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ON FREE PRODUCTS OF COMPLETELY REGULAR SEMIGROUPS

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Abstract

The free product $*_{\mathbf{CR}} S_i$ of an arbitrary family of disjoint completely simple semigroups $\{S_i\}_{i \in I}$, within the variety **CR** of completely regular semigroups, is described by means of a theorem generalizing that of Kaďourek and Polák for free completely regular semigroups. A notable consequence of the description is that all maximal subgroups of $*_{\mathbf{CR}} S_i$ are free, except for those in the factors S_i themselves. The general theorem simplifies in the case of free **CR**-products of groups and, in particular, free idempotent-generated completely regular semigroups.

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Based on fundamental insights of Clifford [1], authors such as Gerhard [3], Trotter [9] and Kaďourek and Polák [8] have offered solutions to the word problem for the free completely regular semigroup $F\mathbf{CR}_X$ on a countably infinite set X. This semigroup is clearly the free product, in the variety **CR** of completely regular semigroups, of a family of infinite cyclic groups. We prove (Theorem 4.1) that, suitably modified, the method of Kaďourek and Polák serves to solve the word problem in the free **CR**-product of any family of disjoint *completely simple semigroups* (modulo effective description of the factors themselves). One notable consequence is that the maximal subgroups of such a free product are always free groups, if disjoint from the original factors.

A slight simplication occurs for the free **CR**-product of *groups*. In particular, the word problem is solved for the *free idempotent-generated* completely regular semigroup (which is a free product of trivial groups) on a set X. This semigroup turns out to be isomorphic with the completely regular subsemigroup of $F\mathbf{CR}_X$ generated by $\{x^0 : x \in X\}$.

The general problem, of describing arbitrary free **CR**-products, seems unassailable at present. See [7] for a review of the current state of affairs on free *band* products. The free product of completely simple semigroups within the *variety* of completely simple semigroups was described in [6]. Free products within some other 'small' subvarieties, such as Clifford semigroups and normal bands of groups are also not difficult to describe.

1. Free CR-products

A semigroup S is *completely regular* if it is a union of its maximal subgroups. For an element x of S, x^{-1} and x^0 denote the inverse of x and the identity element, respectively, in the maximal subgroup H_x. See [2, 5] for basic properties of such semigroups.

Throughout, $\mathbf{U}$ will denote the variety of *unary semigroups* (semigroups equipped with a unary operation $x \to x^{-1}$). The class **CR** of completely regular semigroups forms a (unary) subvariety of $\mathbf{U}$, determined by the identities

$$xx^{-1}x = x, \qquad xx^{-1} = x^{-1}x, \qquad (x^{-1})^{-1} = x.$$

Let $\{S_i\}_{i\in I}$ be a family of disjoint completely regular semigroups; denote their free **CR**-product by $*_{\mathbf{CR}}\{S_i\}_{i\in I}$, or $*_{\mathbf{CR}}S_i$ for short. Thus there exist monomorphisms $\eta_i : S_i \to *_{\mathbf{CR}} S_i$, $i \in I$, and for any $T \in \mathbf{CR}$ and morphisms $\phi_i : S_i \to T$, $i \in I$, there is a unique morphism $\phi : *_{\mathbf{CR}} S_i \to T$ such that $\eta_i \phi = \phi_i$, $i \in I$. (See [4, Section 9]).

In this section we construct $*_{\mathbf{CR}}S_i$ as a quotient of a certain free unary semigroup, which must first be described. For the moment, let $\{S_i\}_{i\in I}$ be any family of disjoint unary semigroups. Let $X = \bigcup_{i\in I} S_i$ and let F be the *free monoid* on the set $X \cup \{(,)^{-1}\}$. Then (see [3]) $F\mathbf{U}_X$ is the smallest subsemigroup of F such that $X \subseteq F\mathbf{U}_X$ and $(w)^{-1} \in F\mathbf{U}_X$ whenever $w \in F\mathbf{U}_X$ (and then, of course, $(w)^{-1}$ is the 'inverse' of w). Let ϵ be the unary congruence on $F\mathbf{U}_X$ generated by $\{(s \cdot t, st) : s, t \in S_i, i \in I\} \cup \{((s)^{-1}, s^{-1}) : s \in S_i, i \in I\}$, where $s \cdot t$ denotes the product in F. The following proposition is easily proved.

PROPOSITION 1.1. *Let $\{S_i\}_{i\in I}$ be a family of disjoint unary semigroups. Put $X = \bigcup_{i\in I} S_i$ and let $U = F\mathbf{U}_X/\epsilon$. Then U (together with the morphisms $\beta_i : x \longrightarrow x\epsilon, x \in S_i, i \in I$) is isomorphic with the free* **U**-*product* $*_{\mathbf{U}} S_i$.

The members of the monoid F are *words* in the *letters* $X\cup\{(,)^{-1}\}$. A *segment* of a word w is a word s such that $w = a\,s\,b$ for some $a, b \in F$; s is *initial* if $a = 1$ and *terminal* if $b = 1$.

PROPOSITION 1.2. *Each ϵ-class of $F\mathbf{U}_X$ contains a unique word w such that*

(a) *successive letters of w do not belong to the same set S_i and*
(b) *w contains no segment of the form $(s)^{-1}$, where s belongs to some S_i.*

PROOF. That each word in $F\mathbf{U}_X$ is ϵ-equivalent to such a word is easily established by an inductive argument, based on lengths of words; a simple confluence argument establishes uniqueness.

In the sequel we generally assume, without comment, that U *consists* of all such words in $F\mathbf{U}_X$, with appropriately modified multiplication and inversion. With this understanding we then refer to *words*, and their *letters*, in U.

PROPOSITION 1.3. *Let $\{S_i\}_{i\in I}$ be a family of disjoint completely regular semigroups. Define ρ to be the semigroup congruence on $U = F\mathbf{U}_X/\epsilon$ generated by the pairs*

$$(u(u)^{-1}u,\ u),\qquad (u(u)^{-1},\ (u)^{-1}u),\qquad (((u)^{-1})^{-1},\ u),\qquad u \in U.$$

Then U/ρ is isomorphic with the free **CR**-*product* $*_{\mathbf{CR}} S_i$.

PROOF. It is sufficient to show that ρ is a *unary* congruence, for then the result follows from standard universal algebraic arguments. So suppose $u, v \in U$ and $u\rho = v\rho$. From the generating relations it is clear that $(u)^{-1}\rho$ is an inverse of $u\rho$ in U/ρ and that $(u)^{-1}\rho\ \mathscr{H}\ u\rho$ in U/ρ. Thus $(u)^{-1}\rho = (u\rho)^{-1}$. Similarly, $(v)^{-1}\rho = (v\rho)^{-1}$ and thus $(u)^{-1}\rho = (v)^{-1}\rho$.

It is well known (and easily proved) that every congruence on a completely regular semigroup is a *unary* congruence.

2. Green's relation $\mathscr{D}$

Throughout this section, S denotes $U/\rho = *_{\mathbf{CR}} S_i$, the free **CR**-product of arbitrary disjoint completely regular semigroups, and β_i denotes the injection $S_i \longrightarrow S$, $i \in I$. Green's relation $\mathscr{D}$ on any completely regular semigroup is the least semilattice congruence [5, Theorem 4.6]. The variety **S** of semilattices is a subvariety of **CR**. Thus $S/\mathscr{D}$ is isomorphic with the free **S**-product of the semilattices $Y_i = S_i/\mathscr{D}$, $i \in I$. We now describe free **S**-products.

Let $\{Y_i\}_{i\in I}$ be any family of disjoint semilattices. Let Y be the direct product of the family $\{Y_i^{(1)}\}_{i\in I}$, where $Y_i^{(1)}$ is obtained from Y_i by adjoining a (new) identity element. Thus Y consists of all functions $f : I \longrightarrow \bigcup_{i\in I} Y_i^{(1)}$ with $if \in Y_i^{(1)}$, $i \in I$. The *support* of such a function f is $\{i \in I : if \neq 1\}$. Let P be the subsemilattice of Y consisting of the functions of *finite nonempty* support. For each $i \in I$, let $\alpha_i : Y_i \longrightarrow P$ be defined by

$$j(x\,\alpha_i) = \begin{cases} x & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases} .$$

The following is folklore:

RESULT 2.1. *The semilattice P, with morphisms α_i defined above, is isomorphic with the free* **S**-*product $*_{\mathbf{S}}\, Y_i$ of the semilattices Y_i, $i \in I$.*

Now let $u \in U$. We define the *content* $c(u)$ as a function in P : for $i \in I$, $ic(u) = D_{s(u)}$, where $s(u)$ is the product of all those letters in u, if any, that belong to S_i (in the order in which they appear, say) or is $1 \in Y_i^{(1)}$ if no such letter belongs to S_i. Clearly the content defines a morphism $c : U \longrightarrow P$. For each $i \in I$ and each $x \in S_i$, $c(x) = c(x\epsilon) = D_x\,\alpha_i$, so $\beta_i c = \mathscr{D}^{\natural}\alpha_i$, where $\mathscr{D}^{\natural}$ is the natural map $S_i \longrightarrow Y_i$. Thus cc^{-1} is the least semilattice congruence on U. Clearly $\rho \subseteq cc^{-1}$, so cc^{-1} induces the least semilattice congruence on $S = U/\rho$. The next proposition therefore describes $\mathscr{D}$ on S.

PROPOSITION 2.2. *Let $u, v \in U$. Then $u\rho\ \mathscr{D}\ v\rho$ in $*_{\mathbf{CR}} S_i$ if and only if $c(u) = c(v)$.*

3. Coproducts of completely simple semigroups: Green's relations

For the remainder of the paper we specialize to the case where each factor S_i belongs to **CS**, the subvariety of **CR** consisting of all completely simple semigroups. At the end of this section we consider some difficulties associated with the general case.

Since, now, each $|Y_i| = |S_i/\mathscr{D}| = 1$, we may redefine $c(u)$ as $\{i \in I : S_i$ contains a letter of $u\}$; c now defines a morphism of U upon the *free semilattice* on I, consisting of the nonempty finite subsets of I, under union.

For each $i \in I$, specify an arbitrary *idempotent* s_i of S_i. Denote by $L^{(i)}$, $R^{(i)}$ and $H^{(i)}$, respectively, the $\mathscr{L}$-, $\mathscr{R}$- and $\mathscr{H}$-class of s_i in S_i. (These play a 'normalizing' role.) For any word $w \in F$, $\hat{w}$ denotes the word of U^1 obtained from w by deleting all unmatched parentheses and choosing the representative of the resulting ϵ-class (see Section 1).

Let $u \in U$, with content A, say, $|A| = n \geq 1$. By analogy with [1, 3, 9], put $u0 = \hat{a}$, where a is the longest initial segment of u such that $c(a) \neq A$. Let y be the next letter of u after the segment a; then $y \notin \{(,)^{-1}\}$, so $y \in S_j$, where $A - c(u0) = \{j\}$. Put $u\sigma = (ys_j)^0$, the idempotent in the $\mathscr{H}$-class $R_y \cap L^{(j)}$ of S_j. Thus $y = (u\sigma)y$ and $u0\,y = u0\,u\sigma\,y$. The product $L(u) = u0\,u\sigma$ is defined to be the *left indicator* of u.

Dually, let $u1 = \hat{b}$, where b is the longest terminal segment of u such that $c(b) \neq A$; let $u\epsilon = (s_j\,x)^0$, the idempotent in the $\mathscr{H}$-class $L_x \cap R^{(k)}$, where $x \in S_k$ is the last letter of u before the segment b, and put $R(u) = u\epsilon\,u1$, the *right indicator* of u. Note that if $c(u) = \{i\}$, that is, u belongs to the factor S_i, then $u0 = 1 = u1$ and $u\sigma$ and $u\epsilon$ are, respectively, the idempotents $(us_i)^0$ and $(s_i\,u)^0$ of S_i.

LEMMA 3.1. *Let $u, v \in U$, with $u\,\rho\,v$. If*

(i) $|c(u)| = 1$ *then* $u = v$;
(ii) $|c(u)| > 1$ *then* $u0\;\rho\;v0$ *and* $u\sigma = v\sigma$, *and dually.*

PROOF. (i) Suppose $u \in S_i$, $i \in I$. By Proposition 2.2, $c(u) = c(v)$, so $v \in S_i$ also. Since S_i embeds in S, $u = v$.

(ii) It suffices to prove the result when v is obtained from u by an elementary transition. Let u_1 be the shortest initial segment of u with content that of u. In the notation above, $u_1 = ay$ and $\hat{u}_1 = \hat{a}y = u0y$. In the free monoid F, $u = u_1u_2$

for some u_2.

Suppose u factors as pqr, in U, with p and/or q possibly 1; note that all matched pairs of parentheses lie within p, q or r. Let $v = pq(q)^{-1}q\,r$. Various possibilities must be treated.

(a) If $a = pqr_1$ and $r = r_1\,y\,u_2$ in U (where r_1 may be 1) then $u = pqr_1yu_2$ and $v = (pq(q)^{-1}qr_1)\,yu_2$; now $v0 = (pq(q)^{-1}qr_1)\hat{} = pq(q)^{-1}q\hat{r}_1\ \rho\ pq\hat{r}_1 = (pqr_1)\hat{} = u0$. Clearly, $u\sigma = v\sigma$.

(b) If $a = pq_1$, $y = q_2r_1$, $q = q_1q_2$ and $r = r_1u_2$, where q_1, r_1 may be 1, then $u = pq_1q_2r_1u_2$ and $v = pq_1q_2\,(q_1q_2)^{-1}\,q_1q_2r_1u_2$. Now $c(q_2) = c(y)$, so $v0 = (pq_1)\hat{} = u0$. Also $y = q_2r_1\,\mathscr{R}\,q_2$ (within some completely simple factor), so $v\sigma = u\sigma$.

(c) In all other cases of this transition, v and u both begin with the segment ay. Thus $v0 = u0$ and $v\sigma = u\sigma$.

The reverse transition is handled similarly. Transitions associated with $(q)^{-1}q \longrightarrow q(q)^{-1}$ and $q \longrightarrow (q^{-1})^{-1}$ and their reverses involve only addition or deletion of parentheses and are handled easily.

LEMMA 3.2. *Let $u \in U$, $|c(u)| > 1$. Let u_1 be an initial segment of u, regarded as a word in F. Then $u\,\rho\,\hat{u}_1\,u_2$, for some $u_2 \in U$. Thus $u\,\rho\,\hat{u}_1(\hat{u}_1)^{-1}\,u$ and, in particular, $u\,\rho\,L(u)u'$ for some $u' \in U$.*

PROOF. We proceed by induction on the number of unmatched left parentheses in u_1. Let $u = u_1w$ in F. If u_1 has no unmatched parenthesis then $\hat{u} = \hat{u}_1\hat{w}$ (as a product in U). Otherwise, write $u_1 = p(q$, where q has no unmatched left parenthesis. Since $u \in U$, $w = r)^{-1}s$ for some r, s. Thus $u = p(qr)^{-1}s$ in U, so $u\,\rho\,v = pqr(qr)^{-1}\,(qr)^{-1}s$. But pq has one fewer unmatched left parenthesis, so $v\rho\,(pq)^\wedge\,u_2$ for some $u_2 \in U$, where $(pq)^\wedge = \hat{u}_1$. This completes the proof of the first statement, the second being an immediate consequence. To prove the third, we use the notation in the preamble to Lemma 3.1: $L(u) = u0u\sigma = \hat{a}g = (ag)^\wedge$, where $u = ays \in ag\cdot ys$, so that ag is an initial segment of u, in F.

For ease of exposition, we extend ρ from U to U^1 by putting $1\rho = \{1\}$.

THEOREM 3.3. *Let $\{S_i\}_{i\in I}$ be a family of disjoint completely simple semigroups, $U = *_{\mathrm{U}}\,S_i$ and $S = U/\rho = *_{\mathbf{CR}}\,S_i$. If $u, v \in U$ then*

(i) *$u\rho\ \mathscr{D}\ v\rho$ if and only if $c(u) = c(v)$;*

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(ii) *$u\rho\ \mathscr{R}\ v\rho$ if and only if $u0\ \rho\ v0$ and $u\sigma = v\sigma$;*
(iii) *$u\rho\ \mathscr{L}\ v\rho$ if and only if $u1\ \rho\ v1$ and $u\epsilon = v\epsilon$.*

PROOF.

(i) In view of the opening remarks of this section, this is a special case of Proposition 2.2.

(ii) We first observe that $u\rho\ \mathscr{R}\ L(u)\rho$. For by the preceding lemma, $\mathscr{R}_{u\rho} \leq R_{L(u)\rho}$; but $L(u)\rho\ \mathscr{D}\ u\rho$ in S, since $c(L(u)) = c(u)$, so equality of $\mathscr{R}$-classes holds, the $\mathscr{D}$-class being completely simple. As a consequence, $u\rho\ \mathscr{R}\ v\rho$ if and only if $L(u)\rho\ \mathscr{R}\ L(v)\rho$. One implication is then immediate. Conversely, if $L(u)\rho\ \mathscr{R}\ L(v)\rho$ then $u0u\sigma\ \rho\ v0v\sigma t$ for some $t \in U$. But $(v0v\sigma t)0 = v0$ and $(v0v\sigma t)\sigma = v\sigma$ (since if t begins with a letter from the same factor as $v\sigma$, then $v\sigma t\ \mathscr{R}\ v\sigma$ and the definitions of $(v0v\sigma)\sigma$ and $(v0v\sigma t)\sigma$ yield $\mathscr{H}$-related idempotents). By Lemma 3.1, $u0\ \rho\ v0$ and $u\sigma = v\sigma$.

(iii) This is dual to (ii).

This result simplifies considerably when the factors are groups—see Section 5.

The proofs of the results in this section have followed closely those of Clifford for free completely regular semigroups [1]. The author [7] has modified these techniques to obtain some properties of free products of *bands*, in the variety **B** of all bands. The difficulties which arise when the factors are no longer completely simple are discussed in detail there. We conclude this section with an example to show that in that case Green's relations $\mathscr{R}$ and $\mathscr{L}$ are not determined in such a simple fashion.

Let S_1 be a trivial semigroup $\{e\}$ and let S_2 be a two-element semilattice $\{f, g\}$, $f > g$. In U, $eg = efg\ \rho\ ef(ef)^{-1}efg = ef(ef)^{-1}eg$. Suppose we were to define $u0$ as $\hat{a}$, where a is the longest initial segment of u involving all the factors but one that appear in u, and define $u\sigma$ as above. Then $(eg)0 = e$ and $(eg)\sigma = g$; also $(ef(ef)^{-1}eg)0 = e$ and $(ef(ef)^{-1}eg)\sigma = f$. Thus Lemma 3.1 would now fail.

A slightly less naive definition for $u0$ would be as $\hat{a}$, with a the longest initial segment whose content is greater than that of u (using the general definition of content in Section 2) and with $u\sigma$ as before. In the given example, $(eg)0 = e$ but $(ef(ef)^{-1}eg)0 = ef(ef)^{-1}e$. Since e and $ef(ef)^{-1}e$ have different contents they cannot be ρ-related. So Lemma 3.1 again fails. Nevertheless, some positive results have been obtained for free band products [7], leaving open the prospect of further progress for completely regular semigroups in general.

4. Coproducts of completely simple semigroups: the word problem

Let $\{S_i\}_{i\in I}$ be a family of disjoint completely simple semigroups; define $U = *_{\mathrm{U}}\, S_i$, recalling the remarks following Proposition 1.2; and let $S = U/\rho = *_{\mathrm{CR}}\, S_i$, as in Section 1. We solve the word problem in S (modulo those in the factors S_i) inductively, based on $|c(u)|$, that is, on the number of factors S_i involved in the word $u \in U$ under consideration.

If $|c(u)| = 1$ and $u\, \rho\, v$ then $u, v \in S_i$ for some $i \in I$ and so $u = v$ in S_i (since S_i embeds in S). The inductive step involves an extension of the notion of characteristic sequence introduced in [8].

Let $|c(u)| = n \geq 2$. The *characteristic sequence* $[u]$ of u, to be constructed below, will have the form

$$(\mu_j e_j u_j g_j)_{0\leq j\leq k} = (\mu_0 e_0 u_0 g_0,\ \mu_1 e_1 u_1 g_1, \dots, \mu_k e_k u_k g_k)$$

where for each j, $\mu_j \in \{1, -1\}$, $c(u_j) = c(u) - \{i_j\}$ for some $i_j \in c(u)$ and e_j and g_j are idempotents of $S^1_{i_j}$. The expression $e_j u_j g_j$ is to be regarded as a product in U (or in F); it is termed a *link* of u. (This term has a more general meaning here than as originally used in [9]). A link $e_j u_j g_j$ is *interior* if neither e_j nor g_j is 1. Let Link (u) be the set of interior links of u. It will be shown that Link $(u) = \{e_j u_j g_j : 0 < j < k\}$ and that $e_0 u_0 g_0$ and $e_k u_k g_k$ are, respectively, the left and right indicators of u.

The characteristic sequence is constructed inductively, as in [8], on the number of segments $(q)^{-1}$ of u with $c(q) = c(u)$.

(i) Suppose u has no such segment. Let $b_0, \dots, b_k$ be the sequence of segments of u, read from left to right, that are maximal such that $|c(b_j)| = |c(u)| - 1$. Put $u_j = \hat{b}_j$, $0 \leq j \leq k$. Let $c(u) - c(b_j) = \{i_j\}$ and let x_j and y_j, respectively, be the letters of u that immediately precede and follow b_j (with value 1 if empty). Let $e_j = (s_{i_j} x_j)^0$ and $g_j = (y_j s_{i_j})^0$, the idempotent in the $\mathscr{H}$-class $R^{(i_j)} \cap L_{x_j}$ or $L^{(i_j)} \cap R_{y_j}$, respectively, of S_{i_j}; or 1 if $x_j = 1$ or $y_j = 1$. Let $[u]$ be the sequence $(+e_j u_j g_j)_{0\leq j\leq k}$ so defined.

(ii) Now suppose $u = p(q)^{-1} r$, where $c(q) = c(u)$. Put

$$[u] = (\langle pq0\rangle_{q\sigma},\ -_{q\epsilon}[q1\,q\,q0]_{q\sigma},\ {}_{q\epsilon}\langle q1r\rangle)$$

$R^{(1)}$

e	b
a	c

$L^{(1)}$

S_1

$R^{(2)}$

f	j
g	h

$L^{(2)}$

S_2

FIGURE 1

where

$$\langle w \rangle = \begin{cases} 1 \cdot w \cdot 1 & \text{if } |c(w)| < |c(u)| \\ [w] & \text{if } |c(w)| = |c(u)| \end{cases},$$

$$-(w_0, \ldots, w_\ell) = (-w_\ell, \ldots, -w_0),$$

and pre-subscripting [post-subscripting] denotes pre-multiplication [post-multiplication] of the first [last] link by the subscript.

In case (i) it is clear from a comparison of the definitions that $\mu_0 e_0 u_0 g_0$ is the left indicator of u, that is, $\mu_0 = 1$, $e_0 = 1$, $u_0 = u0$ and $g_0 = u\sigma$. In fact, a simple induction establishes that this is true for any u. Similarly, $\mu_k e_k u_k g_k$ is the right indicator of u, that is, $\mu_k = 1$, $e_k = u\epsilon$, $u_k = u1$ and $g_k = 1$.

EXAMPLE. Let S_1 and S_2 be the two rectangular bands defined in Figure 1, with designated idempotents $s_1 = e$, $s_2 = f$ and the designated $\mathscr{L}$- and $\mathscr{R}$-classes thus as indicated.

Consider the word $u = ajb\,(ajb)^{-1}af \in U$. Then $(ajb)0 = a$, $(ajb)\sigma = (jf)^0 = f$, $(ajb)1 = b$ and $(ajb)\epsilon = (fj)^0 = j$; note that $(ajb)a = aj(ba) = aje$; similarly $b(ajb) = ejb$. Thus

$$[u] = (\langle aje\rangle_f,\ -_j[eje]_f,\ {}_j\langle ef\rangle).$$

Now $c(aje) = c(u)$, so $\langle aje\rangle = [aje] = (1af,\ eje,\ je\cdot 1)$ and $\langle aje\rangle_f = (1af,\ eje,\ jef)$. Similarly, ${}_j\langle ef\rangle = (jef,\ ef\cdot 1)$. Also $[eje] = (1ef,\ eje,\ je\cdot 1)$, so ${}_j[eje]_f = (jef,\ eje,\ jef)$ and $-{}_j[eje]_f = (-jef,\ -eje,\ -jef)$. Hence

$$[u] = (1af,\ eje,\ jef,\ -jef,\ -eje,\ -jef,\ jef,\ ef\cdot 1).$$

Similarly,

$$\begin{aligned}\left[(ajb)^{-1}aj\,ef\right] &= \left(\langle a\rangle_f,\ -_j[eje]_j,\ _j[ejef]\right)\\ &= (1af,\ -jef,\ -eje,\ -jef,\ jef,\ eje,\ jef,\ ef\cdot 1).\end{aligned}$$

Let Z be a countably infinite set of new variables and let $F\mathbf{G}_Z$ denote the free group on Z. Our main theorem is a close analogue of the Theorem in [8].

THEOREM 4.1. *Let $\{S_i\}_{i\in I}$ be a family of disjoint completely simple semigroups, let $U = *_{\mathrm{U}}S_i$ and $S = U/\rho = *_{\mathbf{CR}}S_i$. Let $u, v \in U$. If $|c(u)| = 1$ then $u\,\rho\,v$ if and only $u = v$; if $|c(u)| > 1$ then $u\,\rho\,v$ if and only if*

(i) $c(u) = c(v)$;
(ii) $u0\,\rho\,v0$ *and* $u\sigma = v\sigma$;
(iii) $u1\,\rho\,v1$ *and* $u\epsilon = v\epsilon$;
(iv) *if* $[u] = (\mu_j e_j u_j g_j)_{0\le j\le k}$ *and* $[v] = (\eta_j f_j v_j h_j)_{0\le j\le \ell}$ *and* ψ *is any map from Link* $(u) \cup$ *Link* (v) *into* Z *such that* $(ewg)\psi = (e'w'g')\psi$ *if and only if* $e = e'$, $w\rho\, w'$ *and* $g = g'$, *then*

$$\prod_{j=1}^{k}((e_ju_jg_j)\psi)^{\mu_j} = \prod_{j=1}^{\ell}((f_jv_jh_j)\psi)^{\eta_j} \quad in\ \ F\mathbf{G}_Z.$$

If Link (u) or Link (v) is empty, the product is interpreted as the identity element of $F\mathbf{G}_Z$. Before its proof, the theorem will be exemplified.

EXAMPLE. In the example above, Link $(ajb(ajb)^{-1}af) = \{eje,\ jef\} =$ Link $((ajb)^{-1}\,ajef)$. We may map eje to z_1 and jef to z_2, say, $z_1 \neq z_2$. Then the product associated with $ajb(ajb)^{-1}af$ is $z_1z_2z_2^{-1}z_1^{-1}z_2^{-1}z_2 = 1$ and that with $(ajb)^{-1}ajef$ is $z_2^{-1}z_1^{-1}z_2^{-1}z_2z_1z_2 = 1$. According to Theorem 4.1, therefore, $ajb(ajb)^{-1}af$ and $(ajb)^{-1}ajef$ represent the same element of S.

PROOF OF NECESSITY. Necessity of (i)–(iii) follows from Theorem 3.3. From the definition of ρ, to prove (iv) it is only necessary to show that

(a) (iv) holds for $(u(u)^{-1}u, u)$, $(u(u)^{-1},\ (u)^{-1}u)$ and $(((u)^{-1})^{-1}, u)$ for all $u \in U$;

(b) if $u\,\rho\,v$ and (iv) holds for (u, v) then (iv) holds for (su, sv) and (us, vs), for any $s \in U$.

We need the following lemma. Observe first that the operators 0 and σ on words in U may be iterated: put $u0^0 = u$ and $u\sigma^0 = 1$ and define $u0^n = (u0^{n-1})0$ and $u\sigma^n = (u0^{n-1})\sigma$ for $n \geq 1$. The operators 1 and ϵ are iterated similarly.

LEMMA 4.2. *Let $s, u \in U$. Then*

(i) *if $c(s)$ and $c(u)$ are incomparable then $[su] = (+e_j w_j g_j)_{0\leq j\leq h}$, where $e_j = s\epsilon^{j\alpha}$, $w_j = s1^{j\alpha}u0^{j\beta}$, $g_j = u\sigma^{j\beta}$, $0\alpha = h\beta = 0$ and for $0 < j < h$, $j\alpha$ and $j\beta$ are positive integers determined by the order in which the factors S_i involved in s last appear and the order in which the factors involved in u first appear, respectively;*

(ii) *if $c(u) \subseteq c(s)$ and $[s] = (\eta_j f_j s_j h_j)_{0\leq j\leq \ell}$ then*

$$[su] = (\eta_0 f_0 s_0 h_0, \dots, \eta_{\ell-1} f_{\ell-1} s_{\ell-1} h_{\ell-1},\ {}_{f_\ell}\langle s_\ell u\rangle).$$

PROOF. This follows straightforwardly from the definition of characteristic sequence.

To prove (a) above, let $u \in U$, $|c(u)| \geq 2$, and suppose $[u] = (\mu_j e_j u_j g_j)_{0\leq j\leq k} = (1u_0g_0,\ T,\ e_ku_k1)$, say. By definition, recalling that $u_0 = u0$, $g_0 = u\sigma$, $e_k = u\epsilon$ and $u_k = u1$,

$$[u(u)^{-1}u] = ([uu_0]_{g_0},\ -_{e_k}[u_k\, u\, u_0]_{g_0},\ {}_{e_k}[u_k u]).$$

By Lemma 4.2 (ii) and its dual,

$$\begin{aligned} [uu_0]_{g_0} &= (1u_0g_0,\ T,\ {}_{e_k}\langle u_k u_0\rangle_{g_0}), \\ {}_{e_k}[u_k u] &= ({}_{e_k}\langle u_k u_0\rangle_{g_0},\ T,\ e_k u_k \cdot 1) \qquad \text{and} \\ {}_{e_k}[u_k\, u\, u_0]_{g_0} &= \left({}_{e_k}\langle u_k u_0\rangle_{g_0},\ T,\ {}_{e_k}\langle u_k u_0\rangle_{g_0}\right). \end{aligned}$$

Substituting these three sequences into the previous one, applying an appropriate map ψ and evaluating in $F\mathbf{G}_Z$ verifies that (iv) holds in this case. The case $(u(u)^{-1},\ (u)^{-1}u)$ is similar. Considering $(((u)^{-1})^{-1},\ u)$ we have, applying the definition,

$$\begin{aligned} [((u)^{-1})^{-1}] &= (\langle u_0\rangle_{g_0},\ -_{e_k}[u_k(u)^{-1}u_0]_{g_0},\ {}_{e_k}\langle u_k\rangle) \\ &= (1u_0g_0,\ -_{e_k}[u_k(u)^{-1}u_0]_{g_0},\ e_k g_k \cdot 1). \end{aligned}$$

Applying the definition once more,

$$[u_k(u)^{-1}u_0] = (\langle u_k u_0\rangle_{g_0},\ -_{e_k}[u_k\, u\, u_0]_{g_0},\ {}_{e_k}\langle u_k u_0\rangle)$$

and so ${}_{e_k}[u_k(u)^{-1}u_0]_{g_0} = \left({}_{e_k}\langle u_k u_0\rangle_{g_0},\ -{}_{e_k}[u_k\, u\, u_0]_{g_0},\ {}_{e_k}\langle u_k u_0\rangle_{g_0}\right)$. Substituting this last sequence in the expression above and using the formulas obtained for the first case, we obtain

$$[((u)^{-1})^{-1}] = (1u_0g_0, -{}_{e_k}\langle u_ku_0\rangle_{g_0}, {}_{e_k}\langle u_ku_0\rangle_{g_0}, T, {}_{e_k}\langle u_ku_0\rangle_{g_0}, -{}_{e_k}\langle u_ku_0\rangle_{g_0}, e_kg_k\cdot 1).$$

The proof may now be easily completed.

The proof of (b) above is by induction on the content. Suppose that (iv) holds for (u, v), $u, v \in U$, $u \rho v$, and let $s \in S$. By Theorem 3.3, $c(u) = c(v)$. Suppose firstly that $c(u)$ and $c(s)$ are incomparable. Then $[su] = (+e_jw_jg_j)_{0\le j\le h}$, in the notation of Lemma 4.2 (i). Similarly, $[sv] = (+f_jz_jh_j)_{0\le j\le \ell}$, where by repeated applications of Lemma 3.1, from $u \rho v$ it follows that $h = \ell$, that $f_j = s1^{j\alpha} = e_j$ and $h_j = v\sigma^{j\beta} = u\sigma^{j\beta} = g_j$, and that $z_j = s1^{j\alpha}v0^{j\beta} \rho$ $s1^{j\alpha}u0^{j\beta} = w_j$, for $0 \le j \le h$. Thus for any appropriate map ψ into Z, $(e_jw_jg_j)\psi = (f_jz_jh_j)\psi, 0 < j < h$, and so $\prod_{j=1}^{h-1}(e_jw_jg_j)\psi = \prod_{j=1}^{h-1}(f_jz_jh_j)\psi$ in $F\mathbf{G}_Z$, that is, (iv) holds for (su, sv).

Next suppose $c(u) \subseteq c(s)$. Let $[s] = (\eta_j f_js_jh_j)_{0\le j\le \ell} = (1s_0h_0,\ S,\ f_\ell s_\ell\cdot 1)$, say. By Lemma 4.2 (ii),

$$[su] = (1s_0h_0,\ S,\ {}_{f_\ell}\langle s_\ell u\rangle) \qquad \text{and}$$
$$[sv] = (1s_0h_0,\ S,\ {}_{f_\ell}\langle s_\ell v\rangle).$$

If $c(u) \subseteq c(s_\ell)$ then ${}_{f_\ell}\langle s_\ell u\rangle = f_\ell(s_\ell u)\cdot 1$ and ${}_{f_\ell}\langle s_\ell v\rangle = f_\ell(s_\ell v)\cdot 1$. Since ψ is only applied to interior links, the result is clear. Otherwise, $c(s_\ell u) = c(su)$ and $\langle s_\ell u\rangle = [s_\ell u]$, $\langle s_\ell v\rangle = [s_\ell v]$. If $c(u)$ and $c(s_\ell)$ are incomparable then the first part of the proof shows that (iv) holds for $(s_\ell u, s_\ell v)$. Moreover, the leading terms of ${}_{f_\ell}[s_\ell u]$ and ${}_{f_\ell}[s_\ell v]$ are ρ-related since they are, respectively, $f_\ell(s_\ell u)\,0\,(s_\ell u)\sigma$ and $f_\ell(s_\ell v)\,0\,(s_\ell v)\sigma$ and applying Lemma 3.1 to $s_\ell u$ and $s_\ell v$, $(s_\ell u)0\ \rho\ (s_\ell v)0$ and $(s_\ell u)\sigma = (s_\ell v)\sigma$. Under an appropriate map ψ, these two terms are therefore equal and (iv) holds for (su, sv). The remaining case has $c(s_\ell) \subsetneq c(u)$ (that is, $c(u) = c(s)$). By the dual of Lemma 4.2 (ii), ${}_{f_\ell}[s_\ell u] = ({}_{f_\ell}\langle s_\ell u0\rangle_{u\sigma}, T, u\epsilon(u1)\cdot 1)$ and ${}_{f_\ell}[s_\ell v] = ({}_{f_\ell}\langle s_\ell v0\rangle_{v\sigma}, V, v\epsilon(v1)\cdot 1)$, where $[u] = (1u0u\sigma, T, u\epsilon(u1)\cdot 1)$ and $[v] = (1v0v\sigma, V, v\epsilon(v1)\cdot 1)$. The case where $c(s_\ell)$ and $c(u0)$ are incomparable is completed similarly to the previously considered case. In the remaining possibility, ${}_{f_\ell}\langle s_\ell u0\rangle_{u\sigma} = f_\ell(s_\ell u0)u\sigma$ and ${}_{f_\ell}\langle s_\ell v0\rangle_{v\sigma} = f_\ell(s_\ell v0)v\sigma$. As above, these links are ρ-related and therefore equal under any appropriate map ψ. It now follows that (iv) holds for (su, sv) in this case also.

The alternative case, $c(s) \subseteq c(u)$, holds similarly. Thus (iv) holds for (su, sv) in all cases. By duality, it holds for (us, vs) also.

PROOF OF SUFFICIENCY. The main tool in the proof of sufficiency in [8] was the 'Decomposition Lemma,' based on arguments in [9]. Its analogue here is Lemma 4.3. Recall from Section 3 that for $i \in I$, s_i denotes the distinguished idempotent of S_i, belonging to the $\mathscr{H}$-class $H^{(i)} = R^{(i)} \cap L^{(i)}$. It will be convenient to assume I is well ordered.

LEMMA 4.3. *Let $u \in U$, $c(u) = A$, say, $|A| \geq 2$. Let $[u] = (\mu_j e_j u_j g_j)_{0 \leq j \leq k}$. Then*

$$u \;\rho\; \omega_L(1u_0g_0) \prod_{j=1}^{k-1} \omega(e_ju_jg_j)^{\mu_j}\; \omega_R(e_ku_k) \tag{1}$$

where for $ewg \in \{e_ju_jg_j : 0 \leq j \leq k\}$,

$$\begin{aligned} \omega_L(ewg) &= ewg\; f\; (fR(w)gf)^{-1} \\ \omega(ewg) &= fewg\; f\; (fR(w)gf)^{-1} \\ \omega_R(ewg) &= fewg \end{aligned}$$

and if $A = \{i_1 < \ldots < i_n\}$, $f = s_{i_1} \ldots s_{i_n} (s_{i_1} \ldots s_{i_n})^{-1} \in U$.

PROOF. For notational convenience, put $I_j = R(u_j)g_j$ $(= u_j\epsilon\, u_j 1 g_j)$, $0 \leq j < k$. Observe that I_j also equals $e_{j+1}L(u_{j+1}) = e_{j+1}u_{j+1}0\,u_{j+1}\sigma$.

First consider the case where u has no segment $(q)^{-1}$ with $c(q) = c(u)$. Let d denote the terminal segment of u that begins with u_1 and put $W = \hat{d}$. If, on the one hand, u_0 was obtained from the initial segment b_0 (see the definition of $[u]$) without deleting any unmatched parentheses then $u = u_0u' = u_0g_0u'$, for some $u' \in U$; thus $W = (u_01)u' = (u_01)g_0u'$. By the dual of Lemma 3.2, $u_0g_0 \;\rho\; u_0g_0(I_0)^{-1}I_0 = u_0g_0(I_0)^{-1}R(u_0)g_0 = u_0g_0(I_0)^{-1}e_1(u_01)g_0$. Therefore $u \,\rho\, u_0g_0(I_0)^{-1}e_1W$.

On the other hand, if some parenthesis was deleted from b_0 we may write $u_0 = p\hat{g}$, where g begins with a left parenthesis and is an initial segment of some segment $(e)^{-1}$ of u. So $u = p(e)^{-1}f$ for some $f \in U'$. Since, by hypothesis, $c(e) \neq c(u)$, and the right parenthesis $)^{-1}$ is to the right of $g_0 = u\sigma$ (see Section 3), it follows that the initial (of g is to the right of $(u0)\epsilon$, whence $u_01 = h\hat{g}$ for some $h \in U^1$. Thus $W = h(e)^{-1}f$ and since by Lemma 3.2, $(e)^{-1} \;\rho\; \hat{g}g_0(\hat{g}g_0)^{-1}(e)^{-1}$, $W \rho\, h\hat{g}g_0\,(\hat{g}g_0)^{-1}(e)^{-1}f = (u_01)\,g_0(\hat{g}g_0)^{-1}(e)^{-1}f$. Similarly

$$\begin{aligned} u = p(e)^{-1}f\,\rho\, p\hat{g}g_0(\hat{g}g_0)^{-1}\,(e)^{-1}f &= u_0g_0\,(\hat{g}g_0)^{-1}\,(e)^{-1}f \\ &\rho u_0g_0\,(I_0)^{-1}\,e_1(u_01)g_0\,(\hat{g}g_0)^{-1}(e)^{-1}f \end{aligned}$$

$$\rho u_0 g_0 (I_0)^{-1} e_1 h\hat{g} g_0 (\hat{g} g_0)^{-1} (e)^{-1} f$$
$$\rho u_0 g_0 (I_0)^{-1} e_1 W,$$

using the proof of the first case in the central step.

In either case, therefore, $u \, \rho \, u_0 g_0 (I_0)^{-1} e_1 W$. Next observe that $(f(fI_0f)^{-1} fI_0)\rho$ is an idempotent in U/ρ which is $\mathscr{L}$-related to $(u_0g_0)\rho$, by Theorem 3.3, since $I_0 = R(u_0g_0)$. Thus

$$u\rho u_0 g_0 \, f(fI_0f)^{-1} \, f \, I_0(I_0)^{-1} e_1 W$$
$$\rho u_0 g_0 \, f(fI_0f)^{-1} \, fe_1 W = \omega_L(1u_0g_0) \, fe_1 \, W,$$

since $I_0 = R(u_0)g_0 = e_1 L(u_1) = L(e_1u_1) = L(e_1W)$.

Now these arguments may be applied to W and repeated until (1) is achieved.

The rest of the proof proceeds inductively on the number of segments of u of the form $(q)^{-1}$ with $c(q) = c(u)$.

Suppose $u = p(q)^{-1}r$. By Lemma 3.2, $L(q)L(q)^{-1}(q)^{-1} \, \rho \, (q)^{-1}$, so $u \, \rho \, pL(q)L(q)^{-1}(q)^{-1}r_0$. The ρ-class of the word $f(fR(q)\,q\,L(q)f)^{-1} f\,R(q)\,q\,L(q)$ is an idempotent in U/ρ that is $\mathscr{L}$-related to $L(q)$, by the same lemma, so

$$u\rho pL(q) \, f(fR(q)\,q\,L(q)f)^{-1} \, fR(q)\,q\,L(q)L(q)^{-1} \, (q)^{-1}r$$
$$\rho pL(q) \, f(fR(q)\,q\,L(q)f)^{-1} \, fR(q)\,q(q)^{-1}r$$

and thus

(2) $$u \, \rho \, pL(q) \, f(fR(q)\,q\,L(q)f)^{-1} \, fR(q)r$$

since $R(q)\,q(q)^{-1} \, \rho \, R(q)$, by the same lemma again. (Compare with [9, Lemma 2.1]).

Note that all the words of the form $f \, w \, f$ that appear in (1) or (2) have content $c(u)$. By Theorem 3.3, therefore, each such $(f \, w \, f)\rho$ belongs to the subgroup $H_{f\rho}$ of U/ρ. Since $fR(L(q))f$ also belongs to this group,

(3) $$u \, \rho \, pL(q) \, f \, (fR(L(q))f)^{-1} \, (fR(q)\,q\,L(q)f \, (fR(L(q))f)^{-1})^{-1} \, fR(q)r.$$

Suppose $c(pq0) \neq c(u)$. Then $\langle pq0 \rangle_{q\sigma} = 1(pq0)\,q\sigma$ $(= 1u_0g_0)$ and

$$\omega_L(1(pq0)q\sigma) = pq0\,q\sigma \; f(fR(pq0)q\sigma f)^{-1}$$
$$= pL(q) \, f \, (fR(L(q))f)^{-1},$$

since $L(q) = q0q\sigma$ and $R(L(q)) = R(q0q\sigma) = R(pq0q\sigma) = R(pq0)q\sigma$.

Otherwise $c(pq0) = c(u)$. Let $[pq0] = (\pi_j f_j r_j h_j)_{0 \le j \le \ell}$. By the inductive hypothesis,

$$pq0 \;\rho\; \omega_L(1r_0h_0) \prod_{j=1}^{\ell-1} \omega(f_jr_jh_j)^{\pi_j} \; \omega_R(f_\ell r_\ell).$$

Since $c(q0) = c(r_\ell)$, we have $R(q0) = R(r_\ell)$ and $R(L(q)) = R(q0)q\sigma = R(r_\ell)q\sigma$.

Hence

$$\omega_R(f_\ell r_\ell 1)\, q\sigma\; f(fR(L(q))f)^{-1} = ff_\ell r_\ell\, q\sigma\; f(fR(r_\ell)\, q\sigma\; f)^{-1} = \omega(f_\ell r_\ell\, q\sigma)$$

and

$$pL(q)f(fR(L(q))f)^{-1} \;\rho\; \omega_L(1r_0h_0) \prod_{j=1}^{\ell-1} \omega(f_jr_jh_j)^{\pi_j} \; \omega(f_\ell r_\ell q\sigma).$$

This covers the first segment of the right hand side of (3). We next treat the large inverted term. Let $[(q1)\, q\, (q0)] = (\eta_j c_j t_j d_j)_{0 \le j \le h}$. By the inductive hypothesis,

$$q1\, q\, q0 \;\rho\; \omega_L(1t_0d_0) \prod_{j=1}^{h-1} \omega(c_jt_jd_j)^{\eta_j} \; \omega_R(c_ht_h).$$

Now $fR(q) = fq\epsilon\, q1$ and $fq\epsilon\, \omega_L(1t_0d_0) = \omega(q\epsilon t_0d_0)$; similarly, $L(q)f\,(fR(L(q))f)^{-1} = q0q\sigma f(fR(L(q))f)^{-1}$ and $\omega_R(c_ht_h1)q\sigma f(fR(L(q))f)^{-1} = \omega(c_ht_hq\sigma)$, since $R(L(q)) = R(q0)q\sigma = R(t_k)q\sigma$, similarly to the earlier argument. Combining these two equations gives

$$fR(q)\, q\, L(q)f(fR(L(q))f)^{-1} \;\rho\; \omega(q\epsilon t_0d_0) \prod_{j=1}^{h-1} \omega(c_jt_jd_j)^{\eta_j} \; \omega(c_ht_hq\sigma).$$

A similar, but simpler, analysis applies to the last segment, $fR(q)r$, of (3). By substituting these equations into (3) and comparing the result with the definition of $[u]$, the proof of the lemma is completed.

To complete the proof of sufficiency, let $u, v \in U$, with $|c(u)| \ge 2$, satisfying the conditions in Theorem 4.1. Let $[u] = (\mu_j e_j u_j g_j)_{0 \le j \le k}$, $[v] = (\eta_j f_j v_j h_j)_{0 \le j \le \ell}$. By Lemma 4.3,

$$u \;\rho \omega_L(1u_0g_0) \prod_{j=1}^{k-1} \omega(e_ju_jg_j)^{\mu_j} \; \omega_R(e_ku_k)$$

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and

$$v\,\rho\omega_L(1v_0h_0)\prod_{j=1}^{\ell-1}\omega(f_jv_jh_j)^{\eta_j}\,\omega_R(f_\ell v_\ell).$$

Let ψ be any map of Link $(u)\cup$ Link (v) into Z, as in the statement of the theorem. Then

$$\prod_{j=1}^{k-1}(e_ju_jg_j)\psi^{\mu_j}=\prod_{j=1}^{\ell-1}(f_jv_jh_j)\psi^{\eta_j}\qquad\text{in }F\mathbf{G}_Z.$$

For any $ewg\in$ Link $(u)\cup$ Link (v), $\omega(ewg)=fewgf(fR(w)gf)^{-1}$. Thus for any ewg, $e'w'g'\in$ Link $(u)\cup$ Link (v), if $(ewg)\psi=(e'w'g')\psi$ in Z then $\omega(ewg)\;\rho\;\omega(e'w'g')$. For by hypothesis $e=e'$, $g=g'$ and $w\,\rho\,w'$, whence, by Lemma 3.1, $R(w)\,\rho\,R(w')$. Since for all such ewg, $(ewg)\rho$ belongs to the subgroup H_{f_ρ} of U/ρ,

$$\prod_{j=1}^{k-1}\omega(e_ju_jg_j)^{\mu_j}\;\rho\;\prod_{j=1}^{\ell-1}\omega(f_jv_jh_j)^{\eta_j}.$$

But by hypothesis (ii), $u_0\,\rho\;\;v_0$ and $g_0=h_0$, so $R(u_0)\,\rho\,R(v_0)$ and $\omega_L(1u_0g_0)\;\rho\;w_L(1v_0h_0)$. Similarly $\omega_R(e_ku_k)\;\rho\;\omega_R(f_\ell v_\ell)$. Hence $u\,\rho\,v$.

COROLLARY 4.4. *Let $\{S_i\}_{i\in I}$ be a disjoint family of completely simple semigroups and put $S=*_{\mathbf{CR}}S_i$. Then the maximal subgroups of S, other than those of the original factors S_i, are free groups.*

PROOF. Let H be a maximal subgroup of S that is disjoint from $\bigcup_{i\in I}S_i$. By Theorem 3.3, H determines a finite subset $A=\{i_1,i_2,\dots,i_n\}$ of I, with $n\geq 2$, namely $c(u)$ for any $u\in H$. Let $\mathrm{Link}_A=\bigcup\{\mathrm{Link}\,(u):u\in U,\,c(u)=A\}$. Let ψ be a map of Link_A into a sufficiently large set Y, such that for ewg, $e'w'g'\in\mathrm{Link}_A$, $(ewg)\psi=(e'w'g')\psi$ if and only if $e=e'$, $w\rho w'$ and $g=g'$.

For $1\leq j\leq n$, let s_{i_j} be the designated idempotent in S_{i_j}, as in Section 3. Put $s=s_{i_1}\dots s_{i_n}s_{i_1}\dots s_{i_{n-1}}$. By Theorem 3.3, $H\subseteq D_{s\rho}$, and so $H\cong H_{s\rho}$. Clearly, $s0=s_{i_1}\dots s_{i_{n-1}}=s1$ and $s\sigma=s\epsilon=s_{i_n}=d$, say.

Note that

$${}_d\langle s1s0\rangle_d=d(s1s0)d=d(s_{i_1}\dots s_{i_{n-1}})^2d\in\mathrm{Link}\,(s_{i_n}s_{i_1}\dots s_{i_{n-1}}s_{i_1}\dots s_{i_n}),$$

so $(d(s1s0)d)\psi$ is well defined.

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Let $u, v \in U$, with $[u] = (\mu_j e_j u_j g_j)_{0\le j\le k} = (1u_0g_0, T, e_k u_k \cdot 1)$ and $[v] = (\eta_j f_j v_j h_j)_{0\le j\le \ell} = (1v_0h_0, V, f_\ell v_\ell \cdot 1)$, say. If $u\rho,\ v\rho \in H_{s\rho}$ then by Theorem 3.3, $u_0 = u0\,\rho\, s0\,\rho\, v0 = v_0$, $g_0 = u\sigma = s\sigma = d = v\sigma = h_0$, $u_k = u1\,\rho\, s1\,\rho\, v1 = v_\ell$ and $g_k = u\epsilon = s\epsilon = d = v\epsilon = h_\ell$. By Lemma 4.2 and its dual $[uv] = (1u_0g_0,\ T,\ {}_d\langle u1v0\rangle_d,\ V,\ f_\ell v_\ell 1)$.

Now $({}_d\langle u1v0\rangle_d)\psi = (ds1s0d)\psi$, since $u1v0\,\rho\, s1s0$. Thus the map

$$u\rho \longmapsto \prod_{j=1}^{k-1} (e_j u_j g_j)\psi^{\mu_j}\,(ds1s0d)\psi$$

is a morphism of $H_{s\rho}$ into $F\mathbf{G}_Y$ which, by Theorem 4.1, is injective. The Schreier subgroup theorem completes the proof.

5. Coproducts of groups

Let $\{G_i\}_{i\in I}$ be a family of disjoint groups. Theorem 4.1 simplifies, to the extent that the solution looks almost identical to that for free completely regular semigroups. For the designated idempotent s_i of G_i is obviously its identity, and $H^{(i)} = L^{(i)} = R^{(i)} = G_i$. Hence if $u \in U = *_{\mathbf{U}} G_i$ and $e_j u_j g_j$ is a link of u, then either $e_j = 1$ or e_j is the identity of G_{i_j}, where $c(u) - c(u_j) = \{i_j\}$, and similarly for g_j. Thus we may omit mention of e_j and g_j altogether. Theorem 3.3 now specializes as follows.

PROPOSITION 5.1. *Let $\{G_i\}_{i\in I}$ be a family of disjoint groups, $U = *_{\mathbf{U}} G_i$ and $S = U/\rho = *_{\mathbf{CR}}\, G_i$. If $u, v \in U$ then*

(i) *$u\rho\ \mathscr{D}\ v\rho$ if and only if $c(u) = c(v)$;*
(ii) *$u\rho\ \mathscr{R}\ v\rho$ if and only if $c(u) = c(v)$ and $u0\,\rho\, v0$;*
(iii) *$u\rho\ \mathscr{L}\ v\rho$ if and only if $c(u) = c(v)$ and $u1\,\rho\, v1$.*

The definition of the characteristic sequence $[u]$ of u may be simplified: $[u] = (\mu_j u_j)_{0\le j\le k}$, where

(i) if u has no segment of the form $(q)^{-1}$ with $c(q) = c(u)$, then $\mu_j = 1$ and $u_j = \hat{b}_j$, $0 \le j \le k$, as before, and

(ii) if $u = p(q)^{-1}r$, where $c(q) = c(u)$, then $[u] = (\langle pq0\rangle, -[q1\,q\,q0], \langle q1r\rangle)$, where

$$\langle w\rangle = \begin{cases} w & \text{if } c(w) \ne c(u) \\ [w] & \text{if } c(w) = c(u). \end{cases}$$

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Now Theorem 4.1 becomes the following (cf. [8, Theorem]).

COROLLARY 5.2. *Let* $\{G_i\}_{i\in I}$ *be a family of disjoint groups and let* $U = *_{\mathbf{U}} G_i$. *Let* $u, v \in U$, $|c(u)| \geq 2$. *Then* $u \,\rho\, v$ *if and only if*

(i) $c(u) = c(v)$,
(ii) $u0 \,\rho\, v0$,
(iii) $u1 \,\rho\, v1$,
(iv) *if* $[u] = (\mu_j u_j)_{0\leq j\leq k}$ *and* $[v] = (\eta_j v_j)_{0\leq j\leq \ell}$ *and* ψ *is any map from* $\{u_1, \dots, u_{k-1}\} \cup \{v_1, \dots, v_{\ell-1}\}$ *to* Z *such that for* s, t *in the domain,* $s\psi = t\psi$ *if and only if* $s \,\rho\, t$, *then* $\prod_{j=1}^{k-1} (u_j\psi)^{\mu_j} = \prod_{j=1}^{\ell-1} (v_j\psi)^{\eta_j}$ *in* $F\mathbf{G}_Z$.

By standard arguments, when each group G_i is infinite cyclic, $*_{\mathbf{CR}} G_i \cong F\mathbf{CR}_I$ and Corollary 5.1 becomes precisely the Theorem in [8]. A more novel special case is that when each group G_i is *trivial*, $G_i = \{e_i\}$, say. Let $\overline{I} = \{e_i : i \in I\}$. Then $\overline{I}$ is a set of idempotents of $S = *_{\mathbf{CR}} \{e_i\}$ which generates S and is bijective with I.

PROPOSITION 5.3. *The free product* $S = *_{\mathbf{CR}} \{e_i\}$ *of trivial groups* $\{e_i\}_{i\in I}$, *together with the injection* $\theta : i \mapsto e_i$, $i \in I$, *is the free idempotent-generated completely regular semigroup* $FI\mathbf{CR}_I$ *on* I, *in the following sense. For any map* ϕ *of* I *into the set* E_T *of idempotents of a completely regular semigroup there is a unique morphism* $\bar{\phi} : S \to T$ *such that* $\theta\bar{\phi} = \phi$.

PROOF. Let ϕ be as in the statement of the theorem. Then for each i, the restriction ϕ_i of ϕ to $\{i\}$ induces the monomorphism $\bar{\phi}_i$ of the group $\{e_i\}$ into T given by $e_i\bar{\phi}_i = i\phi$. These monomorphisms extend uniquely to a morphism $\bar{\phi} : S \to T$ such that $e_i\bar{\phi} = e_i\bar{\phi}_i$ for all $i \in I$. Thus $\theta\bar{\phi} = \phi$.

Corollary 5.2 therefore solves the word problem in $FI\mathbf{CR}_A$, for any set A. In particular, by Corollary 4.4, the nontrivial subgroups of $FI\mathbf{CR}_A$ are free. In the next section we show that $FI\mathbf{CR}_A$ is isomorphic with the completely regular subsemigroup of $F\mathbf{CR}_A$ generated by $\{a^0 : a \in A\}$.

6. Subsemigroups of the free product

In this section we show that if for each $i \in I$, T_i is a (completely) regular subsemigroup of the completely simple semigroup S_i then the completely regular subsemigroup of the free product $S = *_{\mathbf{CR}} S_i$ generated by $\bigcup_{i\in I} T_i$ is isomorphic to $*_{\mathbf{CR}} T_i$. (It was shown in [7] that for bands, at least, this need not happen if complete simplicity of the factors is relaxed. We do not know if the same is true for completely regular semigroups.)

Recall from Section 1 the notation X, for $\bigcup_{i\in I} S_i$, and F, for the free monoid on $X \cup \{(,)^{-1}\}$. Now let $Y = \bigcup_{i\in I} T_i \subseteq X$ and let F_Y be the submonoid of F generated by $Y \cup \{(,)^{-1}\}$. Clearly, F_Y is freely generated by this set. Also $F\mathbf{U}_Y$, the free unary semigroup on Y, is (isomorphic to) the smallest subsemigroup of F_Y such that $Y \subseteq F\mathbf{U}_Y$ and $(w)^{-1} \in F\mathbf{U}_Y$ whenever $w \in F\mathbf{U}_Y$; thus $F\mathbf{U}_Y$ is the unary subsemigroup of $F\mathbf{U}_X$ generated by Y. Let ϵ_Y denote the unary congruence on $F\mathbf{U}_Y$ generated by

$$\{(s \cdot t, st) : s, t \in T_i,\ i \in I\} \cup \{((t)^{-1}, t^{-1}) : t \in T_i,\ i \in I\}.$$

Suppose $v, w \in F\mathbf{U}_Y$ and $v \,\epsilon\, w$ (in $F\mathbf{U}_X$). Let $\bar{v}$ and $\bar{w}$, respectively, denote the unique words in $F\mathbf{U}_Y$ that are ϵ_Y-related to v and w according to Proposition 1.2, as applied to Y. Then applying the same proposition to X we see that since $\epsilon_Y \subseteq \epsilon_X$, $\bar{v} = \bar{w}$ and so $v \,\epsilon_Y\, w$. Hence ϵ restricts to ϵ_Y on $F\mathbf{U}_Y$ and we may identify the free unary product $*_{\mathbf{U}} T_i$ with a unary subsemigroup U_Y, say, of $U = *_{\mathbf{U}} cp S_i$. Furthermore, we may suppose that U_Y comprises those 'reduced' words of U whose letters (other than (and $)^{-1}$) belong to Y (see the remarks following Proposition 1.2).

All the discussion so far has been valid in general, for T_i a unary subsemigroup of S_i, $i \in I$. Now let each S_i be completely simple. Denote by ρ_Y the congruence on U_Y that induces $*_{\mathbf{CR}} T_i$, as in Proposition 1.3. In the notation of Section 3, for each $i \in I$, we may choose the designated idempotent s_i of S_i from T_i. For $u \in U$, if $u \in U_Y$ it is clear that $u0$, $u1 \in U_Y$ also. Further, $u\sigma = (ys_j)^0 \in U_Y$ and $u\epsilon = (s_j x)^0 \in U_Y$ (see Section 3 for the definition of x and y). In fact, if $[u] = \{\mu_j e_j u_j g_j\}_{0\le j\le k}$ denotes the characteristic sequence (Section 4) of u, regarded as an element of U, then similar reasoning shows that each $e_j, g_j \in T_{i_j}$ for some $i_j \in I$, and each $u_j \in U_Y$. Hence $[u]$ is also the characteristic sequence of u, when u is regarded as an element of U_Y.

Now denote by $\langle \bigcup_{i\in I} T_i \rangle$ the completely regular subsemigroup of $S = *_{\mathbf{CR}} S_i$ generated by the subsemigroups T_i, $i \in I$. Clearly, $\langle \bigcup_{i\in I} T_i \rangle$ is the image of U_Y under the projection of U on $U/\rho = S$. From Theorem 4.1 and the discussion in the preceding paragraph, it is now evident that for words u, v in U, if $u, v \in U_Y$ and $u \,\rho\, v$ then $u \,\rho_Y\, v$; that is, ρ restricts to ρ_Y on U_Y. Hence $\langle \bigcup_{i\in I} T_i \rangle \cong *_{\mathbf{CR}} T_i$. This completes the proof of the main result of the section.

THEOREM 6.1. *Let $\{S_i\}_{i\in I}$ be a family of completely simple semigroups and let T_i be a regular subsemigroup of S_i, $i \in I$. Then the unary subsemigroup of $*_{\mathbf{CR}} S_i$ generated by the semigroups T_i, $i \in I$, is isomorphic with $*_{\mathbf{CR}} T_i$.*

COROLLARY 6.2. *The free idempotent-generated completely regular semigroup $FI\mathbf{CR}_A$ on a set A is isomorphic with the unary subsemigroup of $F\mathbf{CR}_A$ generated by $\{a^0 : a \in A\}$.*

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