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WARPED PRODUCTS IN RIEMANNIAN MANIFOLDS

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Abstract

In this paper we prove two inequalities relating the warping function to various curvature terms, for warped products isometrically immersed in Riemannian manifolds. This extends work by B. Y. Chen ['On isometric minimal immersions from warped products into real space forms', *Proc. Edinb. Math. Soc.* (2) **45**(3) (2002), 579–587 and 'Warped products in real space forms', *Rocky Mountain J. Math.* **34**(2) (2004), 551–563] for the case of immersions into space forms. Finally, we give an application where the target manifold is the Clifford torus.

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1. Introduction

Let (B, g_B) and (F, g_F) be Riemannian manifolds, where g_B and g_F are Riemannian metrics on manifolds B and F, respectively. Let f be a positive differentiable function on B. Consider the product manifold $B \times F$ with the natural projections $\pi_1 : B \times F \mapsto B$ and $\pi_2 : B \times F \mapsto F$. The *warped product manifold* $M = B \times_f F$ is the product manifold $B \times F$ equipped with the Riemannian metric g such that

$$\|X\|^2 = \|d\pi_1 X\|^2 + f^2(\pi_1(x))\|d\pi_2 X\|^2$$

for any tangent vector $X \in T_x M$, $x \in M$. Thus, we get $g = g_B + f^2 g_F$. The function f is called the *warping function* of the warped product manifold M [4].

Warped product manifolds play important roles in differential geometry and in physics, particularly in general relativity, and there are many papers on this topic (see, for example, [4] and the references therein). According to the result of Nash [9] which says that every Riemannian manifold can be isometrically embedded in some Euclidean space, every warped product can be isometrically embedded in some Euclidean space. The main results of this paper, Theorems 3.1 and 3.4, generalise Chen's results [2, 3] by giving upper and lower bounds for $\triangle f/f$.

The paper is organised as follows. In Section 2 we recall some notions which are needed in later sections. In Section 3 we obtain the main results from which we can see both the upper bound and the lower bound of the function $\triangle f/f$. In Section 4 we give some applications.

2. Preliminaries

Let (M, g) be an m-dimensional Riemannian manifold and N an n-dimensional submanifold of (M, g). Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of N and M, respectively. The *Gauss* and *Weingarten formulas* are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\overline{\nabla}_X Z = -A_Z X + D_X Z,$$

respectively, for tangent vector fields $X, Y \in \Gamma(TN)$ and a normal vector field $Z \in \Gamma(TN^{\perp})$, where h denotes the *second fundamental form*, D the *normal connection*, and A the *shape operator* of N in M.

The second fundamental form and the shape operator are related by

$$\langle A_Z X, Y\rangle = \langle h(X, Y), Z\rangle,$$

where $\langle\ ,\ \rangle$ denotes the induced metric on N as well as the Riemannian metric g on M. Choose a local orthonormal frame $\{e_1, \dots, e_m\}$ of TM such that $e_1, \dots, e_n$ are tangent to N and $e_{n+1}, \dots, e_m$ are normal to N. Then the *mean curvature vector* $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = \frac{1}{n}\text{trace } h = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$$

and the *squared mean curvature* is given by $H^2 := \langle\overrightarrow{H}, \overrightarrow{H}\rangle$.

The *squared norm of the second fundamental form h* is given by

$$\|h\|^2 = \sum_{i,j=1}^{n} \langle h(e_i, e_j), h(e_i, e_j)\rangle.$$

Let $h^r_{ij} := \langle h(e_i, e_j), e_r\rangle$ for $1 \le i, j \le n$ and $n + 1 \le r \le m$.

A submanifold N is said to be *totally geodesic* in M if the second fundamental form of N in M vanishes identically. Denote by $K(\pi)$ and $\overline{K}(\pi)$ the sectional curvatures of a plane $\pi \subset T_pN$, $p \in N$, in N and in M, respectively. That is, if the plane π is spanned by vectors $X, Y \in T_pN$, then

$$K(\pi) = \frac{\langle R(X, Y)Y, X\rangle}{\langle X, X\rangle \cdot \langle Y, Y\rangle - \langle X, Y\rangle^2} \quad \text{and} \quad \overline{K}(\pi) = \frac{\langle \overline{R}(X, Y)Y, X\rangle}{\langle X, X\rangle \cdot \langle Y, Y\rangle - \langle X, Y\rangle^2},$$

where R and $\overline{R}$ are the Riemann curvature tensors of N and M, respectively.

The *scalar curvature* τ of N is defined by

$$\tau = \sum_{1\le i<j\le n} K(e_i \wedge e_j),$$

where $K(e_i \wedge e_j) = \langle R(e_i, e_j)e_j, e_i\rangle$ for $1 \le i, j \le n$.

Let

$$(\inf \overline{K})(p) := \inf\{\overline{K}(\pi) \mid \pi \subset T_pN, \dim \pi = 2\}$$

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and

$$(\sup \overline{K})(p) := \sup\{\overline{K}(\pi) \mid \pi \subset T_pN, \dim \pi = 2\}$$

for $p \in N$.

The *Gauss equation* is given by

$$R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle \tag{2.1}$$

for tangent vectors $X, Y, Z, W \in T_pN$, $p \in N$, where $R(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle$ and $\overline{R}(X, Y, Z, W) = \langle \overline{R}(X, Y)Z, W\rangle$.

Then we easily obtain

$$\sum_{i,j=1}^{n} \overline{K}(e_i \wedge e_j) = 2\tau + \|h\|^2 - n^2H^2.$$

The *Laplacian* of a differentiable function f on N is defined by

$$\triangle f = \sum_{i=1}^{n}((\nabla_{e_i} e_i)f - e_i^2 f).$$

3. Some inequalities

Let (B, g_B) be an n_1-dimensional Riemannian manifold and (F, g_F) an n_2-dimensional Riemannian manifold, with $n = n_1 + n_2$. Let (M, g) be a warped product manifold of (B, g_B) and (F, g_F) such that $M = B \times_f F$ and $g = g_B + f^2 g_F$ with the projections $\pi_1 : B \times F \mapsto B$ and $\pi_2 : B \times F \mapsto F$, and $(\overline{M}, \overline{g})$ an m-dimensional Riemannian manifold. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ denote the distributions in M obtained from the vectors tangent to the horizontal lifts of B and F, respectively.

Let $\phi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion. We choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ of the tangent bundle TM of M such that $e_1, \ldots, e_{n_1} \in \Gamma(\mathcal{D}_1)$ and $e_{n_1+1}, \ldots, e_n \in \Gamma(\mathcal{D}_2)$. For convenience, we identify $d\phi(e_i)$ with e_i for $1 \le i \le n$. We also choose a local orthonormal frame $\{e_{n+1}, \ldots, e_m\}$ of the normal bundle $TM^\perp$ of M in $\overline{M}$ via ϕ such that e_{n+1} is in the direction of the mean curvature vector field.

Then we have

$$\triangle f = \sum_{i=1}^{n_1}((\nabla_{e_i} e_i)f - e_i^2 f). \tag{3.1}$$

Denote by $\operatorname{tr} h_1$ and $\operatorname{tr} h_2$ the trace of h restricted to B and F, respectively. that is,

$$\operatorname{tr} h_1 = \sum_{i=1}^{n_1} h(e_i, e_i) \quad \text{and} \quad \operatorname{tr} h_2 = \sum_{j=n_1+1}^{n} h(e_j, e_j).$$

Given unit vector fields $X, Y \in \Gamma(TM)$ such that $X \in \Gamma(\mathcal{D}_1)$ and $Y \in \Gamma(\mathcal{D}_2)$, we easily obtain

$$\nabla_X Y = \nabla_Y X = (X \ln f)Y,$$

where ∇ is the Levi-Civita connection of (M, g), so that

$$\begin{aligned} K(X \wedge Y) &= \langle \nabla_Y \nabla_X X - \nabla_X \nabla_Y X, Y \rangle \\ &= \frac{1}{f}((\nabla_X X) f - X^2 f). \end{aligned}$$

Hence,

$$\frac{\triangle f}{f} = \sum_{i=1}^{n_1} K(e_i \wedge e_j)$$

for each $j = n_1 + 1, \dots, n$.

The map ϕ is called *mixed totally geodesic* if $h(e_i, e_j) = 0$ for $1 \le i \le n_1$ and $n_1 + 1 \le j \le n$.

Then we get the following theorem.

THEOREM 3.1. *Let $(M = B \times_f F, g)$ be a warped product manifold of Riemannian manifolds (B, g_B) and (F, g_F) with the warping function f, and $(\overline{M}, \overline{g})$ a Riemannian manifold. Let $\phi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion. Then we obtain*

$$\frac{\triangle f}{f} \le \frac{n^2}{4n_2} H^2 + n_1 \sup \overline{K}, \tag{3.2}$$

where $n_1 = \dim B$ and $n_2 = \dim F$, with $n = n_1 + n_2$. The equality case of (3.2) holds identically if and only if ϕ is a mixed totally geodesic immersion such that $\operatorname{tr} h_1 = \operatorname{tr} h_2$ and $\overline{K}(X \wedge Y) = \sup \overline{K}$ for unit vectors $X \in \Gamma(\mathcal{D}_1)$ and $Y \in \Gamma(\mathcal{D}_2)$.

PROOF. Given a local orthonormal frame $\{e_1, \dots, e_n\}$ of TM such that $e_1, \dots, e_{n_1} \in \Gamma(\mathcal{D}_1)$ and $e_{n_1+1}, \dots, e_n \in \Gamma(\mathcal{D}_2)$, we have

$$\frac{\triangle f}{f} = \sum_{i=1}^{n_1} K(e_i \wedge e_j) \tag{3.3}$$

for each $j = n_1 + 1, \dots, n$.

By the Gauss equation, we get

$$2\tau = n^2 H^2 - \|h\|^2 + \sum_{i,j=1}^{n} \overline{K}(e_i \wedge e_j),$$

where $\overline{K}(e_i \wedge e_j) = \overline{g}(\overline{R}(e_i, e_j) e_j, e_i)$ and $\overline{R}$ is the Riemann curvature tensor of $\overline{M}$.

Let

$$\delta := 2\tau - \sum_{i,j=1}^{n} \overline{K}(e_i \wedge e_j) - \frac{n^2}{2} H^2. \tag{3.4}$$

Then we obtain

$$n^2 H^2 = 2\delta + 2\|h\|^2. \tag{3.5}$$

Given a local orthonormal frame $\{e_{n+1}, \ldots, e_m\}$ of the normal bundle such that e_{n+1} is in the direction of the mean curvature vector field, from (3.5) we have

$$\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 = 2\Big(\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\Big). \tag{3.6}$$

Let $a_1 := \sum_{i=1}^{n_1} h_{ii}^{n+1}$ and $a_2 := \sum_{i=n_1+1}^{n} h_{ii}^{n+1}$.

Using the relation

$$a_1^2 + a_2^2 \geq \tfrac{1}{2}(a_1 + a_2)^2,$$

by (3.6) we obtain

$$\sum_{1\leq j<k\leq n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n_1+1\leq s<t\leq n} h_{ss}^{n+1}h_{tt}^{n+1} \geq \frac{1}{2}\delta + \sum_{1\leq i<j\leq n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2. \tag{3.7}$$

From (3.3) and the Gauss equation (2.1), we get

$$\begin{aligned}
\frac{n_2\triangle f}{f} &= \tau - \sum_{1\leq i<j\leq n_1} K(e_i\wedge e_j) - \sum_{n_1+1\leq s<t\leq n} K(e_s\wedge e_t)\\
&= \tau - \sum_{1\leq i<j\leq n_1} \overline{K}(e_i\wedge e_j) - \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n_1}(h_{ii}^r h_{jj}^r - (h_{ij}^r)^2)\\
&\quad - \sum_{n_1+1\leq s<t\leq n} \overline{K}(e_s\wedge e_t) - \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n}(h_{ss}^r h_{tt}^r - (h_{st}^r)^2)\\
&= \tau - \frac{1}{2}\sum_{i,j=1}^{n}\overline{K}(e_i\wedge e_j) + \sum_{\substack{1\leq i\leq n_1\\ n_1+1\leq s\leq n}}\overline{K}(e_i\wedge e_s)\\
&\quad - \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n_1}(h_{ii}^r h_{jj}^r - (h_{ij}^r)^2)\\
&\quad - \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n}(h_{ss}^r h_{tt}^r - (h_{st}^r)^2).
\end{aligned} \tag{3.8}$$

By (3.4), (3.7), and (3.8), we obtain

$$\begin{aligned}
\frac{n_2\triangle f}{f} &\leq \tau - \frac{1}{2}\sum_{i,j=1}^{n}\overline{K}(e_i\wedge e_j) + \sum_{\substack{1\leq i\leq n_1\\ n_1+1\leq s\leq n}}\overline{K}(e_i\wedge e_s)\\
&\qquad -\frac{1}{2}\delta - \sum_{\substack{1\leq i\leq n_1\\ n_1+1\leq s\leq n}}(h_{is}^{n+1})^2 - \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2
\end{aligned}$$

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$$
\begin{aligned}
&- \sum_{r=n+2}^{m} \sum_{1 \le i < j \le n_1} (h^r_{ii} h^r_{jj} - (h^r_{ij})^2) \\
&- \sum_{r=n+2}^{m} \sum_{n_1+1 \le s < t \le n} (h^r_{ss} h^r_{tt} - (h^r_{st})^2) \\
\le{}& \tau - \frac{1}{2} \sum_{i,j=1}^{n} \overline{K}(e_i \wedge e_j) + n_1 n_2 \sup \overline{K} - \frac{1}{2}\delta \\
&- \sum_{r=n+1}^{m} \sum_{\substack{1 \le i \le n_1 \\ n_1+1 \le s \le n}} (h^r_{is})^2 - \frac{1}{2} \sum_{r=n+2}^{m} \Big(\sum_{1 \le i \le n_1} h^r_{ii} \Big)^2 \\
&- \frac{1}{2} \sum_{r=n+2}^{m} \Big(\sum_{n_1+1 \le j \le n} h^r_{jj} \Big)^2 \\
\le{}& \tau - \frac{1}{2} \sum_{i,j=1}^{n} \overline{K}(e_i \wedge e_j) + n_1 n_2 \sup \overline{K} - \frac{1}{2}\delta \\
={}& \frac{n^2}{4} H^2 + n_1 n_2 \sup \overline{K}. \qquad (3.9)
\end{aligned}
$$

Therefore, we have

$$
\frac{\triangle f}{f} \le \frac{n^2}{4n_2} H^2 + n_1 \sup \overline{K}.
$$

Similarly, using [2, Theorem 1.4], from (3.7) and (3.9), we see that the equality sign in (3.2) holds if and only if the immersion ϕ is mixed totally geodesic such that $\operatorname{tr} h_1 = \operatorname{tr} h_2$ and $\overline{K}(X \wedge Y) = \sup \overline{K}$ for unit vectors $X \in \Gamma(\mathcal{D}_1)$ and $Y \in \Gamma(\mathcal{D}_2)$. Notice that we can choose e_1 and e_{n_1+1} such that the plane spanned by e_1 and e_{n_1+1} is equal to the plane spanned by X and Y so that $\overline{K}(e_1 \wedge e_{n_1+1}) = \overline{K}(X \wedge Y)$. □

REMARK 3.2. (1) If $(\overline{M}, \overline{g})$ is a Riemannian manifold of constant sectional curvature c, then (3.2) becomes (1.2) of [2, Theorem 1.4].

(2) Let $\overline{M}_{m_1,m_2} := S^{m_1}(\sqrt{m_1/m}) \times S^{m_2}(\sqrt{m_2/m}) \subset S^{m+1}(1)$ be the Clifford torus, where $m = m_1 + m_2$, $m \ge 4$, and $2 \le m_1 \le m - 2$ [6, 8]. As we know, $\overline{M}_{m_1,m_2}$ is a compact minimal hypersurface in $S^{m+1}(1)$ and has only two distinct principal curvatures $\sqrt{m_2/m_1}$, $-\sqrt{m_1/m_2}$ with multiplicities m_1, m_2, respectively. The squared norm of the second fundamental form of $\overline{M}_{m_1,m_2}$ in $S^{m+1}(1)$ is equal to m so that by using the Gauss equation, the scalar curvature of $\overline{M}_{m_1,m_2}$ is equal to $m(m-2)/2$ [7, 10].

Moreover, if we take two unit vectors $e_1, e_2 \in T_p\overline{M}_{m_1,m_2}$, $p \in \overline{M}_{m_1,m_2}$ such that $A_p e_1 = \sqrt{m_2/m_1}\, e_1$ and $A_p e_2 = -\sqrt{m_1/m_2}\, e_2$, then, given $x, y \in \mathbb{R}$ with $x^2 + y^2 = 1$, we have

$$
\operatorname{Ric}(xe_1 + ye_2) = m - 1 - \Big(\frac{m_2}{m_1} x^2 + \frac{m_1}{m_2} y^2 + \frac{m^2}{m_1 m_2} x^2 y^2 \Big),
$$

where Ric denotes the Ricci curvature of $\overline{M}_{m_1,m_2}$. With a simple computation, we easily obtain

$$\mathrm{Ric} \geq m - 1 - \Big(\frac{m_2}{m_1} + \frac{m_1}{m_2}\Big)$$

with equality holding if and only if $(x, y) = (\pm\sqrt{m_2/m}, \pm\sqrt{m_1/m})$ [5, 11].

In particular, we know that $\overline{M}_{m_1,m_2}$ is not a constant curvature space but has only three types of sectional curvatures $\{0, m/m_1, m/m_2\}$ so that when we consider an isometric immersion $\phi : M \mapsto \overline{M}_{m_1,m_2}$, Theorem 3.1 will be useful [2].

To prove the next theorem, we need to introduce the following lemma, which we get from [1].

Lemma 3.3. *Let $a_1, \ldots, a_n, c$ be any real numbers with $n \geq 2$ such that*

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^{n} a_i^2 + c\Big).$$

Then

$$2a_1a_2 \geq c$$

with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Theorem 3.4. *Let $(M = B \times_f F, g)$ be a warped product manifold of Riemannian manifolds (B, g_B) and (F, g_F) with the warping function f, and $(\overline{M}, \overline{g})$ a Riemannian manifold. Let $\phi : (M, g) \mapsto (\overline{M}, \overline{g})$ be an isometric immersion. Then*

$$\frac{\triangle f}{f} \geq \frac{n_1 n^2}{2(n-1)} H^2 - \frac{n_1}{2}\|h\|^2 + n_1 \inf \overline{K}, \tag{3.10}$$

where $n_1 = \dim B$ and $n_2 = \dim F$, with $n = n_1 + n_2$.

Proof. We choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ of TM such that $e_1, \ldots, e_{n_1} \in \Gamma(\mathcal{D}_1)$ and $e_{n_1+1}, \ldots, e_n \in \Gamma(\mathcal{D}_2)$, and a local orthonormal frame $\{e_{n+1}, \ldots, e_m\}$ of the normal bundle such that e_{n+1} is in the direction of the mean curvature vector field.

Using the Gauss equation, we get

$$2\tau = n^2H^2 - \|h\|^2 + \sum_{i,j=1}^{n} \overline{K}(e_i \wedge e_j). \tag{3.11}$$

Let

$$\delta := 2\tau - \frac{n^2(n-2)}{n-1} H^2 - \sum_{i,j=1}^{n} \overline{K}(e_i \wedge e_j). \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$n^2H^2 = (n-1)\|h\|^2 + (n-1)\delta \tag{3.13}$$

so that

$$\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 = (n-1)\Big(\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 + \delta\Big). \tag{3.14}$$

Applying Lemma 3.3 to (3.14) with $a_1 = h_{11}^{n+1}$ and $a_2 = h_{n_1+1n_1+1}^{n+1}$, we have

$$2h_{11}^{n+1}h_{n_1+1n_1+1}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 + \delta$$

so that

$$\begin{aligned} K(e_1 \wedge e_{n_1+1}) &\geq \sum_{r=n+1}^{m}\sum_{j\in S_{1n_1+1}}\{(h_{1j}^{r})^2 + (h_{n_1+1j}^{r})^2\} \\ &\quad + \frac{1}{2}\sum_{\substack{i,j\in S_{1n_1+1}\\ i\neq j}}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j\in S_{1n_1+1}}(h_{ij}^{r})^2 \\ &\quad + \frac{1}{2}\sum_{r=n+2}^{m}(h_{11}^{r} + h_{n_1+1n_1+1}^{r})^2 + \frac{\delta}{2} + \inf \overline{K} \\ &\geq \frac{\delta}{2} + \inf \overline{K}, \end{aligned}$$

where $S_{1n_1+1} = \{1,\ldots,n\} - \{1, n_1+1\}$.

Similarly, we obtain

$$K(e_i \wedge e_{n_1+1}) \geq \frac{\delta}{2} + \inf \overline{K} \tag{3.15}$$

for $1 \leq i \leq n_1$.

Since $K(e_i \wedge e_{n_1+1}) = 1/f((\nabla_{e_i}e_i)f - e_i^2 f)$ for $1 \leq i \leq n_1$, by (3.1), (3.13) and (3.15), we get

$$\begin{aligned} \frac{\triangle f}{f} &\geq \frac{n_1}{2}\delta + n_1 \inf \overline{K} \\ &\geq \frac{n_1 n^2}{2(n-1)}H^2 - \frac{n_1}{2}\|h\|^2 + n_1 \inf \overline{K}. \end{aligned}$$

Therefore, we have the result. □

REMARK 3.5. (1) In a similar way, using [3, Theorem 1], we can also give a condition for equality to hold in (3.10). This is just the same as the conditions of [3, Theorem 1], except for the additional condition $\overline{K}(X \wedge Y) = \inf \overline{K}$ for $X \in \Gamma(\mathcal{D}_1)$ and $Y \in \Gamma(\mathcal{D}_2)$.

(2) From Theorems 3.1 and 3.4, we obtain both the upper bound and the lower bound of the function $\triangle f/f$:

$$\frac{n_1 n^2}{2(n-1)}H^2 - \frac{n_1}{2}\|h\|^2 + n_1 \inf \overline{K} \leq \frac{\triangle f}{f} \leq \frac{n^2}{4n_2}H^2 + n_1 \sup \overline{K}.$$

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4. Applications

Let $M(c_1, c_2) := M(c_1) \times M(c_2)$ be the product manifold of Riemannian manifolds $M(c_1)$ and $M(c_2)$, where $M(c_i)$ is a constant curvature space of constant sectional curvature c_i for $i = 1, 2$. Then we know that $M(c_1, c_2)$ has only three types of sectional curvatures $\{c_1, c_2, 0\}$.

Let $c := \min\{c_1, c_2, 0\}$ and $\bar{c} := \max\{c_1, c_2, 0\}$. Using Theorem 3.1, we easily get the following corollary.

COROLLARY 4.1. *Let $(B \times_f F, g)$ be a warped product manifold of Riemannian manifolds (B, g_B) and (F, g_F) with the warping function f such that $n_1 = \dim B$, $n_2 = \dim F$, and $n = n_1 + n_2$. Let ϕ be an isometric immersion from the warped product manifold $(B \times_f F, g)$ to the product manifold $M(c_1, c_2)$. Then*

$$\frac{\triangle f}{f} \leq \frac{n^2}{4n_2} H^2 + n_1 \bar{c}.$$

REMARK 4.2. Let $(B \times_f F, g)$ be a warped product manifold of Riemannian manifolds (B, g_B) and (F, g_F) with the warping function f such that $n_1 = \dim B$, $n_2 = \dim F$, and $n = n_1 + n_2$. Let ϕ be an isometric minimal immersion from the warped product manifold $(B \times_f F, g)$ to the product manifold $M(c_1, c_2)$. Then we obtain

$$\frac{\triangle f}{f} \leq n_1 \bar{c}.$$

By Remark 4.2, we get the following theorem.

THEOREM 4.3. *Let $(M = B \times_f F, g)$ be a warped product manifold of Riemannian manifolds (B, g_B) and (F, g_F) with the warping function f such that $n_1 = \dim B$, $n_2 = \dim F$, and $n = n_1 + n_2$, and let $M(c_1, c_2)$ be the product manifold of constant curvature spaces $M(c_1)$ and $M(c_2)$. Then there does not exist an isometric minimal immersion ϕ from the warped product manifold (M, g) to the product manifold $M(c_1, c_2)$ such that $(\triangle f / f)(\pi_1(p)) > n_1 \bar{c}$ for some $p \in M$.*

Using Theorem 3.4, we have the following corollary.

COROLLARY 4.4. *Let $(B \times_f F, g)$ be a warped product manifold of Riemannian manifolds (B, g_B) and (F, g_F) with the warping function f such that $n_1 = \dim B$, $n_2 = \dim F$, and $n = n_1 + n_2$. Let ϕ be an isometric immersion from the warped product manifold $(B \times_f F, g)$ to the product manifold $M(c_1, c_2)$. Then we obtain*

$$\frac{\triangle f}{f} \geq \frac{n_1 n^2}{2(n-1)} H^2 - \frac{n_1}{2} \|h\|^2 + n_1 c.$$

From Remark 3.2, we know that the Clifford torus $\overline{M}_{m_1, m_2}$ is a product manifold of spheres $S^{m_1}(\sqrt{m_1/m})$ and $S^{m_2}(\sqrt{m_2/m})$. That is, $\overline{M}_{m_1, m_2}$ is a product manifold of constant curvature spaces $S^{m_1}(\sqrt{m_1/m})$ and $S^{m_2}(\sqrt{m_2/m})$, where $S^{m_1}(\sqrt{m_1/m})$ and $S^{m_2}(\sqrt{m_2/m})$ are constant curvature spaces of constant sectional curvatures m/m_1 and m/m_2, respectively.

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Thus, by using Corollary 4.1, we immediately obtain the following result.

COROLLARY 4.5. *Let* $(B \times_f F, g)$ *be a warped product manifold of Riemannian manifolds* (B, g_B) *and* (F, g_F) *with the warping function* f *such that* $n_1 = \dim B$, $n_2 = \dim F$, *and* $n = n_1 + n_2$. *Let* ϕ *be an isometric immersion from the warped product manifold* $(B \times_f F, g)$ *to the Clifford torus* $\overline{M}_{m_1,m_2}$ *with* $2 \leq m_1 \leq m/2$, $m = m_1 + m_2$, *and* $m \geq 4$. *Then*

$$\frac{\triangle f}{f} \leq \frac{n^2}{4n_2} H^2 + \frac{n_1 m}{m_1}.$$

REMARK 4.6. Let $(B \times_f F, g)$ be a warped product manifold of Riemannian manifolds (B, g_B) and (F, g_F) with the warping function f such that $n_1 = \dim B$, $n_2 = \dim F$, and $n = n_1 + n_2$. Let ϕ be an isometric minimal immersion from the warped product manifold $(B \times_f F, g)$ to the Clifford torus $\overline{M}_{m_1,m_2}$ with $2 \leq m_1 \leq m/2$, $m = m_1 + m_2$, and $m \geq 4$. Then we obtain

$$\frac{\triangle f}{f} \leq \frac{n_1 m}{m_1}.$$

By Remark 4.6, we get the following theorem.

THEOREM 4.7. *Let* $(M = B \times_f F, g)$ *be a warped product manifold of Riemannian manifolds* (B, g_B) *and* (F, g_F) *with the warping function* f *such that* $n_1 = \dim B$, $n_2 = \dim F$, *and* $n = n_1 + n_2$, *and let* $\overline{M}_{m_1,m_2}$ *be the Clifford torus* $S^{m_1}(\sqrt{m_1/m}) \times S^{m_2}(\sqrt{m_2/m})$ *such that* $2 \leq m_1 \leq m/2$, $m = m_1 + m_2$, *and* $m \geq 4$. *Then there does not exist an isometric minimal immersion* ϕ *from the warped product manifold* (M, g) *to the Clifford torus* $\overline{M}_{m_1,m_2}$ *such that* $(\triangle f/f)(\pi_1(p)) > n_1 m/m_1$ *for some* $p \in M$.

Using Corollary 4.4, we have the following result.

COROLLARY 4.8. *Let* $(B \times_f F, g)$ *be a warped product manifold of Riemannian manifolds* (B, g_B) *and* (F, g_F) *with the warping function* f *such that* $n_1 = \dim B$, $n_2 = \dim F$, *and* $n = n_1 + n_2$. *Let* ϕ *be an isometric immersion from the warped product manifold* $(B \times_f F, g)$ *to the Clifford torus* $\overline{M}_{m_1,m_2}$ *with* $2 \leq m_1 \leq m/2$, $m = m_1 + m_2$, *and* $m \geq 4$. *Then we obtain*

$$\frac{\triangle f}{f} \geq \frac{n_1 n^2}{2(n-1)} H^2 - \frac{n_1}{2} \|h\|^2.$$

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