o_{i+1}o_{i+1}o_{i+1})(o_io_io_i)}C_{o_io_io_i}C_{o_{i+1}o_{i+1}o_{i+1}}$, etc. Then further axioms about the notion of sense are adjoined according to either of two alternative heuristic principles: (1) two names are assumed to have different senses in all cases where it is not already a consequence that the senses are the same; (2) $\boldsymbol{A}_\alpha$ and $\boldsymbol{B}_\alpha$ are assumed to have the same sense if and only if $\boldsymbol{A}_\alpha = \boldsymbol{B}_\alpha$ is logically valid. Alternative (1) seems to be that intended by Frege. (2) leads to notions of necessity and strict implication akin to those of Lewis.

Added April 29, 1946. From the foregoing it follows that, if $\boldsymbol{A}_\alpha$ conv $\boldsymbol{B}_\alpha$ (i.e., if $\boldsymbol{A}_\alpha$ can be changed to $\boldsymbol{B}_\alpha$ by a chain of applications of Rules I–III), and if there are no free variables, then $\boldsymbol{A}_\alpha$ and $\boldsymbol{B}_\alpha$ must have the same sense. This accords well with the tendency of Alternative (2) just mentioned, but not with that of (1). Therefore, (only) for the case that the direction of Alternative (1) is followed, the author now wishes to make the following amendments.

The primitive symbol λ is replaced by an infinite list of primitive symbols λ_i, of which λ_0 is abbreviated as λ and interpreted as an abstraction operator. If $\boldsymbol{M}_{\alpha_i}$ is well-formed (is a constant or variable name) and is of type α_i, if x_{β_i} is a variable of type β_i, then $(\lambda_i x_{\beta_i}\boldsymbol{M}_{\alpha_i})$ is well-formed of type $\alpha_i\beta_i$, and in it x_{β_i} is a bound variable. Notation is such that a (constant) name becomes a name of its sense by increasing subscripts in every type symbol and after every λ by 1. In the definitions given above λ is replaced throughout by λ_i, and in the third definition α is replaced by α_i. In Rules II, III, λ is restricted to be λ_0. Then (1) involves the following principle, capable of expression as an axiom of the system: If the (constant) names $\boldsymbol{F}_{\alpha\beta}\boldsymbol{A}_\beta$ and $\boldsymbol{G}_{\alpha\beta}\boldsymbol{B}_\beta$ have the same sense, then $\boldsymbol{F}_{\alpha\beta}$ and $\boldsymbol{G}_{\alpha\beta}$ have the same sense, and $\boldsymbol{A}_\beta$ and $\boldsymbol{B}_\beta$ have the same sense.

By the argument of the Richard paradox, not every namable infinite sequence of natural numbers can have a name in the logistic system. *A fortiori* it is false that for every sense there is a name in the system having that sense. Nor can this situation be remedied by any extension of the system. At appropriate points the preceding discussion must therefore be applied to names (and their senses) which might be added to the system, as well as to names in it.

https://doi.org/10.2307/2269172 Published online by Cambridge University Press