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PENTAGON-GENERATED TRIVALENT GRAPHS WITH GIRTH 5

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1. Fundamentals. The terminology of [**1**] will be assumed in what follows. Let $P_5(G)$ stand for the set of pentagons in the graph G. Call a graph *pentagon-generated* when it is the union of its contained pentagons. Let $P_{5,3}$ be the class of connected trivalent pentagon-generated graphs with girth 5. These graphs form a family including the Petersen graph and the graph of the dodecahedron. They are studied here and completely classified in terms of a decomposition which all but some specifically determined indecomposable graphs admit.

Assume henceforth that $H \in P_{5,3}$. Let $E_k(H)$ be the set of edges in exactly $k \geqq 0$ pentagons of H. Clearly $E_k(H) = \emptyset$ if $k \neq 1, 2, 3, 4$ and $|E_1(H) \cap E(P)| \leqq 2$, for all $P \in P_5(H)$. $P \in P_5(H)$ is *singular* when $|E_1(H) \cap E(P)| = 2$. Then, the link graph $I \subseteq P$ whose ends are incident with the two members of $E_1(H) \cap E(P)$ is called a *pivot*. We will also call the $A \in E(I)$ a *pivot edge* and any $x \in V(I)$ a *pivot vertex*. Each pivot I is contained in exactly two pentagons P, Q of H. These P, Q are singular, have I as pivot, and $P \cap Q = I$. Pivots are thus disjoint.

We say that $P, Q \in P_5(H)$ are *related* when $Q_0, Q_1, \ldots, Q_n \in P_5(H)$ exist, with $P = Q_0$, $Q = Q_n$, such that $Q_{i-1} \cap Q_i$ is neither null nor a pivot of H, for $i = 1, \ldots, n$. This is an equivalence relation on $P_5(H)$. H is *decomposable* if it has a singular pentagon and *indecomposable* otherwise. The *constituents* of H are the unions of the pentagons in its equivalence classes of related pentagons. By definition, constituents are non-separable and pentagon-generated.

Suppose that G and G_1 are unions of constituents of H and have no common pentagon. Then the components of $G \cap G_1$ are the pivots of H in one pentagon of G and one of G_1. To see this, note that the valencies of $a \in V(G \cap G_1)$ in G, G_1, and H ensure the existence of an incident $A \in E(G \cap G_1)$. Pentagons $P \subseteq G$, $Q \subseteq G_1$ containing A also exist and, not being related, must be singular and such that $P \cap Q$ is a pivot. Pivots are disjoint, and so $G \cap G_1$ is as claimed. A constituent is thus joined to the rest of H by pivots. When H is indecomposable, it has only one constituent. Figure 1A shows that the converse statement is false.

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$H \cdot (E_1(H) \cup E_3(H))$ is clearly a divalent subgraph of H. Its components are the *structure polygons* of H. If a structure polygon contains more than one pivot vertex, the residual arcs of its pivot vertices are the *structure arcs* of H. Each structure arc, or structure polygon with at most one pivot vertex, is in one constituent of H because the pentagons of H containing its edges are clearly related.

A function $f\colon X \rightarrow Y$ is k-to-1 when each $y \in fX$ is the image of exactly k distinct $x \in X$. Let $f\colon L \rightarrow H$ be a graph mapping [**1**, Chapter 6] and $M \subseteq H$. f is 2-to-1 *on* M and one-to-one *off* M when f maps vertices to vertices, forming a vertex function, and edges to edges, forming an edge function, and these functions are 2-to-1 on and one-to-one off their elements in M.

The constituents G of H can be described in terms of a slightly simpler class of graphs. A *part* L of H is a pentagon-generated graph for which there exists a mapping $f\colon L \rightarrow H$ such that $fL = G$ is a constituent of H and f is 2-to-1 on and one-to-one off the pivots of H whose singular pentagons are in G. Then L is said to *represent* G under $f\colon L \rightarrow H$ and we write $L \rightarrow G$, specializing to $L \cong G$ when f is an isomorphism. Such a mapping is illustrated in Figure 1A. Labels a, b determine the 2-to-1 restriction of $f\colon L \rightarrow H$, hence the whole mapping.

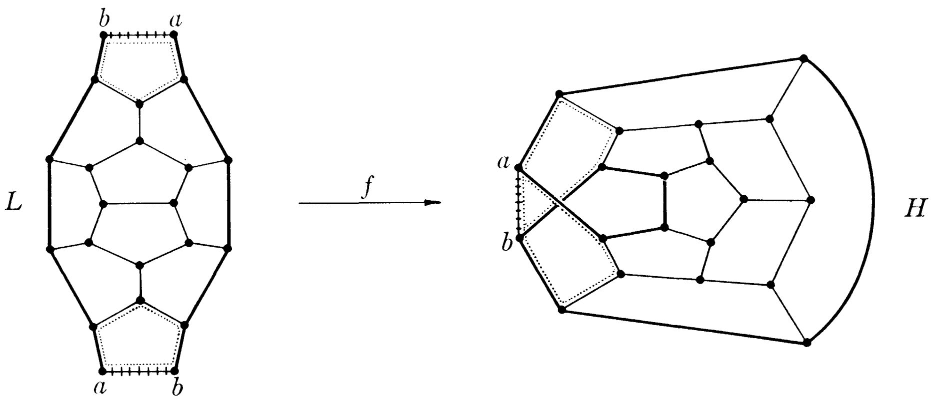

FIGURE 1A

Two problems are solved here. The first is to find a minimum set W of parts for all H and the second is to show how these parts combine to produce the decomposable H. The diagrams in § 2 provide a set W and this fact is verified in § 3. In § 4 each decomposable H is assigned a map (drawn on a closed surface), with vertices corresponding to the constituents of H, labelled appropriately from W, and edges the pivots of H. The surface determines how the parts combine to produce H. These labelled maps are intrinsically characterized.

2. Representative parts for $P_{5,3}$. The graphs C_i are drawn in Figure 2A as though embedded in a cylinder or Moebius band, the dotted line AB to the left in the figure being identified with the dotted line AB immediately above C_i. In

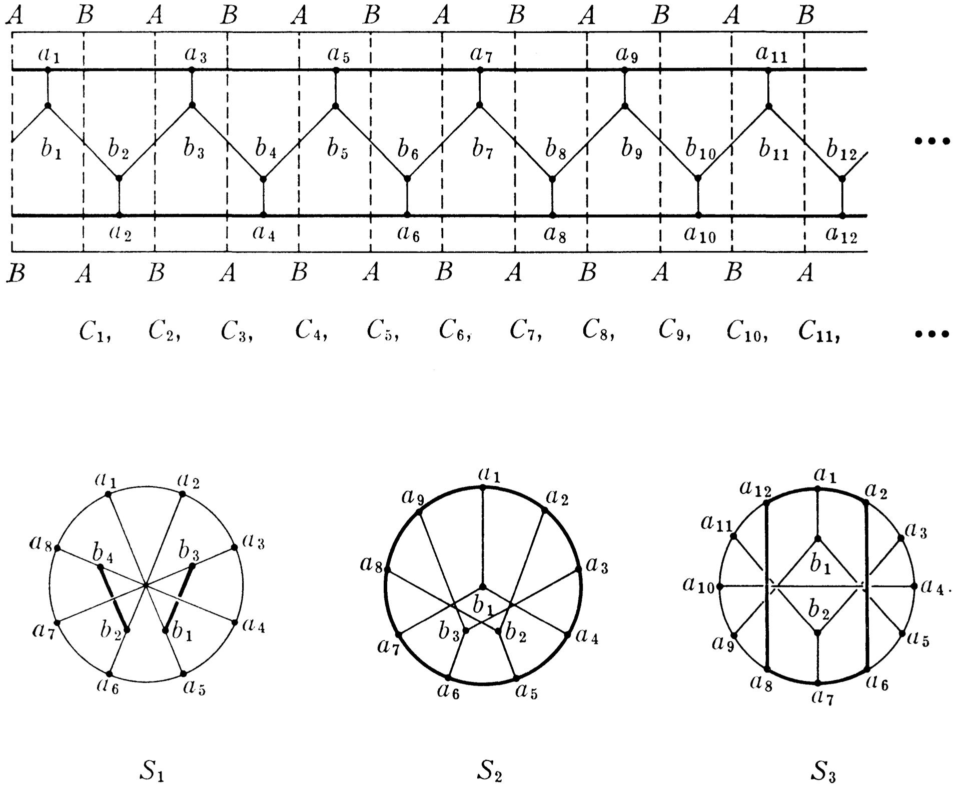

FIGURE 2A

this scheme the C_i, for odd $i \geqq 5$ and even $i \geqq 10$, together with S_1, S_2, S_3, make up a set of representative indecomposable graphs. All structure polygons of C_i, for odd $i > 5$ and even $i > 10$, and S_2, S_3 are distinguished, as are the edges of S_1 in $E_4(S_1)$. C_5, C_{10}, and S_1 have no structure polygons.

An infinite sequence $D_1, D_2, \ldots$ of graphs is suggested in Figure 2B, with a finite sequence $T_1, T_2, \ldots, T_7$. These graphs act as representative parts for the constituents of any decomposable H.

Let W be the set of graphs defined by the diagrams in Figures 2A and 2B (deleting D_1 because $T_1 \cong D_1$).

THEOREM 2.1. *If G is a constituent of $H \in P_{5,3}$, then a unique $L \in W$ exists with $L \rightarrow G$.*

This is proved in § 3, with the fact that W is minimal. It is evident that any two such W are equivalent, within isomorphisms of their members.

Suppose that $L \in W$ represents a constituent G of H under a mapping f: $L \rightarrow H$. In L the divalent vertices form the ends of disjoint link graphs, each

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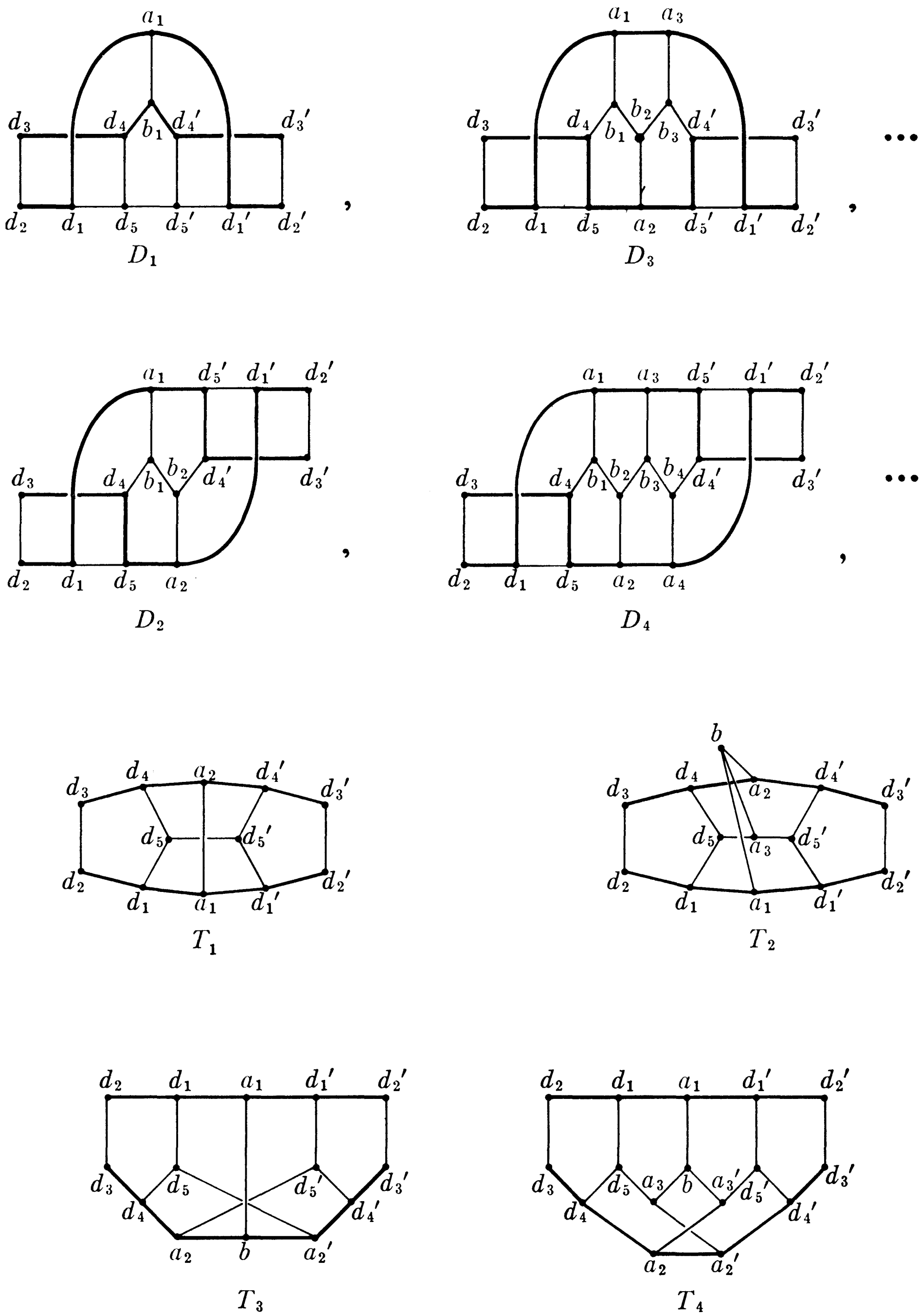

FIGURE 2B (*Continued*)

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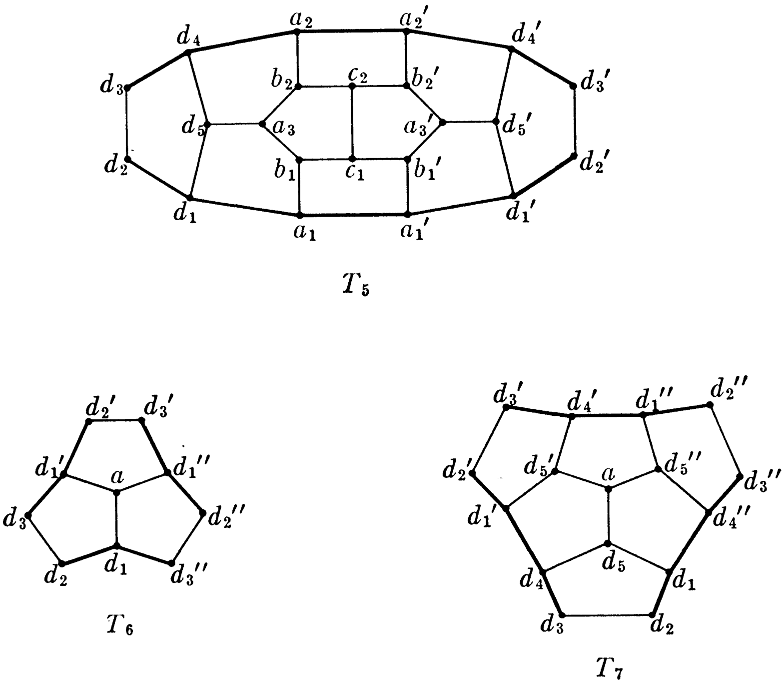

FIGURE 2B

contained in exactly one pentagon. The mapping ρ: $P_5(L) \rightarrow P_5(G)$, defined by $\rho P = fP$, for $P \in P_5(L)$, is an isomorphism. This follows easily, because f is one-to-one off the pivots of G and the above-mentioned link graphs map under f onto the pivots of G, while the pentagons containing them map under ρ onto the singular pentagons of G in a one-to-one manner. This justifies calling corresponding objects the *pivots*, *singular pentagons*, and *structure polygons* or *structure arcs* in L and G. These objects are distinguished in Figure 1A. The parts of decomposable H in W contain no structure polygons, although structure arcs in L can map onto structure polygons in H.

Each diagram in Figure 2B has a number of distinguished arcs, ending on pivot vertices, called its *angles*. Except for T_1 (and D_1), these are just its structure arcs. Angles in D_k, for odd $k \geqq 3$, and T_3, T_4 are not symmetrical. There the shorter angle is called the *top* angle of the part. When $L \rightarrow G$, the subgraphs of G corresponding to angles of L will also be called *angles*.

3. Verification of the standard forms. Suppose that $A \in E(H)$ is not a pivot edge of H and that G is the unique constituent of H containing A. Let H_A be the union of the 2-arcs in H having a common end with A.

PROPOSITION 3.1. *H_A is isomorphic to one of Figures* 3A(**B**)–(**F**) (*denoted throughout by $H_A \cong$* (**B**), . . . , *$H_A \cong$* (**F**), *respectively*).

Proof. Label H_A according to Figure 3A(**A**). Girth $\gamma(H) = 5$ implies that $x_1, x_2, y_1, y_2, y_3, y_4, z_1, z_3, z_5, z_7$ are distinct, although each z_i for i even may coincide with a z_i for i odd (henceforth referred to as even z_i and odd z_i). It is routine to verify, within symmetries of H_A, that $z_1 = z_6$; $z_1 = z_6, z_2 = z_7$; $z_1 = z_6, z_3 = z_8$; $z_1 = z_6, z_2 = z_7, z_3 = z_8$, and $z_1 = z_6, z_2 = z_7, z_3 = z_8, z_4 = z_5$ enumerates possible coincidences, yielding Figures 3A(**B**)–(**F**).

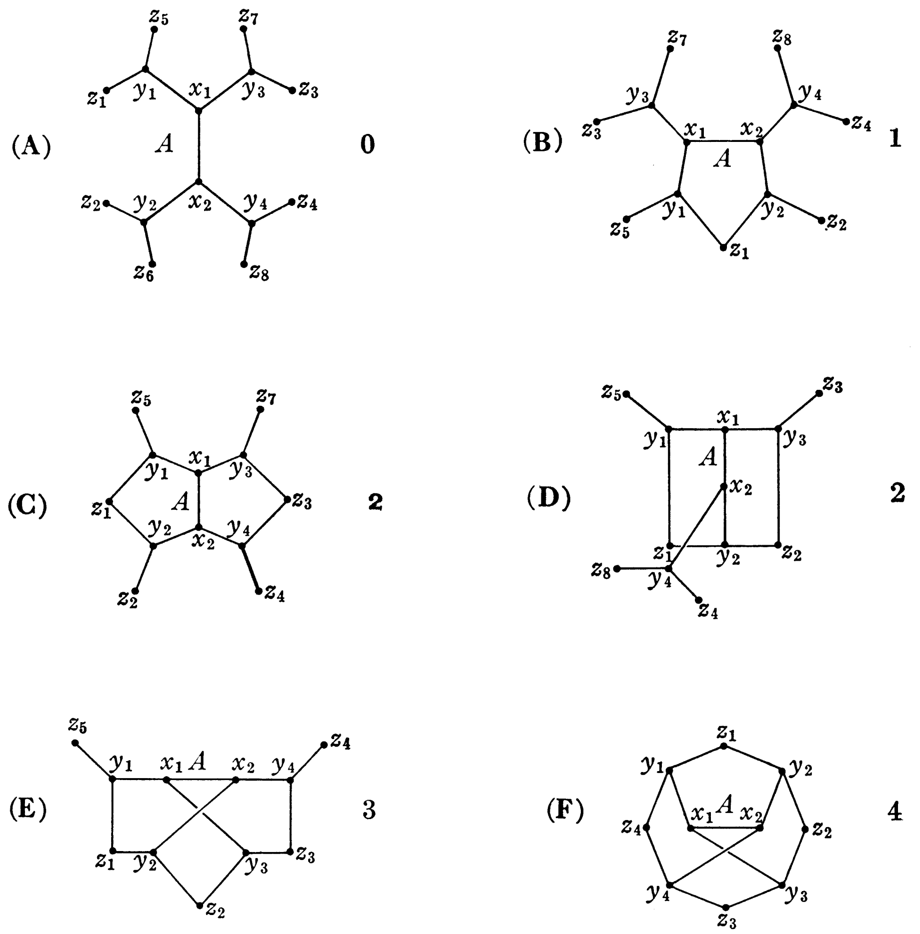

FIGURE 3A

When G is simple and $A \in E(G)$ has ends x, y, we can write $A = xy$ and $G \cdot \{A\} = [x, y]$. Denote by $L = [a_0, a_1, \ldots, a_n]$ the *arc* in G with $V(L) = \{a_0, a_1, \ldots, a_n\}$ and $E(L) = \{a_{i-1}a_i: i = 1, 2, \ldots, n\}$. If, also $a_0a_n \in E(G)$, we may speak of the *polygon* $[a_0, a_1, \ldots, a_n]$.

To prove Theorem 2.1 we consider sequences $G_0, G_1, \ldots, G_k \subseteq H$ with $G_0 = H_A$, $G \subseteq G_k$, and $G_i = G_{i-1} \cup [a_i, b_i]$, where $a_i \in V(G_{i-1})$ and

$A_i = a_ib_i \notin E(G_{i-1})$, for $i = 1, \ldots, k$, to show that for any G some $L \in W$ exists with $L \rightarrow G$. Symmetries of the G_i will be used to eliminate redundancies. Similarly, routine use of the definition of $P_{5,3}$ (H connected, $\gamma(H) = 5$, $\text{val}(H, x) \equiv 3$ and $H = \bigcup P_5(H)$) in proofs will be left to the reader. When $x, y \in V(G_i)$ exist at distance $d(x, y) = 3$ in G_i, then distinct

$$P, Q \in P_5(H)\backslash P_5(G_i)$$

exist, each containing x or y. Call this statement $(*)$ in what follows. The proof of Theorem 2.1 falls into three (disjoint) cases, $H_A \cong$ (**F**), $H_A \cong$ (**E**), and $G \subseteq H \cdot (E_1(H) \cup E_2(H))$. It is easy to verify that the following propositions cover the alternatives for these cases.

Case 1. *Assume that* $H_A \cong$ (**F**) *and let* $H_A = G_0$.

PROPOSITION 3.2. $C_5 \cong G$, $S_1 \cong G$, *or* $T_2 \cong G$.

Proof. If $A_1 \in E(H)\backslash E(H_A)$ exists with both ends in H_A, then

$$A_1 = z_1z_3 \in E(H)$$

can be assumed. Then $A_2 = z_2z_4 \in E(H)$, forming $G_2 \cong C_5$. Otherwise, there exist $u_i \notin V(H_A)$ such that $A_i = z_iu_i \in E(H)$, for $i = 1, 2, 3, 4$. The even u_i are distinct from the odd u_i.

When the u_i are not distinct, we can assume that $u_1 = u_3 = u$. Then $uu_2, uu_4 \in E(H)$ so that $u_2 = u_4 = u'$ and $A_5 = uu' \in E(H)$. Now $S_1 \cong G_5 = H$, with $A, A_5 \in E_4(H)$. If the u_i are distinct, then $A_i \in E_1(H) \cup E_2(H)$ for $i = 1, 2, 3, 4$. If $A_i \in E_1(H)$, for $i = 1, 2, 3, 4$, then $A_5 = u_1u_2$, $A_6 = u_3u_4 \in E(H)$, and $u_2u_3, u_4u_1 \notin E(H)$ can be assumed. Then $T_2 \cong G_6 = G$, with A corresponding to a_3b in T_2. Alternatively, $A_2 \in E_2(H)$ can be assumed, with $A_5 = u_1u_2$, $A_6 = u_2u_3 \in E(H)$. $A_4 \notin E_0(H)$, and so $A_7 = u_3u_4 \in E(H)$ can be chosen. This is contrary to $(*)$, and hence cannot occur.

Case 2. *Assume that* $H_A \cong$ (**E**) *and let* $H_A = G_0$. *Then there exist edges* $A_1 = z_1u_1$, $A_2 = z_2u_2$, *and* $A_3 = z_3u_3$, *each belonging to* $E(H)\backslash E(H_A)$.

PROPOSITION 3.3. $A_1 \neq A_3$.

Proof. If $A_1 = A_3$, then $z_2z_4, z_2z_5 \in E(H)$, contrary to the trivalency of z_2.

PROPOSITION 3.4. *If* $u_2 = z_4$ *or* $u_2 = z_5$, *then* $T_1 \cong G$.

Proof. Without loss of generality, assume that $u_2 = z_4$. Then

$$A_4 = z_4u_4 \in E(H)\backslash E(G_3)$$

exists and, before further assumptions are introduced, G_4 has a symmetry fixing x_2y_4 and sending $z_5, y_1, x_1, y_3, z_3, u_3$ to $u_1, z_1, y_2, z_2, z_4, u_4$, respectively. If u_1, u_3, u_4, and z_5 are distinct, then $A_5 = u_3u_4 \in E(H)$, the pentagons of G_5 containing y_1z_1 and u_3u_4 are singular and $T_1 \cong G \subseteq G_5$. Otherwise $u_3 = z_5$ can be assumed and $A_5 = z_5u_5 \in E(H)\backslash E(G_5)$ exists. $A_4 \notin E_0(H)$ implies that $u_4 = u_1$ or $u_4 = u_5$, contrary to $(*)$, ruling out this possibility.

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PROPOSITION 3.5. *If* $u_2 \notin V(H_A)$ *and* $u_1 = z_4$ *or* $u_3 = z_5$, *then* $S_2 \cong G$ *or* $T_3 \cong G$.

Proof. Without loss of generality, assume that $u_1 = z_4$. Then z_3, u_2z_2 and z_4, z_5y_1 are symmetrical in G_2. $T \in P_5(H)$ and $z_5y_1 \in E(T)$ imply that $T \cap G_2$ is the unique 3-arc or 4-arc joining z_5 to z_4 or z_3, respectively. If $u_3 = z_5$, then $A_4 = u_2z_4, A_5 = u_2z_5 \in E(H)$ exist, and $S_2 \cong H = G_5$. Otherwise,

$$u_2z_4,\ z_3z_5 \notin E(H)$$

can be assumed. Then $u_2z_2, z_5y_1 \in E_1(H)$, implying that $A_4 = z_4u_4, A_5 = u_2u_3$, $A_6 = z_5u_4$ exist, with $u_4 \notin V(G_3)$. By (*), u_2 and u_3 are not joined to u_4 and z_5; thus $A_3, A_4 \in E_1(H)$ and $T_3 \cong G = G_6$. The angles of G are $[z_5, y_1, x_1, x_2, y_2, z_2, u_2]$ and $[u_3, z_3, y_4, z_4, u_4]$.

PROPOSITION 3.6. *If* $u_1, u_2, u_3 \notin V(H_A)$, *then* u_1, u_2, *and* u_3 *are distinct.*

Proof. If they are not distinct, then $u_1 = u_3$. $A_2 \notin E_0(H)$ implies $A_4 = u_1u_2 \in E(H)$. Similarly, $z_4u_2\, z_5u_2 \in E(H)$, contrary to the trivalency of u_2.

PROPOSITION 3.7. *If* $u_1, u_2, u_3 \notin V(H_A)$ *and* $A_2 \notin E_1(H)$, *then* $S_3 \cong G$ *or* $T_4 \cong G$.

Proof. By hypothesis, $A_2 \in E_2(H)$ and $A_4 = u_1u_2$, $A_5 = u_2u_3 \in E(H)$. $y_4z_4 \notin E_0(H)$ implies that $A_6 = u_3u_4$, $A_7 = u_4z_4 \in E(H)$ exist for some $u_4 \in V(H)$. If $u_4 \in V(G_5)$, then $u_4 = z_5$ and $A_8 = u_1z_4 \in E(H)$. Thus $S_3 \cong H = G_8$, with $x_1x_2, A_7 \in E_3(H)$. If $u_4 \notin V(G_5)$, there exists $u_5 \notin V(G_7)$ such that $A_8 = u_1u_5, A_9 = u_5z_5 \in E(H)$. By (*), $u_4u_5 \notin E(H)$, so that $T_4 \cong G = G_9$. The angles of G_9 are $[u_5, u_1, u_2, u_3, u_4]$ and $[z_5, y_1, x_1, x_2, y_4, z_4]$.

PROPOSITION 3.8. *If* $u_1, u_2, u_3 \notin V(H_A)$ *and* $A_2 \in E_1(H)$, *then* $D_k \rightarrow G$ *for some* $k \geqq 2$.

Proof. $A_4 = u_1u_2 \in E(H)$ and $u_2u_3 \notin E(\mathrm{H})$ can be assumed since $A_2 \in E_1(H)$. Let M_2 be the pentagon-generated subgraph of G_4, changing labels $x_1, x_2, y_1, y_2, y_3, y_4, z_1, z_2, z_3, u_1, u_2$ to $d_5, d_4, a_2, b_1, d_1, d_3, b_2, a_1, d_2, b_3, a_3$, respectively, as in D_3. Suppose, inductively, that $M_k \subseteq G$ is labelled $d_1, d_2, d_3, d_4, d_5, a_1, b_1, a_2, b_2, \ldots, a_{k+1}, b_{k+1}$, as in D_{k+1}. Any pentagon of H not in M_k but containing a_k can be written $L_k = [a_k, b_k, b_{k+1}, b_{k+2}, a_{k+2}]$ or $N_k = [a_k, b_k, b_{k+1}, a_{k+1}, d_1']$ because $d_1d_2, d_3d_4 \in E_1(H)$.

If $L_k \subseteq G$, then $M_{k+1} = M_k \cup L_k$ is as in D_{k+2}, for otherwise $a_{k+2}b_{k+2} = d_2d_3$ and M_{k+1} has only one non-trivalent vertex. Thus $N_k \subseteq G$, for some $k \geqq 2$, because H is finite. Then $d_1' \notin V(M_k)$ and $N_k' = [d_1', d_2', d_3', d_4', d_5'] \in P_5(H)$ exist, with $N_k \notin P_5(M_k \cup N_k)$ and $a_{k+1} = d_5', b_{k+1} = d_4'$. N_k' is the only such pentagon containing d_1' or b_{k+1}, hence is singular in H. Then

$$D_k \rightarrow G = M_k \cup N_k \cup N_k'.$$

When labels are not distinct, $d_2d_3 = d_2'd_3'$. Then $k \geqq 2$ and $k > 2$ if also $d_2 = d_2'$, $d_3 = d_3'$.

Case 3. *Suppose that* $G \subseteq E_1(H) \cup E_2(H)$ *and let* $H_A = G_0$.

PROPOSITION 3.9. $H_A \cong$ **(C)** *or* $H_A \cong$ **(B)**, *for all* $A \in E(G)$.

Proof. Otherwise, by Proposition 3.1, $H_A \cong$ **(D)**, for some $A \in E(G)$. Then $x_2y_2 \in E(G)$ is in two pentagons of Figure 3A(**D**) plus all pentagons of H containing x_2y_4.

Let $w \in V(H)$ be an *a-vertex* when it is in a structure polygon and a *b-vertex* otherwise. Here, a-vertices are joined only by edges of $E_1(H)$ or pivot edges.

PROPOSITION 3.10. *If H is decomposable, then* $T_5 \rightarrow G$, $T_6 \cong G$ *or* $T_7 \cong G$.

Proof. G has a singular pentagon $P = [x_1, x_2, z_1, y_2, y_1]$ with unique arcs $X_i = [x_1, x_2, \ldots, x_i]$, $Y_i = [y_1, y_2, \ldots, y_i]$ of a-vertices, for $2 \leqq i \leqq 6$. Set $K_1 = P \cup X_4 \cup Y_4$ and $K_{i+1} = K_i \cup Q_i$ for certain $Q_i \in P_5(G)$. The labels on $V(K_1)$ are distinct, except possibly when $x_4 = y_4 = z$. If

$$V(Q_i) \cap \{x_1, y_1\} \neq \emptyset,$$

then Q_i is singular and $P \cap Q_i = [x_1, y_1]$.

$\gamma(H) = 5$ and $G = \bigcup P_5(G)$ imply that $z_2, u_1, u_2 \notin V(K_1)$ exist with $Q_1 = [x_3, x_2, z_1, z_2, u_1]$, $Q_2 = [y_3, y_2, z_1, z_2, u_2]$, and $H_A \cong$ **(C)**, for $A = z_1z_2$. If z_2 is an a-vertex, then Q_1 and Q_2 are singular and $T_6 \cong G \subset K_3$. Otherwise, $Q_3 = [u_1, z_2, u_2, u_4, u_3]$, $Q_4 = [x_4, x_3, u_1, u_3, u_5]$, and $Q_5 = [y_4, y_3, u_2, u_4, u_6]$ exist. Q_3 is not singular, and so $u_3, u_4 \notin V(K_3)$. Pentagons meet in at most one edge and z is trivalent if it exists; thus x_4, y_4, u_5, u_6 are distinct with $u_5, u_6 \notin V(K_4)$. If u_3 is an a-vertex, then $T_7 \cong G = K_6$. Otherwise $Q_6 = [u_5, u_3, u_4, u_6, z_3]$, $Q_7 = [x_5, x_4, u_5, z_3, z_4]$, $Q_8 = [y_5, y_4, u_6, z_3, z_4]$ exist, none singular. Thus $z_3 \notin V(K_6)$, $H_B \cong$ **(C)** for $B = z_3z_4$ and $x_5, z_4, y_5 \notin V(K_7)$. Finally, $Q_9 = [x_6, x_5, z_4, y_5, y_6]$ exists and is singular. Then $T_5 \rightarrow G = K_{10}$, with possibly $x_6 = y_1$ and $y_6 = x_1$.

PROPOSITION 3.11. *If H is indecomposable, then* $C_k \cong G$ *for odd* $k \geqq 7$ *and even* $k \geqq 10$.

Proof. Suppose that $P \in P_5(H)$ exists with $E(P) \subseteq E_2(H)$. Write

$$P = [a_1, a_3, a_5, a_7, a_9]$$

and define $Q = [a_2, a_4, a_6, a_8, a_{10}]$, $Q_i = [a_{i-1}, b_{i-1}, b_i, b_{i+1}, a_{i+1}]$, i taken modulo 10. If the Q_i all exist and the vertices are distinct, then $C_{10} \cong H$. By assumption, the even Q_i exist, using $\gamma(H) = 5$ and Proposition 3.9, the odd a_i and b_i are distinct and disjoint and the even b_i are distinct, non-adjacent and disjoint from the odd a_i and b_i. The even $a_ib_i \in E(H)$ exist with the a_i disjoint from the even Q_i. Since H is indecomposable and pentagon-generated, Q_1, Q_3, Q_5, and Q_7 can be assumed to exist. $\gamma(H) = 5$ implies that the even a_i are distinct; thus $a_8a_{10} \in E(H)$ and $C_{10} \cong H$.

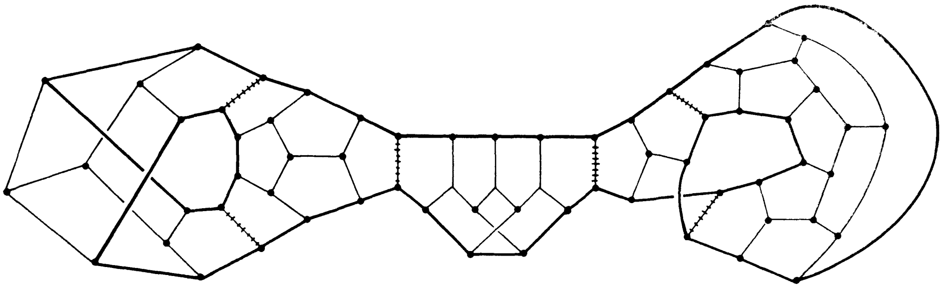
Now each $P \in P_5(H)$ can be assumed to have exactly one edge of $E_1(H)$. Let $B = [b_1, b_2, \ldots, b_n]$ be a component of the divalent subgraph of H generated by its b-vertices and let $a_1, a_2, \ldots, a_n$ be the respective adjacent a-vertices. No a-vertex is joined to two b-vertices, and so these a-vertices are distinct. Each 1-arc of B is in two pentagons and each 3-arc is in 0 pentagons hence each 2-arc of B is in exactly one pentagon. Then $a_i a_{i+2} \in E(H)$, for $i = 1, 2, \ldots, n \pmod n$. The a-vertices generate an n-gon, for odd $n > 5$, and two $(n/2)$-gons, for even $n > 10$, proving that $C_n \cong H$ for all n required.

4. The structure of decomposable graphs. Define a *map* $M = (R, U)$, where R is a connected trivalent graph and $U = (U_0, U_1, U_2)$ is a triple of edge-disjoint spanning subgraphs of valency 1 such that the components of $U_2 \cup U_0$ are quadrilaterals. Then the *vertex set* $V(M)$, *edge set* $E(M)$, and *face set* $F(M)$ are the sets of components of $U_0 \cup U_1$, $U_2 \cup U_0$, and $U_1 \cup U_2$, respectively. The members of $V(R)$, $E(U_0)$, $E(U_1)$, and $E(U_2)$ are termed the *corners*, *ties*, *angles*, and *sides* of M, in that order. The *valency* of a vertex or face of M is the number of angles it contains. The *graph* of M, $G(M)$, is the one with the above vertex and edge sets, its incidence determined by the ties in common to such vertices and edges. M has a *dual map* $M^* = (R, U^*)$, where $U^* = (U_2, U_1, U_0)$, and *dual graph* $G^*(M) = G(M^*)$. Clearly $M^{**} = M$, and $G^{**}(M) = G(M)$. It is not hard to see intuitively the equivalence of these maps with those defined in the standard way (i.e., with connected graph and simply connected faces) on closed surfaces.

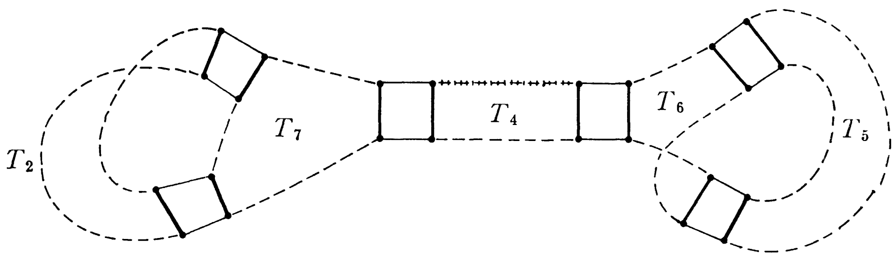
Suppose that H is decomposable. The *structure map* of H, $M(H) = (R, U)$, is such that $V(R)$ is the set of $E_1(H)$ edges in the singular pentagons of H, $E(U_0)$ is the set of singular pentagons in H, $E(U_1)$ is the set of angles for the constituents in H, and $E(U_2)$ is the set of pivot vertex graphs in H. Subgraphs are used to ensure that the $E(U_i)$ are disjoint. $A \in V(R)$ is incident in R with the singular pentagon and angle containing it and its incident pivot vertex graph.

For present classification purposes, a *labelled map* (M, l, m) is composed of a map M, a function l: $V(M) \to W$ such that $a \in V(M)$ and $fa \in W$ have the same number of angles, and a function m, such that ma is an angle of $a \in V(M)$, defined when la has a top angle. (M, l, m) is *properly labelled* if and only if there exists an isomorphism θ: $M \to M(H)$, for some $H \in P_{5,3}$ such that when $a \in V(M)$, la represents the constituent of H containing the vertices and edges of θa, and θma, if defined, is the top angle of θa. These easily characterized properly labelled maps illustrate the abundance of distinct graphs in $P_{5,3}$. Figure 4A contains a properly labelled map and corresponding H.

A graph K can be built from a labelled map (M, l, m) in the following way. Let D be a graph with components $D_a \cong la$, for $a \in V(M)$, such that $D_a \cap D_b = \Omega$ (the null graph) when $a \neq b$. Let L be a graph with link graph components L_A, for $A \in E(M)$, disjoint from the non-pivot vertices and edges of D and such that $L_A \cap L_B = \Omega$ when $A \neq B$. Identify the angles, corners, and ties of D_a with those of a, in their natural cyclic order, so that when D_a has



H



(M, l, m)

FIGURE 4A

a top angle it is identified with ma. Identify the vertices of L_A with the sides of A. Form K using L and the non-pivot vertices, edges, and incidences of D. Set corners of D and sides of L incident, when their counterparts in R (where $M = (R, U)$) are incident, to complete the definition. Any two graphs so produced are easily seen to be isomorphic. When H is decomposable it is associated with $M(H)$ in this manner. (M, l, m) is clearly properly labelled provided that $\gamma(K) = 5$, $E(L) \subseteq E_2(K)$ and if $P \in P_5(K)$ and $P \cap L = \Omega$, then $P \in P_5(D)$.

THEOREM 4.1. *Necessary and sufficient conditions that a labelled map* (M, l, m) *be properly labelled are:*

(1) *If* $a \in V(M)$ *is incident with a loop, then* $G(M)$ *is a loop graph. When* M *has only one face,* $la = T_5$ *or* D_k *for* $k \geqq 3$. *When* M *has two faces,* $la = D_k$ *for* $k \geqq 4$*;*

(2) *Suppose that* $a, b \in V(M)$ *are the distinct trivalent ends of distinct* $A, B \in E(M)$. *If a divalent* $f \in F(M)$, *with sides in* A, B *exists, then* $la = lb = T_7$. *If* $f \in F(M)$, *with three consecutive sides in* A, B *exists, then* la *or* $lb = T_7$.

Proof. Section 3 implies that $L \rightarrow G$ and $L \not\cong G$ for some constituent G of a decomposable H and $L \in W$, if and only if $L = T_5$ or D_k, for $k \geqq 3$. Then only one $T_5 \rightarrow G$ or $D_3 \rightarrow G$ is possible, and both $D_k \rightarrow G$, for $k \geqq 4$, are possible. Otherwise, $M(H)$ has no loop because all $L \rightarrow G$ are isomorphisms.

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We can suppose that K is built from (M, l, m), with $G(M)$ loopless, which is not properly labelled. Then a polygon $P \subseteq K$ exists, with girth $\leqq 5$, formed from (a) some L_A and two angles in distinct D_a, $D_b \cong T_6$ or (b) two angles in distinct D_a, $D_b \cong T_6$ or $D_a \cong T_6$, $D_b \cong T_7$. These are exactly the cases excluded by condition (2).

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