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Abstract

In this article we construct a p-adic three-dimensional eigenvariety for the group $U(2,1)(E)$, where E is a quadratic imaginary field and p is inert in E. The eigenvariety parametrizes Hecke eigensystems on the space of overconvergent, locally analytic, cuspidal Picard modular forms of finite slope. The method generalized the one developed in Andreatta, Iovita and Stevens [*p-adic families of Siegel modular cuspforms* Ann. of Math. (2) **181**, (2015), 623–697] by interpolating the coherent automorphic sheaves when the ordinary locus is empty. As an application of this construction, we reprove a particular case of the Bloch–Kato conjecture for some Galois characters of E, extending the results of Bellaiche and Chenevier to the case of a positive sign.

Contents

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1. Introduction

Families of automorphic forms have been a rather fruitful area of research since their introduction by Hida [Hid86] in 1986 for ordinary modular forms and their generalizations, notably the Coleman–Mazur eigencurve, but also to other groups than GL_2. Among examples of applications we can for example cite some cases of the Artin conjecture, for many modular forms the parity conjecture, and generalizations to a bigger class of automorphic representations of instances of Langlands' philosophy (together with local–global compatibility).

The goal of this article is to present a new construction of what is called an 'eigenvariety', i.e. a p-adically rigid-analytic variety which parameterizes *Hecke eigensystems*. More precisely, the idea is to construct families of eigenvalues for an appropriate Hecke algebra acting on certain rather complicated cohomology groups which are large $\mathbb{Q}_p$-Banach spaces, into which we can identify *classical* Hecke eigenvalues. For example Hida and Emerton consider for these cohomology groups some projective systems of étale cohomology on a tower of Shimura varieties, whereas Ash and

Stevens [AS08] and Urban [Urb11] instead consider cohomology of a large system of coefficients on a Shimura variety. Another construction which was introduced for GL_2 by Andreatta, Iovita and Stevens [AIS14] and Pilloni [Pil13] was to construct large coherent Banach sheaves on some open neighborhood of the rigid modular curve (more precisely on strict neighborhoods of the ordinary locus at p) indexed by p-adic weights and that vary p-adically. Their approach was then improved in [AIP15, ABI+16] to treat the case of Siegel and Hilbert modular forms, still interpolating classical automorphic sheaves by large (coherent) Banach sheaves. This method relies heavily on the construction of the Banach sheaves for which the theory of the canonical subgroup is central. For example in the case of GL_2, the idea is to construct a fibration in open ball centered in the images through the Hodge–Tate map of generators of the (dual of the) canonical subgroup inside the line bundle associated to the conormal sheaf ω on the modular curve $X_0(p)$. This rigid sub-bundle has then more functions but as the canonical subgroup does not exist on the entire modular curve this fibration in open balls only exists on a strict neighborhood of the ordinary locus. Following the strategy of [AIP15, ABI+16], Brasca [Bra16] extended this eigenvariety construction to groups that are associated to PEL Shimura varieties whose ordinary locus (at p) is non-empty, still using the canonical subgroup theory as developed in [Far11].

As soon as the ordinary locus is empty, the canonical subgroup theory gives no information and without a generalization of it the previous strategy seems vacuous. To my knowledge no eigenvarieties has been constructed using coherent cohomology when the ordinary locus is empty. Fortunately we developed in [Her19] a generalization of this theory, called the canonical filtration, for (unramified at p) PEL Shimura varieties. The first example when this happens is the case of $U(2,1)_{E/\mathbb{Q}}$, where E is a quadratic imaginary field, as the associated Picard modular surface has a non-empty ordinary locus if and only if p splits in E. In this article we present a construction of an eigenvariety interpolating p-adically (cuspidal) Picard modular forms when p is inert in E. The strategy is then to construct new coherent Banach sheaves on strict neighborhoods of the μ-ordinary locus using the (two-step) canonical filtration, and we get the following result,

Theorem 1.1. *Let E be a quadratic imaginary field and $p \neq 2$ a prime, inert in E. Fix a neat level K outside p, and a type[1] (K_J, J) outside p, i.e. a compact open K_J trivial at p, together with J a finite dimensional complex representation of K_J, and we assume moreover that $K \subset \operatorname{Ker} J \subset K_J$. Let N be the places where K is not hyperspecial (or very special) and I_p a Iwahori subgroup at p. There exists two equidimensional of dimension-3 rigid spaces,*

$$\mathcal{E} \xrightarrow{\kappa} \mathcal{W},$$

with κ locally finite, together with dense inclusions $\mathbb{Z}^3 \subset \mathcal{W}$ and $\mathcal{Z} \subset \mathcal{E}$ such that $\kappa(\mathcal{Z}) \subset \mathbb{Z}^3$, and each $z \in \mathcal{Z}$ coincides with Hecke eigensystem for $\mathcal{H} = \mathcal{H}^{Np} \otimes \mathbb{Z}[U_p, S_p]$ acting on cuspidal Picard modular forms of weight $\kappa(z)$, type $(K_J I_p, J)$ that are finite slope for the action of U_p. More precisely, we have a map $\mathcal{H} \longrightarrow \mathcal{O}(\mathcal{E})$, which induces an injection,

$$\mathcal{E}(\overline{\mathbb{Q}_p}) \hookrightarrow \operatorname{Hom}(\mathcal{H}, \overline{\mathbb{Q}_p}) \times \mathcal{W}(\overline{\mathbb{Q}_p}),$$

and such that for all $w \in \mathcal{W}(\overline{\mathbb{Q}_p})$, $\kappa^{-1}(w)$ is identified by the previous map with eigenvalues for $\mathcal{H}$ on the space of cuspidal, overconvergent and locally analytic Picard modular forms of weight w, type (K_J, J) which are finite slope for the action of U_p. When $z \in \mathcal{Z}$ is of weight w (necessarily in $\mathbb{Z}^3$), then the system of Hecke eigenvalues for $\mathcal{H}$ moreover coincide with one of $\mathcal{H}$ acting on classical *previous such forms.*

[1] In first approximation we can simply think of (K_J, J) being $(K, 1)$. Forms of type $(K, 1)$ are simply forms of level K.

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In order to get the previous result we need to have a control on the global sections of the Banach sheaves mentioned before the theorem. A general strategy to prove such a control result is developed in [AIP15] and rely on a vanishing theorem for cuspidal functions on the toroidal and minimal compactifications of the corresponding Shimura varieties. This kind of vanishing results have been proven in great generality in [Lan13] (though it does not apply here directly), but in the simpler case of $U(2,1)$, as the boundary of the toroidal compactification is quite simple, we managed to simplify a part of the argument of [AIP15]. In a forthcoming work, we will use this method together with the technics developed in [Her19] to construct eigenvarieties for more general PEL Shimura datum.

The second part of this article focuses on a very nice application of eigenvarieties to construct Galois extensions in certain Selmer groups. The method follows the strategy initiated by Ribet [Rib76] in the case of unequal characteristics to prove the converse to Herbrand theorem. It was then understood by Mazur and Wiles how to apply this technic in equal characteristics using Hida families to prove Iwasawa's main conjecture. In his PhD [Bel02], Bellaïche understood that using a certain endoscopic representation together with a generalization of Ribet's lemma he could produce some extensions of Galois representations, and then how to delete the *wrong* extensions to only keep the one predicted by the Bloch–Kato conjecture. This method was then improved using p-adic families and Kisin's result on triangulations of modular forms to construct the desired extensions in Selmer groups as in [BC04] for imaginary quadratic Hecke character and [SU02] for modular forms using Saito–Kurokawa lifts to GSp_4. In the previous constructions, it seemed necessary that the sign at the center of the functional equation is -1, in order to get an endoscopic automorphic representation for a bigger group ($U(3)$ in [BC04], GSp_4 in [SU02]) related the one we started with. In this article, we study the simplest case with a sign $+1$.

Let χ be an algebraic Hecke character of E satisfying the polarization

$$\chi^{\perp} := (\chi^c)^{-1} = \chi|.|^{-1}$$

and let $L(\chi, s)$ be its L-function. In particular $\chi_\infty(z) = z^a \overline{z}^{1-a}$ for all $z \in \mathbb{C}^\times$ where $a \in \mathbb{Z}$. Denote by $\chi_p : G_E \longrightarrow F^\times$, where $F/\mathbb{Q}_p$ is a finite extension, the associated p-adic Galois character, and $H^1_f(E, \chi_p)$ the Selmer group of χ_p, which is the sub-F-vector space of $H^1(E, \chi_p)$ generated by the extensions,

$$0 \longrightarrow \chi_p \longrightarrow V \longrightarrow 1 \longrightarrow 0,$$

such that for $v \not| p$

$$\dim V^{I_v} = 1 + \dim \rho^{I_v},$$

and for $v|p$

$$\dim D_{\mathrm{cris},v}(V) := \dim D_{\mathrm{cris}}(V_{|G_v}) = 1 + \dim D_{\mathrm{cris},v}(\rho).$$

We will say that such an extension has *good reduction everywhere*. The conjecture of Bloch and Kato predicts the equality $\mathrm{ord}_{s=0} L(\chi, s) = \dim_F H^1_f(E, \chi_p)$, and in particular the following result, due to Rubin [Rub91].

THEOREM 1.2 (Rubin). *Suppose $p \not| |\mathcal{O}_E^\times|$. If $L(\chi, 0) = 0$, then $H^1_f(E, \chi_p) \neq \{0\}$.*

The previous result follows from Rubin's work on Iwasawa's main conjecture for elliptic curves with complex multiplication (CM) and its proof uses Euler systems. In particular this construction proves that the Selmer group is non-trivial but does not really construct a particular extension. For example it is not possible to know if the extension that exists is *geometric* or *automorphic* in nature. Another proof of this result [BC04] uses families of modular forms given by the corresponding eigenvarieties, a particular case of transfer as predicted by Langland's philosophy, together with a generalization of Ribet's 'change of lattice' lemma. More precisely, if p is split in E, $p \nmid \mathrm{Cond}(\chi)$, and the order of vanishing $\mathrm{ord}_{s=0} L(\chi, s)$ is odd, then Bellaïche and Chenevier can construct the predicted extension in $H^1_f(E, \chi_p)$ by deformation of a non-tempered automorphic form $\pi^n(\chi)$ for $U(3)$, the compact at infinity unitary group in three variables. It is a natural question to ask why this condition of the order of vanishing being odd is necessary. If the order of vanishing is even, following multiplicity results on automorphic representations for unitary groups on three variables of Rogawski [Rog92, Rog90], there exists a non-tempered automorphic representation $\pi^n(\chi)$ for $U(2,1)$ with Galois representation $\rho_{\pi^n(\chi),p} = 1 \oplus \chi_p \oplus \chi_p^\perp$. In this article we check that we can indeed deform this representation such that the associate Galois deformation is generically irreducible, and that we can control the reduction at each place, thus constructing an extension in the Selmer group. More precisely we can reprove the following case of Rubin's result.

Theorem 1.3. *Let p a prime, unramified in E, such that $p \nmid \mathrm{Cond}(\chi)$. Suppose moreover that $p \neq 2$ if p is inert. If $L(\chi, 0) = 0$ and $\mathrm{ord}_{s=0} L(\chi, s)$ is even, then*

$$H^1_f(E, \chi_p) \neq 0.$$

In particular, we can extend the result of [BC04] when the order of vanishing is even, and also to the case of an inert prime p using the corresponding eigenvariety (in this case the ordinary locus is empty). An advantage of the construction of the eigenvariety presented here is that if a Hecke eigensystem appears in the classical cuspidal global sections of a coherent automorphic sheaf, then there is an associated point on the eigenvariety. This argument might be more complicated with other constructions, as the representation $\pi^n(\chi)$ is not a regular discrete series (it does not even *appear* in the cohomology in middle degree). Another advantage of using coherent cohomology is that we can also deal with the limit case where$\pi^n(\chi)$ does not appear in the étale cohomology[2] (but it was known to Bellaïche [Bel12] how to get this limit case). Apart from this fact, the deformation when p is split follows the lines of [BC04], whereas when p is inert the geometry of the eigenvariety is quite different. In particular, there are fewer *refinements* (and thus only one point on $\mathcal{E}$ corresponding to $\pi^n(\chi)$ instead of three) and we need a bit more care to isolate the *right* extension. We also need to be slightly more careful with p-adic Hodge theory to understand the local–global compatibility at p and a generalization of Kisin's result on triangulation of refined families as provided by [Liu15]. Let us also remark that a consequence of this construction and Chenevier's method [Che05] to compare the eigenvarieties for $U(3)$ and $U(2,1)$ (say when p splits) is that the point $\pi^n(\chi)$ when the sign at infinity is $+1$ together with its *good* refinement also appears in the eigenvariety of $U(3)$, despite *not* being a classical point for this group.

A lot of the ideas developed in this article extend to more general Shimura varieties, and in particular to Picard modular varieties associated to a general CM field E. More precisely, the result on the canonical filtration (Theorem 5.8) in this last case remains true as soon as $p \geqslant 3$. For more general groups the bound on the prime p will grow with the group as predicted in [Her19]. This is also why we cannot use $p = 2$ in this article. With this restriction on the prime, we should also be able to construct the associated eigenvariety, and we hope to come back on this matter in a following article. For the second part, which deals with the Bloch–Kato conjecture, it is less clear what to expect. Already for $U(2,1)_{E/F}$ with E a CM extension, we can use the work

[2] That is, when $a \in \{0, 1\}$, i.e. $\chi_\infty(z) = z$ or $\chi_\infty(z) = \overline{z}$.

of Rogawski and prove that the Galois representation passing through a particular refinement is irreducible; unfortunately, just as in [BC09], we cannot get rid of the bad extension in $H^1_f(E,\omega_p)$ anymore (see § 11.5). For higher-dimensional groups, we could probably have similar results as the one in the book [BC09], but, in addition to the previous remark, results in this case would be conditional on Arthur's conjecture, as in [BC09].

2. Shimura datum

2.1 Global datum

Let $E/\mathbb{Q}$ a quadratic imaginary field and denote by $\overline{\bullet}$ the complex conjugation of E. Let $(V = E^3, \psi)$ be the hermitian space of dimension 3 over E, of signature $(2,1)$ at infinity given by the matrix

$$J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Let us then denote by

$$\begin{aligned} G &= GU(V,\psi) = GU(2,1) \\ &= \{(g,c(g)) \in \mathrm{GL}(V) \times \mathbb{G}_{m,\mathbb{Q}} : \forall x,y \in V, \psi(gx,gy) = c(g)\psi(x,y)\} \subset \mathrm{GL}_V \times \mathbb{G}_m \end{aligned}$$

the reductive group over $\mathbb{Q}$ of unitary similitudes of (V,ψ).

Let p be a prime number, unramified in E. If $p = v\overline{v}$ is split in E, then

$$V \otimes_{\mathbb{Q}} \mathbb{Q}_p = V \otimes_E E_v \oplus \overline{V \otimes_E E_v},$$

where the action of E_v is by $\overline{v}$ on $\overline{V \otimes_E E_v}$. Moreover, the complex conjugation exchanges $V \otimes_E E_v$ and $\overline{V \otimes_E E_v}$. In particular, $G \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathrm{GL}(V \otimes_E E_v) \times \mathbb{G}_m$ (this isomorphism depends on the choice of v over p).

We will be particularly interested in the case where p is inert in E; the case when p split has been studied before (see for example [Bra16]).

Remark 2.1. We could more generally work in the setting of $(B,\star)$ a simple E-algebra of rank 9 with an involution of the second kind, such that $(B \otimes \mathbb{Q}_p, \star)$ is isomorphic to $(\mathrm{End}(V \otimes \mathbb{Q}_p), \star_V)$,[3] and replace G with the group,

$$G_B = \{g \in B^\times : g^\star g = c(g) \in \mathbb{G}_{m,\mathbb{Q}}\}.$$

The construction of the eigenvarieties in the case where B is not split is easier as the associated Shimura varieties are compact, but some non-tempered automorphic form, for example the one constructed by Rogawski and studied in [BC04] and the second part of this article, will never be automorphic for such non-split B.

The Shimura datum we consider is given by

$$h : \begin{array}{ccc} \mathbb{S} & \longrightarrow & G_{\mathbb{R}} \\ z = x+iy & \longmapsto & \begin{pmatrix} x & & iy \\ & z & \\ iy & & x \end{pmatrix} \end{array}.$$

[3] Here $\star_V(g) = J^t\overline{g}J$.

2.2 Complex Picard modular forms and automorphic forms

Classically, Picard modular forms are introduced using the unitary group $U(2,1)$, but we can treat the case of $GU(2,1)$ similarly. Let $G(\mathbb{R}) = GU(2,1)(\mathbb{R})$ the group stabilizing (up to scalar) the signature matrix J and let

$$Y = \{z = (z_1, z_2) \in \mathbb{C}^2 : 2\Im(z_1) + |z_2|^2 < 0\}$$

be the symmetric space associated to $G(\mathbb{R})$, it is isomorphic to the two-dimensional complex unit ball

$$B = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 < 1\}.$$

On Y, there is an action of $G(\mathbb{R})$ through

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} z = \frac{1}{c \cdot z + d}(Az + b) \in B, \quad A \in M_{2\times 2}(\mathbb{C}).$$

Remark 2.2. It is known that $U(2,1)(\mathbb{R})$ stabilizes Y, and $GU(2,1)(\mathbb{R})$ stabilizes X too as if ${}^t\overline{A}JA = cJ$ with $c \in \mathbb{R}^\times$, we get $\overline{\det A} \det A = |\det A|^2 = c^3$ thus $c > 0$.

This action is transitive and identifies Y with $G(\mathbb{R})/K_\infty$ where $K_\infty = \mathrm{Stab}((i,0)) \subset \{(A,e) \in GU(2)(\mathbb{R}) \times GU(1)(\mathbb{R})\}$ can be identified with $\{(A,e) \in GU(2)(\mathbb{R}) \times \mathbb{C}^\times : c(A) = N(e)\}$. We write $(i,0) = x_0$.

The subgroup K_∞ is not compact but can be written $Z(\mathbb{R})^0(U(2)(\mathbb{R}) \times U(1)(\mathbb{R}))$, with Z the center of $GU(2,1)$. Let L be the $\mathbb{C}$-points of K_∞. Then $L \simeq (\mathrm{GL}_2 \times \mathrm{GL}_1) \times \mathrm{GL}_1(\mathbb{C})$ is the Levi of a parabolic P in $\mathrm{GL}_3 \times \mathrm{GL}_1(\mathbb{C})$. For any $\kappa = (k_1, k_2, k_3, r) \in \mathbb{Z}^4$ such that $k_1 \geqslant k_2$, there is an associated (irreducible) representation $S_\kappa(\mathbb{C})$ of P, of highest weight

$$\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}, c \in (\mathrm{GL}_2 \times \mathrm{GL}_1) \times \mathrm{GL}_1(\mathbb{C}) \longmapsto t_1^{k_1} t_2^{k_2} t_3^{k_3} c^r.$$

The subgroup K_∞ embeds in $L \subset P$ by $(A,e) \mapsto ((\overline{A}, e), N(e))$.

Following [Har90b, Har84, Mil88], such a representation gives Ω^κ a locally free sheaf with $G(\mathbb{C})$-action on $G(\mathbb{C})/P$, whose structure as sheaf does not depend on r. Restricting it to $G(\mathbb{R})/K_\infty = Y$ we get a sheaf Ω^κ whose sections over X can be seen as holomorphic functions,

$$f : G(\mathbb{R})/K_\infty \longmapsto S_\kappa(\mathbb{C}),$$

such that $f(gk) = \rho_\kappa(k)^{-1} f(g)$, for $g \in G(\mathbb{R}), k \in K_\infty$, which we call (meromorphic at infinity) modular forms of weight κ. In an informal way, the choice of the previous integer r normalizes the action of the Hecke operators and corresponds to normalizing the (norm of the) central character of the modular forms. We will not use this description of the sheaves, and instead introduced a modular description of these automorphic sheaves.

Fix $\tau_\infty : E \longrightarrow \mathbb{C}$ an embedding, and $\sigma \neq 1 \in \mathrm{Gal}(E/\mathbb{Q})$; thus $\sigma\tau_\infty = \overline{\tau_\infty}$ is the other embedding of E. Over $\mathbb{C}$, for any sufficiently small compact open K in $G(\mathbb{A}_f)$, the Picard variety $Y_K(\mathbb{C})$ of level K can be identified with a (disjoint union of some) quotient of $B = GU(2,1)/K_\infty$, but also with the moduli space parametrizing quadruples $(A, \iota, \lambda, \eta)$ where A is an abelian scheme of genus 3, $\iota : \mathcal{O}_E \longrightarrow \mathrm{End}(A)$ is an injection, λ is a polarization for which Rosati involution corresponds to the conjugation $\bar{\cdot}$ on $\mathcal{O}$, and η is a type-K-level structure such that the action of

$\mathcal{O}_E$ on the conormal sheaf $\omega_A = e^*_{A/S}\Omega^1_{A/S}$[4] decomposes under the embeddings $\tau_\infty, \sigma\tau_\infty$ into two direct factors of respective dimensions 1 and 2. This is done, for example, explicitly in [dSG16, § 1.2.2–1.2.4], and we will be especially interested in the description by 'moving lattice' given in [dSG16, § 1.2.4]. Thus, to every $x = (z_1, z_2) \in B$, we can associate a complex abelian variety,

$$A_x = \mathbb{C}^3/L_x,$$

where L_x is the $\mathcal{O}_E$-module given in [dSG16, (1.25)], and the action of $\mathcal{O}_E$ on A_x is given by

$$a \in \mathcal{O}_E \longmapsto \begin{pmatrix} \overline{\tau}_\infty(a) & & \\ & \overline{\tau}_\infty(a) & \\ & & \tau_\infty(a) \end{pmatrix} \in M_3(\mathbb{C}).$$

There is, moreover, η_x a canonical (K-orbit of) level-N-structure (for $K(N) \subset K \subset G(\mathbb{A}_f)$). Over $Y_K(\mathbb{C})$ we thus have a sheaf ω_A that can be decomposed $\omega_{\tau,A} \oplus \omega_{\sigma\tau,A}$ according to the action of $\mathcal{O}_E$, and we can consider the sheaf

$$\omega^\kappa := \mathrm{Sym}^{k_1-k_2}\, \omega_{\sigma\tau,A} \otimes \det{}^{\otimes k_2}\omega_{\sigma\tau,A} \otimes \det{}^{\otimes k_3}\omega_{\tau,A},$$

for (k_1, k_2, k_3) a dominant (i.e. $k_1 \geqslant k_2$) weight. Using the previous description, if we denote by $\zeta_1, \zeta_2, \zeta_3$ the coordinates on $\mathbb{C}^3$, $\omega_{A_x,\sigma\tau}$ is generated by $d\zeta_1$, $d\zeta_2$ and $\omega_{A_x,\tau}$ by $d\zeta_3$.

There is also $X_K(\mathbb{C})$ a toroidal compactification of $Y_K(\mathbb{C})$, [Lar92] and [Bel06b], on which ω^κ extends as ω^κ (the *canonical* sheaf of Picard modular forms) and $\omega^\kappa(-D)$ (the sheaf of cuspidal forms) where D here is the closed subscheme $X_K \backslash Y_K$ together with its reduced structure.

Definition 2.3. We call the module $H^0(X_K(\mathbb{C}), \omega^\kappa)$ (respectively $H^0(X_K(\mathbb{C}), \omega^\kappa(-D)) =: H^0_{\mathrm{cusp}}(X_K(\mathbb{C}), \omega^\kappa)$) the space of (respectively, cuspidal) Picard modular forms of level K and weight κ.

We sometimes say 'classical' if we want to emphasis the difference with overconvergent modular forms defined later. Denote also by V^κ the representation of $\mathrm{GL}_2 \times \mathrm{GL}_1$ given by

$$(A, e) \longmapsto \mathrm{Sym}^{k_1-k_2}(\overline{A}) \otimes \det{}^{k_2}\overline{A} \otimes \det{}^{k_3} e.$$

Definition 2.4. For all $g \in G(\mathbb{R}) = GU(2,1)(\mathbb{R})$, write

$$g = \begin{pmatrix} A & b \\ c & d \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

and for $x = (z_1, z_2) \in B$, following [Shi78], define

$$\kappa(g, x) = \begin{pmatrix} \overline{a_1} - \overline{a_3} z_1 & \overline{c_2} z_1 - \overline{c_1} \\ \overline{a_3} z_2 - \overline{b_1} & \overline{d} - \overline{c_2} z_2 \end{pmatrix}, \quad \text{and} \quad j(g, x) = (cx + d).$$

Finally, define

$$J(g, x) = (\kappa(g, x), j(g, x)) \in \mathrm{GL}_2 \times \mathrm{GL}_1(\mathbb{C}).$$

The following proposition is well known (see [Hsi14, Lemma 3.7]) and probably already in [Shi78], but we rewrite it to fix the notations,

Proposition 2.5. *There is a bijection between $H^0(Y_K(\mathbb{C}), \omega^\kappa)$ and functions $F : B \times G(\mathbb{A}_{\mathbb{Q},f}) \longrightarrow V^\kappa$ such that:*

[4] Here $e_{A/S}$ denotes the unit section of $A \longrightarrow S$.

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(i) *for all* $\gamma \in G(\mathbb{Z}), F(\gamma x, \gamma k) = J(\gamma, x) \cdot F(x,k)$;
(ii) *for all* $k' \in K, F(x, kk') = F(x,k)$

given by $F(x,k) = f(A_x, \eta_x \circ k^\sigma, (d\zeta_1, d\zeta_2, d\zeta_3))$.

Proof. For all $\gamma \in G(\mathbb{Z})$, there is an isomorphism between (A_x, η_x) and $(A_{\gamma x}, \eta_{\gamma x} \circ \gamma^\sigma)$, for example described in [dSG16, § 1.2.2] or in [Gor92], which sends $(d\zeta_1, d\zeta_2, d\zeta_3)$ to $\gamma^*(d\zeta_1, d\zeta_2, d\zeta_3) = (\gamma^*(d\zeta_1, d\zeta_2), \gamma^* d\zeta_3)$ as γ preserve the action of $\mathcal{O}_E$; $\gamma^* d\zeta_3$ is calculated in [dSG16, Proposition 1.15] and given by

$$\gamma^* d\zeta_3 = j(\gamma, x)^{-1} d\zeta_3.$$

Moreover, by the Kodaira–Spencer isomorphism $\omega_{A_x,\tau} \otimes \omega_{A_x,\sigma\tau} = \Omega^1$ (see [dSG16, Proposition 1.22] for example), we only need to determine the action of γ on dz_1, dz_2. But this is done in [Shi78, 1.15] (or an explicit calculation), given by $c(\gamma)^t\kappa^{-1}(\gamma, x) j(\gamma, x)$, and we get

$$\gamma^*(d\zeta_1, d\zeta_2) = {}^t\kappa(\gamma, x)^{-1}(d\zeta_1, d\zeta_2).$$

Thus, setting $F(x,k) = f(A_x, \eta_x \circ k^\sigma, (d\zeta_1, d\zeta_2, d\zeta_3))$, we get,

$$F(\gamma x, \gamma k) = \mathrm{Sym}^{k_1-k_2}({}^t\kappa(\gamma, x)^{-1})((\det \kappa(\gamma, x)))^{-k_2} j(\gamma, x)^{-k_3} F(x,k). \qquad \square$$

Thus, to $f \in H^0(Y_K(\mathbb{C}), \omega^\kappa)$ we can associate a function, $\Phi_f : G(\mathbb{Q})\backslash G(\mathbb{A}) \longrightarrow V^\kappa$, by

$$\Phi_f(g) = c(g_\infty)^{-k_1-k_2-k_3} J(g_\infty, x_0)^{-1} \cdot F(g_\infty x_0, g_f),$$

where the action $\cdot$ is the one on V^κ, and we use the decomposition $g = g_{\mathbb{Q}} g_\infty g_f \in G(\mathbb{Q})G(\mathbb{R})G(\mathbb{A}_f)$. We can check that this expression does not depend on the choice in the decomposition. This association commutes with Hecke operators. The function Φ_f does not necessarily have a unitary central character, but it is nevertheless normalized by κ. Indeed, for $z_\infty \in \mathbb{C}^\times = Z(\mathbb{R})$,

$$\Phi_f(z_\infty g) = N(z_\infty)^{-k_1-k_2-k_3} \overline{z_\infty}^{k_1+k_2} z_\infty{}^{k_3} \Phi_f(g) = \overline{z}_\infty^{-k_3} z_\infty{}^{-k_1-k_2} \Phi_f(g).$$

Let $L : V^\kappa \longrightarrow \mathbb{C}$ a non-zero linear form. Define the injective map of right-K_∞-modules,

$$\mathcal{L} : \begin{array}{ccc} V^\kappa & \longrightarrow & \mathit{Fonct}(K_\infty, \mathbb{C}) \\ v & \longmapsto & L(J(k, x_0)^{-1} v). \end{array}$$

We denote by $L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \mathbb{C})$ the set of cuspidal L^2-automorphic forms for G. We have the following well-known proposition

Proposition 2.6. *The map $f \mapsto \varphi_f = L \circ \Phi_f$ is an isometry from $H^0_{\mathrm{cusp}}(X_K(\mathbb{C}), \omega^\kappa) = H^0(X_K(\mathbb{C}), \omega^\kappa(-D))$ to the subspace of $L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \mathbb{C})$ of functions φ, C^∞ in the real variable, such that:*

(i) *for all $g \in G(\mathbb{A})$, the function $\varphi_g : k \in K_\infty \mapsto \varphi(gk)$ is in $\mathcal{L}(V^\kappa)$, and in particular φ is right K_∞-finite;*
(ii) *for all $k \in K, \varphi(gk) = \varphi(g)$;*
(iii) *for all $X \in \mathfrak{p}^-_{\mathbb{C}}$, $X\varphi = 0$, i.e. φ is holomorphic;*
(iv) *φ is cuspidal, i.e. for all unipotent radical N of a proper parabolic P of G,*

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\, dn = 0.$$

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This isometry is equivariant under the Hecke action of $\mathcal{H}^N$ $(K(N) \subset K)$.

Using the previous proposition, to every $f \in H^0_{\mathrm{cusp}}(X_K, \omega^\kappa)$, an eigenvector for the Hecke algebra, we will be able to attach a automorphic form φ_f, and an automorphic representation Π_f (with same central character).

2.3 Local groups

In this subsection, we describe the local group at an inert prime. Let p be a prime, inert in E. Let $E_p \supset \mathbb{Q}_p$ its p-adic completion. Recall that $V \otimes \mathbb{Q}_p = E_p^3$ and that the hermitian form is given by the matrix,

$$J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

The diagonal maximal torus T of $G_{\mathbb{Q}_p}$ is isomorphic to $E_p^\times \times E_p^\times$,

$$T(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a & & \\ & e & \\ & & N(e)\overline{a}^{-1} \end{pmatrix}, \quad a, e \in E_p^\times \right\}$$

and contains T^1, isomorphic to $E_p^\times \times E_p^1$, where $E_p^1 = \{x \in E_p : x\overline{x} = 1\} = (\mathcal{O}_{E_p})^1$, the torus of $U(E^3, J)$,

$$T^1(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a & & \\ & e & \\ & & \overline{a}^{-1} \end{pmatrix}, \quad a \in E_p^\times, e \in E_p^1 \right\}.$$

We also have the Borel subgroups $B = B_{\mathrm{GL}_3(E)} \cap G_{\mathbb{Q}_p}$ of upper-triangular matrices,

$$B(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a & x & y \\ & e & \overline{x}\overline{a}^{-1}e \\ & & N(e)\overline{a}^{-1} \end{pmatrix}, \quad a, e, x, y \in E_p^\times \text{ and } \mathrm{Tr}(\overline{a}^{-1}y) = N(a^{-1}x) \right\},$$

and B^1 the corresponding Borel subgroup, for $U(E^3, J)$,

$$B^1(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a & x & y \\ & e & \overline{x}\overline{a}^{-1}e \\ & & \overline{a}^{-1} \end{pmatrix}, \quad a, x, y \in E_p^\times, e \in E_p^1 \text{ and } \mathrm{Tr}(\overline{a}^{-1}y) = N(a^{-1}x) \right\}.$$

3. Weight space

Write $\mathcal{O} = \mathcal{O}_{E_p}$, and denote by $T^1(\mathbb{Z}_p)$ the torus $\mathcal{O}^\times \times \mathcal{O}^1$ over $\mathbb{Z}_p$. It is the $\mathbb{Z}_p$-points of a maximal torus of $U(2,1)$. Denote by $T = T^1 \otimes_{\mathbb{Z}_p} \mathcal{O}$ the split torus over $\mathrm{Spec}(\mathcal{O})$.

Definition 3.1. The weight space $\mathcal{W}$ is the rigid space over $\mathbb{Q}_p$ given by $\mathrm{Hom}_{\mathrm{cont}}(T^1(\mathbb{Z}_p), \mathbb{G}_m)$. It coincides with Berthelot's rigid fiber [Ber96], of the formal scheme (called the *formal weight space*) given by

$$\mathfrak{W} := \mathrm{Spf}(\mathbb{Z}_p[[T^1(\mathbb{Z}_p)]]).$$

If K is an extension of $\mathbb{Q}_p$, the K-points of $\mathcal{W}$ are given by

$$\mathcal{W}(K) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}^\times \times \mathcal{O}^1, K^\times).$$

The weight space $\mathcal{W}$ is isomorphic to a union of $(p+1)(p^2-1)$ open balls of dimension 3 (see Appendix A, compare with [Urb11, § 3.4.2]),

$$\mathcal{W} \simeq \coprod_{(\mathcal{O}^\times \times \mathcal{O}^1)^{\mathrm{tors}}} B_3(0,1).$$

There is also a universal character,

$$\kappa^{\mathrm{un}} : T^1(\mathbb{Z}_p) \longrightarrow \mathbb{Z}_p[[T^1(\mathbb{Z}_p)]],$$

which is locally analytic. We can write $\mathcal{W} = \bigcup_{w>0} \mathcal{W}(w)$ as an increasing union of affinoids using the analycity radius (see Appendix A).

Definition 3.2. To $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ is associated a character,

$$\underline{k} : \begin{array}{ccc} \mathcal{O}^\times \times \mathcal{O}^1 & \longrightarrow & \mathbb{Q}_{p^2} \\ (x,y) & \longmapsto & (\sigma\tau)(x)^{k_1}\tau(x)^{k_3}(\sigma\tau)(y)^{k_2}. \end{array}$$

Characters of this form are called *algebraic*, or classical if moreover $k_1 \geqslant k_2$. They are analytic and Zariski dense in $\mathcal{W}$.

4. Induction

Set $U = U(2,1)/\mathbb{Z}_p$, T^1 its maximal torus, $K = \mathbb{Q}_{p^2}$ and $\mathcal{O} = \mathcal{O}_K$. We have $U \otimes_{\mathbb{Z}_p} \mathcal{O} \simeq \mathrm{GL}_3/\mathcal{O}$, and we denote by T its torus, and $\mathrm{GL}_2 \times \mathrm{GL}_1 \subset P$ the Levi of the standard parabolic of $\mathrm{GL}_3/\mathcal{O}$. Let $T \subset B$ the upper triangular Borel subgroup of $\mathrm{GL}_2 \times \mathrm{GL}_1$ and U its unipotent radical.

Definition 4.1. Let $\kappa \in X^+(T)$. Then there exists an (irreducible) algebraic representation of $\mathrm{GL}_2 \times \mathrm{GL}_1$ (of highest weight κ) given by

$$V_\kappa = \{f : \mathrm{GL}_2 \times \mathrm{GL}_1 \longrightarrow \mathbb{A}^1 : f(gtu) = \kappa(t)f(g), t \in T, u \in U\},$$

where $\mathrm{GL}_2 \times \mathrm{GL}_1$ acts by translation on the left (i.e. $gf(x) = f(g^{-1}x)$). The representation V_κ is called the *algebraic induction of highest weight* κ of $\mathrm{GL}_2 \times \mathrm{GL}_1$.

Let $I = I_1$ be the Iwahori subgroup of $\mathrm{GL}_2(\mathcal{O}) \times \mathrm{GL}_1(\mathcal{O})$, i.e. matrices that are upper-triangular modulo p. Let I_n be the subset of matrices in B modulo p^n, i.e. of the form

$$\begin{pmatrix} a & b & \\ p^n c & d & \\ & & e \end{pmatrix}, \quad a,b,c,d \in \mathcal{O}.$$

Denote by N^0 the opposite unipotent of U, and N^0_n the subgroup of N^0 of elements reducing to identity modulo p^n. These are seen as subgroups of $\mathrm{GL}_2(\mathcal{O}) \times \mathrm{GL}_1(\mathcal{O})$ rather than formal or rigid spaces, but could be identified with $\mathcal{O}$-points of rigid spaces. Precisely,

$$N^0 = \left\{ \begin{pmatrix} 1 & 0 & \\ x & 1 & \\ & & 1 \end{pmatrix}, \quad x \in \mathcal{O} \right\} \quad \text{and} \quad N^0_n = \left\{ \begin{pmatrix} 1 & 0 & \\ p^n x & 1 & \\ & & 1 \end{pmatrix}, \quad x \in \mathcal{O} \right\}.$$

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We identify N_n^0 with $p^n\mathcal{O} \subset (\mathbb{A}^1_{\mathcal{O}})^{\mathrm{an}}$. For $\varepsilon > 0$, denote by $N^0_{n,\varepsilon}$ the affinoid,

$$N^0_{n,\varepsilon} = \bigcup_{x\in p^n\mathcal{O}} B(x,\varepsilon) \subset (\mathbb{A}^1_{\mathcal{O}})^{\mathrm{an}}.$$

The affinoid $N^0_{n,\varepsilon}$ is a rigid space. Actually we could define a rigid space $\mathcal{N}^0$ (which would just be $\mathbb{A}^1$ here) such that $\mathcal{N}^0(\mathcal{O}) = N^0$, $N^0_{n,\varepsilon}$ is an affinoid in $\mathcal{N}^0$ and a neighborhood of N^0_n. For L an extension of $\mathbb{Q}_p$, denote by $\mathcal{F}^{\varepsilon\text{-an}}(N_n^0, L)$ the set of functions $N_n^0 \longrightarrow L$ which are restriction of analytic functions on $N^0_{n,\varepsilon}$. Let $\varepsilon > 0$ and $\kappa \in \mathcal{W}_\varepsilon(L)$ an ε-analytic character; we write

$$V^{\varepsilon\text{-an}}_{\kappa,L} = \{f : I \longrightarrow L : f(ib) = \kappa(b)f(i) \text{ and } f_{N^0} \in \mathcal{F}^{\varepsilon\text{-an}}(N^0, L)\}.$$

We also write, for $\varepsilon > 0$ and $k = \lfloor -\log_p(\varepsilon)\rfloor$,

$$V^{\varepsilon\text{-an}}_{0,\kappa,L} = \{f : I_k \longrightarrow L : f(ib) = \kappa(b)f(i) \text{ and } f_{N^0_k} \in \mathcal{F}^{\varepsilon\text{-an}}(N^0_k, L)\},$$

where $\lfloor . \rfloor$ denotes the previous integer, and

$$V^{l\text{-an}}_{\kappa,L} = \bigcup_{\varepsilon>0} V^{\varepsilon\text{-an}}_{\kappa,L} \quad \text{and} \quad V^{l\text{-an}}_{0,\kappa,L} = \bigcup_{\varepsilon>0} V^{\varepsilon\text{-an}}_{0,\kappa,L}.$$

Concretely, $V^{\varepsilon\text{-an}}_{0,\kappa,L}$ is identified with analytic functions on $B(0, p^{\lfloor -\log_p \varepsilon\rfloor})$ (a ball of dimension 1).

We can identify $V^{l\text{-an}}_{\kappa,L}$ with $\mathcal{F}^{l\text{-an}}(p\mathcal{O}, L)$ by restricting $f \in V^{l\text{-an}}_{\kappa,L}$ to N^0. We can also identify $V^{l\text{-an}}_{0,\kappa,L}$ to the germ of locally analytic function on 0.

Let

$$\delta = \begin{pmatrix} p^{-1} & & \\ & 1 & \\ & & 1 \end{pmatrix},$$

which acts on $\mathrm{GL}_2 \times \mathrm{GL}_1$, stabilizes the Borel subgroup $B(K)$, and defines an action on $V_{\kappa,L}$ for $\kappa \in X^+(T)$, via $(\delta \cdot f)(g) = f(\delta g \delta^{-1})$. The action by conjugation of δ on I does not stabilize it, but it stabilizes N^0. We can thus set, for $j \in I$, $j = nb$ the Iwahori decomposition of j, and set

$$\delta \cdot f(j) = f(\delta n \delta^{-1} b).$$

We can thus make δ act on $V^{\varepsilon\text{-an}}_{\kappa,L}, V^{l\text{-an}}_{\kappa,L}, V^{\varepsilon\text{-an}}_{0,\kappa,L}, V^{l\text{-an}}_{0,\kappa,L}$. Via the identification $V^{\varepsilon\text{-an}}_{\kappa,L} \simeq \mathcal{F}^{l\text{-an}}(p\mathcal{O}, L)$, $\delta \cdot f(z) = f(pz)$. Thus δ improves the analycity radius. Moreover, its supremum norm is negative.

PROPOSITION 4.2. *Let $f \in V^{l\text{-an}}_{0,\kappa,L}$. Suppose f is of finite slope under the action of δ, i.e. $\delta\cdot f = \lambda f$, $\lambda \in L^\times$. Then f comes (by restriction) from a (unique) $f \in V^{\mathrm{an}}_{\kappa,L}$.*

Proof. We have that $f \in V^{p^n-an}_{0,\kappa,L}$ for a certain n; in particular, it defines a function

$$f : \begin{pmatrix} a & u & \\ p^n\mathcal{O} & b & \\ & & c \end{pmatrix} = I_n \longrightarrow L$$

that is identified to a function in $\mathcal{F}^{\mathrm{an}}(p^n\mathcal{O}, L)$. But f is an eigenform for δ with eigenvalue $\lambda \neq 0 \in L$, and thus $f = \lambda^{-1}\delta(f)$. But if $f = f(z)$, with the identification to $\mathcal{F}^{\mathrm{an}}(p^n\mathcal{O}, L)$, δf is identified with $f(pz)$, thus $f \in \mathcal{F}^{\mathrm{an}}(p^{n-1}\mathcal{O}, L)$, i.e., δ^{-1} strictly increases the analyticity radius, and, by iterating, $f \in \mathcal{F}^{\mathrm{an}}(p\mathcal{O}, L)$, and thus $f \in V^{\mathrm{an}}_{\kappa,L}$. $\square$

Proposition 4.3. *For $\kappa=(k_1,k_2,r)\in X^+(T)$, there is an inclusion,*

$$V_{\kappa,L}\subset V_{\kappa,L}^{\mathrm{an}},$$

which under the identification of $V_{\kappa,L}^{\mathrm{an}}$ with $\mathcal{F}^{\mathrm{an}}(p\mathcal{O},L)$ identifies V_κ with polynomial functions of degree less or equal than k_1-k_2.

Proposition 4.4. *Let $\kappa=(k_1,k_2,r)\in X^+(T)$. The following sequence is exact,*

$$0\longrightarrow V_{\kappa,L}\longrightarrow V_{\kappa,L}^{\mathrm{an}}\xrightarrow{d_\kappa}V_{(k_2-1,k_1+1,r),L}^{\mathrm{an}},$$

where d_κ is given by

$$f\in V_{\kappa,L}^{\mathrm{an}}\longmapsto X^{k_1-k_2+1}f,$$

and

$$Xf(g)=\left(\frac{d}{dt}f\left(g\begin{pmatrix}1&&\\-t&1&\\&&1\end{pmatrix}\right)\right)_{t=0}.$$

Proof. Let us first check that d_κ is well defined. Indeed, using (k_1-k_2+1)-times the formula

$$\begin{aligned}(Xf)&\left(g\begin{pmatrix}t_1&&\\&t_2&\\&&1\end{pmatrix}\right)\\&=\left(\frac{d}{dt}f\left(g\begin{pmatrix}t_1&&\\&t_2&\\&&1\end{pmatrix}\begin{pmatrix}1&&\\-t&1&\\&&1\end{pmatrix}\begin{pmatrix}t_1^{-1}&&\\&t_2^{-1}&\\&&1\end{pmatrix}\begin{pmatrix}t_1&&\\&t_2&\\&&1\end{pmatrix}\right)\right)_{t=0}\\&=\left(\frac{d}{dt}f\left(g\begin{pmatrix}1&&\\-t_2t_1^{-1}t&1&\\&&1\end{pmatrix}\right)t_1^{k_1}t_2^{k_2}\right)_{t=0}=t_2t_1^{-1}(Xf)(g)t_1^{k_1}t_2^{k_2}=t_1^{k_1-1}t_2^{k_2+1}(Xf)(g)\end{aligned}$$

(and the corresponding formula for the action of $\left(\begin{smallmatrix}1&u&\\&1&\\&&1\end{smallmatrix}\right)$), we deduce that $d_\kappa f$ has the right weight.

We can check (evaluating on $\left(\begin{smallmatrix}1&&\\u&1&\\&&1\end{smallmatrix}\right)$) that on $\mathcal{F}^{\mathrm{an}}(p\mathcal{O},L)$ d_κ correspond to $(d/dz)^{k_1-k_2+1}$, where z is the variable on $p\mathcal{O}$. Thus, using the previous identification with V_κ and polynomials of degree less or equal than k_1-k_2, we deduce can check that V_κ is exactly the kernel of d_κ. □

Remark 4.5. A more general version of the previous proposition has been developed by Urban, [Urb11, Proposition 3.2.12], and Jones [Jon11].

5. Hasse Invariants and the canonical subgroups

Let p be a prime. Fix $E\subset\overline{E}\subset\mathbb{C}$ an algebraic closure of E and fix an isomorphism $\mathbb{C}\simeq\overline{\mathbb{Q}_p}$. Call $\tau,\sigma\tau$ the two places of $\overline{\mathbb{Q}_p}$ that correspond respectively to $\tau_\infty,\sigma\tau_\infty$ through the previous isomorphism (sometimes if p splits in E we will instead write $v=\tau$ and $\overline{v}=\sigma\tau$ following the notation of [BC04]). Suppose now p is inert in E. Let us take $K=K^pK_p\subset G(\mathbb{A}_f)$ a sufficiently small compact open, such that K_p is hyperspecial, and write $Y=Y_K/\operatorname{Spec}(\mathcal{O})$ the integral model of Kottwitz [Kot92] of the Picard Variety associated to the Shimura datum of the first section and the level K (recall $\mathcal{O}=\mathcal{O}_{E,p}$). Denote by $\mathcal{I}=\operatorname{Hom}(\mathcal{O},\mathbb{C}_p)=\{\tau,\sigma\tau\}$ the set of embeddings of $\mathcal{O}$ into an algebraic closure of $\mathbb{Q}_p$ ($\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$), where σ is the Frobenius morphism of $\mathcal{O}$, which acts transitively on $\mathcal{I}$, and $\mathrm{Gal}(E/\mathbb{Q}) = \{\mathrm{id}, \sigma\}$.

Recall the (toroidal compactification of the) Picard modular surface $X = X_K$ is the (compactified) moduli space of principally polarized abelian varieties $\mathcal{A} \longrightarrow S$ of genus 3, endowed with an action of $\mathcal{O}_E$, and a certain level structure K^p, and such that, up to extending scalars of S, we can decompose the conormal sheaf of $\mathcal{A}$ under the action of $\mathcal{O} = \mathcal{O}_{E,p}$,

$$\omega_{\mathcal{A}} = \omega_{\mathcal{A},\tau} \oplus \omega_{\mathcal{A},\sigma\tau},$$

and we assume $\dim_{\mathcal{O}_S} \omega_{\mathcal{A},\tau} = 1$ (and thus $\dim_{\mathcal{O}_S} \omega_{\mathcal{A},\sigma\tau} = 2$). These two sheaves extend from Y to X. One reason is that there is a semi-abelian scheme on X together with an action of $\mathcal{O}_E$, thus its conormal sheaf extend ω_A to the boundary, and the $\mathcal{O}_E$ action allows the splitting to make sense on the boundary too (see for example [Lan13, Theorem 6.4.1.1]).

Remark 5.1. If p splits in E, there is also a integral model of the Picard Surface, which is above $\mathrm{Spec}(\mathbb{Z}_p)$, and it has a similar description (of course in this case $\mathcal{O}_{E,p} \simeq \mathbb{Z}_p \times \mathbb{Z}_p$).

Remark 5.2. The case of toroidal compactifications of the Picard modular surface is particularly simple. Contrary to the case of other (PEL) Shimura varieties, there is a unique and thus canonical choice of a rational polyhedral cone decomposition Σ, and thus a unique toroidal compactification. This cone decomposition is the one of $\mathbb{R}_+$, it is smooth and projective. As a result, the toroidal compactification is smooth and projective.

5.1 Classical modular sheaves and geometric modular forms

On X, there is a sheaf ω, the conormal sheaf of $\mathcal{A}$, the universal (semi-)abelian scheme, along its unit section, and $\omega = \omega_\tau \oplus \omega_{\sigma\tau}$.

For any $\kappa = (k_1, k_2, k_3) \in \mathbb{Z}^3$ such that $k_1 \geqslant k_2$, there is associated a 'classical' modular sheaf,

$$\omega^\kappa = \mathrm{Sym}^{k_1-k_2}\, \omega_{\sigma\tau} \otimes (\det \omega_{\sigma\tau})^{k_2} \otimes \omega_\tau^{k_3}.$$

Write $\kappa' = (-k_2, -k_1, -k_3)$; this is still a dominant weight, and $\kappa \mapsto \kappa'$ is an involution. There is another way to see the classical modular sheaves.

Write $\mathcal{T} = \mathrm{Hom}_{X,\mathcal{O}}(\mathcal{O}_X^2 \otimes \mathcal{O}_X, \omega)$ where $\mathcal{O}$ acts by $\sigma\tau$ on the first two-dimensional factor and τ on the other one. Write $\mathcal{T}^\times = \mathrm{Isom}_{X,\mathcal{O}}(\mathcal{O}_X^2 \otimes \mathcal{O}_X, \omega)$, the $\mathrm{GL}_2 \times \mathrm{GL}_1$-torsor of trivializations of ω as a $\mathcal{O}$-module. There is an action on $\mathcal{T}$ of $\mathrm{GL}_2 \times \mathrm{GL}_1$ by $g \cdot w = w \circ g^{-1}$.

Denote by $\pi : \mathcal{T}^\times \longrightarrow X$ the projection. For any dominant κ as before, define

$$\omega^\kappa = \pi_* \mathcal{O}_{\mathcal{T}^\times}[\kappa'],$$

the subsheaf of κ'-equivariant functions for the action of the upper triangular Borel subgroup $B \subset \mathrm{GL}_2 \times \mathrm{GL}_1$. As the notation suggests, there is an isomorphism, if $\kappa = (k_1, k_2, k_3)$,

$$\pi_* \mathcal{O}_{\mathcal{T}^\times}[\kappa'] \simeq \mathrm{Sym}^{k_1-k_2}\, \omega_{\sigma\tau} \otimes (\det \omega_{\sigma\tau})^{k_2} \otimes \omega_\tau^{k_3}.$$

DEFINITION 5.3. Recall that X is the (compactified) Picard variety of level $K = K_p K^p$. The global sections $H^0(X, \omega^\kappa)$ is the module of Picard modular forms of level K and weight κ. If D denotes the (reduced) boundary of X, the submodule $H^0(X, \omega^\kappa(-D))$ is the submodule of Picard cusp-forms.

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In the sequel we will be interested in the case $K_p = I$, the Iwahori subgroup when we speak about modular forms on the rigid space associated to X. But for moduli-theoretic reason we will not directly consider an integral model of X_I^{rig}, but only an integral model of a particular open subset (the one given by the canonical filtration). Indeed, in this particular setting ($U(2,1)$ with p inert), the canonical filtration identifies a strict neighborhood of the μ-ordinary locus in X with an open in the level I-Picard variety.

Remark 5.4. There is a more general construction of automorphic sheaves $\omega^{k_1,k_2,k_3,r}$ given in [Har84], they are independent of r as sheaves on the Picard variety, and only the G-equivariant action (and thus the action of the Hecke operators) depends on r. We will only use the previous definition of the sheaves. We could get more automorphic forms by twisting by the norm character (which would be equivalent to twist the action of the Hecke operators).

5.2 Local constructions

Let G be the p-divisible group of the universal abelian scheme over $Y \subset X$. Later we will explain how to extend our construction to all X. The group G is endowed with an action of $\mathcal{O}$, and we have that its signature is given by

$$\begin{cases} p_\tau = 1, & q_\tau = 2, \\ p_{\sigma\tau} = 2, & q_{\sigma\tau} = 1, \end{cases}$$

which means that if we write $\omega_G = \omega_{G,\sigma\tau} \oplus \omega_{G,\tau}$, the two pieces have respective dimensions $p_{\sigma\tau} = 2$ and $p_\tau = 1$. Moreover, G carries a polarization λ, such that $\lambda : G \xrightarrow{\sim} G^{D,(\sigma)}$ is $\mathcal{O}$-equivariant.

The main result of [Her18], see also [GN17], is the following,

Definition 5.5. There exists sections,

$$\widetilde{\mathrm{ha}_{\sigma\tau}} \in H^0(Y \otimes \mathcal{O}/p, \det(\omega_{G,\sigma\tau})^{\otimes(p^2-1)}) \quad \text{and} \quad \widetilde{\mathrm{ha}_\tau} \in H^0(Y \otimes \mathcal{O}/p, (\omega_{G,\tau})^{\otimes(p^2-1)}),$$

such that $\widetilde{\mathrm{ha}_\tau}$ is given by (the determinant of) V^2,

$$\omega_{G,\tau} \xrightarrow{V} \omega_{G,\sigma\tau}^{(p)} \xrightarrow{V} \omega_{G,\tau}^{(p^2)},$$

and $\widetilde{\mathrm{ha}_\sigma \tau}$ is given by a division by p on the Dieudonne crystal of G of V^2, restricted to a lift of the Hodge Filtration $\omega_{G^D,\sigma\tau}$.

On $Y \otimes \mathcal{O}/p$, the sheaves $\omega_{G,\tau}$ and ω_τ (respectively $\omega_{G,\sigma\tau}$ and $\omega_{\sigma\tau}$) are isomorphic. In fact, the previous sections extends to all X, for example by Koecher's principle, see [Lan17, Theorem 8.7].

Remark 5.6. (i) These sections are Cartier divisors on X, i.e. they are invertible on an open and dense subset (cf. [Her18, Proposition 3.22] and [Wed99]).

(ii) Because of the $\mathcal{O}$-equivariant isomorphism $\lambda : G \simeq G^{D,(\sigma)}$, and the compatibility of $\widetilde{\mathrm{ha}_\tau}$ with duality (see [Her18, § 1.10]), we deduce that

$$\widetilde{\mathrm{ha}_\tau}(G) = \widetilde{\mathrm{ha}_\tau}(G^D) = \widetilde{\mathrm{ha}_\tau}(G^{(\sigma)}) = \widetilde{\mathrm{ha}_{\sigma\tau}}(G).$$

Thus, we could use only $\widetilde{\mathrm{ha}_{\sigma\tau}}$ or $\widetilde{\mathrm{ha}_\tau}$ and define it in this case without using any crystalline construction. We usually write ${}^\mu\mathrm{ha} = \widetilde{\mathrm{ha}_\tau} \otimes \widetilde{\mathrm{ha}_{\sigma\tau}}$, but because of the this remark, we will only use $\widetilde{\mathrm{ha}_\tau}$ in this article (which is then reduced, see the appendix).

(iii) We use the notation $\widetilde{\bullet}$ to denote the previous global sections, but if we have $G/\operatorname{Spec}(\mathcal{O}_{\mathbb{C}_p}/p)$ a p-divisible $\mathcal{O}$-module of signature $(2,1)$, we will also use the notation $\mathrm{ha}_\tau(G) = v(\widetilde{\mathrm{ha}_\tau(G)})$, where the valuation v on $\mathcal{O}_{\mathbb{C}_p}$ is normalized such that $v(p) = 1$ and truncated by 1.

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Definition 5.7. We denote by $\overline{X}^{\mathrm{ord}}$ the μ-ordinary locus of $\overline{X} = X \otimes_{\mathcal{O}} \mathcal{O}/p$, which is $\{x \in \overline{X} : \widetilde{\mathrm{ha}_\tau}$ is invertible$\}$. It is open and dense (see [Wed99]).

Let us recall the main theorem of [Her19] in the simple case of Picard varieties. Recall that p still denotes a prime, inert in E, and suppose $p > 2$.

Theorem 5.8. *Let $n \in \mathbb{N}^\times$. Let $H/\operatorname{Spec}(\mathcal{O}_L)$, where L is a valued extension of $\mathbb{Q}_p$, a truncated p-divisible $\mathcal{O}$-module of level $n+1$ and signature $(p_\tau = 1, p_{\sigma\tau} = 2)$. Suppose*

$$\mathrm{ha}_\tau(H) < \frac{1}{4p^{n-1}}.$$

Then there exists a unique filtration (so-called 'canonical' of height n) of $H[p^n]$,

$$0 \subset H_\tau^n \subset H_{\sigma\tau}^n \subset H[p^n],$$

by finite flat sub-$\mathcal{O}$-modules of $H[p^n]$, of $\mathcal{O}$-heights n and $2n$ respectively. Moreover,

$$\deg_{\sigma\tau}(H_{\sigma\tau}^n) + p\deg_\tau(H_{\sigma\tau}^n) \geqslant n(p+2) - \frac{p^{2n}-1}{p^2-1}\,\mathrm{ha}_\tau(H),$$

and

$$\deg_\tau(H_\tau^n) + p\deg_{\sigma\tau}(H_\tau^n) \geqslant n(2p+1) - \frac{p^{2n}-1}{p^2-1}\,\mathrm{ha}_\tau(H).$$

In particular, the groups H_τ^n and $H_{\sigma\tau}^n$ are of high degree. In addition, points of H_τ^n coincide with the kernel of the Hodge–Tate map $\alpha_{H[p^n],\tau,n-((p^{2n}-1)/(p^2-1))\,\mathrm{ha}_\tau(H)}$ and $H_{\sigma\tau}^n$ with the one of $\alpha_{H[p^n],\sigma\tau,n-((p^{2n}-1)/(p^2-1))\,\mathrm{ha}_\tau(H)}$. They also coincide with steps of the Harder–Narasimhan filtration and are compatible with p^s-torsion ($s \leqslant n$) and quotients.

Proof. This is exactly [Her19] if $p \geqslant 5$. Here the bound on p is slightly better than the one of [Her19]. The reason is that we construct $H_{\sigma\tau}^n$ using [Her19, Théorème 8.3 and Remarque 8.4] (with our bound for ha_τ). We can then construct $D_\tau^n \subset G[p^n]^D$ of $\mathcal{O}$-height $2n$ by the same reason. Then setting

$$H_\tau^n := (D_\tau^n)^\perp = (G[p^n]^D/D_\tau^n)^D \subset G[p^n],$$

we can have the asserted bound on $\deg_\tau H_\tau^n$. For the assertion on the Kernel of the Hodge–Tate map, we get an inclusion with the bound on the degree. We can then use [Her19, Proposition 7.6]. Then by [Her19, Remarque 8.4], these groups are steps of the Harder–Narasimhan filtration (the classical one), and thus we get the assertion of the inclusion $H_\tau^n \subset H_{\sigma\tau}^n$. By unicity of the Harder–Narasimhan filtration, we get that $(H_\tau^n)^\perp = H_{\sigma\tau}^n$. $\square$

Definition 5.9. Let $H/\operatorname{Spec}(\mathcal{O}_L)$ as before, with $n = 2m$. Then we can consider inside $H[2m]$ the finite flat subgroup,

$$K_m = H_\tau^{2m} + H_{\sigma\tau}^m.$$

It coincides, after reduction to $\operatorname{Spec}(\mathcal{O}_L/\pi_L)$ (the residue field of L) with the kernel of F^{2m} of $H[p^{2m}]$ (see [Her19, § 2.9.1]).

Recall that we denoted by $X/\operatorname{Spec}(\mathcal{O})$ the (schematic) Picard surface. Denote by X^{rig} the associated rigid space over E_p, there is a specialization map,

$$\mathrm{sp} : X^{\mathrm{rig}} \longrightarrow \overline{X},$$

and we denote by $X^{\mathrm{ord}} \subset X^{\mathrm{rig}}$ the open subspace defined by $\mathrm{sp}^{-1}(\overline{X}^{\mathrm{ord}})$.

Let us denote, for $v \in (0,1]$,

$$X(v) = \{x \in X^{\mathrm{rig}} : \mathrm{ha}_\tau(x) = v(\widetilde{\mathrm{ha}_\tau}(G_x)) < v\} \quad \text{and} \quad X(0) = X^{\mathrm{ord}},$$

the strict neighborhoods of X^{ord}. The previous theorem and technics introduced in [Far10] (see [Her19, § 2.9]) implies, if $v \leqslant 1/4p^{n-1}$, that we have a filtration in families over the rigid space $X(v)$,

$$0 \subset H_\tau^n \subset H_{\sigma\tau}^n \subset G[p^n].$$

Let us explain how to get this result. On $Y^{\mathrm{rig}}(v)$ this is simply [Her19, Théorème 9.1] which is essentially [Far10, Théorème 4] (again, see proof of Theorem 5.8 about the bound). The problem is that the p-divisible group G does not extend to the boundary. But by results of Stroh [Str10, § 3.1], there exists

$$\overline{U} \longrightarrow X^{\mathrm{rig}},$$

which is an étale covering of X^{rig} (actually $\overline{U}$ is algebraic and exists also integrally) together with $\overline{R} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \overline{U}$ étale maps, such that

$$X^{\mathrm{rig}} \simeq [\overline{U}/\overline{R}].$$

Over $\overline{U}$ there is a Mumford 1-motive $M = [L \longrightarrow \widetilde{G}]$ such that $M[p^n] = A[p^n]$ (A is the semi-abelian scheme), and thus there is a canonical $\widetilde{G}$ semi-abelian scheme with an action of $\mathcal{O}_E$ of (locally) constant toric rank and thus $\widetilde{G}[p^n]$ is finite flat. Thus applying to $\widetilde{G}[p^n]$ the results of [Her19], there exists on $\overline{U}(v) = \overline{U} \times_X X(v)$ the two groups $H_\tau^n \subset H_{\sigma\tau}^n$. Moreover, over $R(v) \underset{p_2}{\overset{p_1}{\rightrightarrows}} U(v)$, we have $pr_1^* H_\star^n = pr_2^* H_\star^n$ as $H_\star^n$ descend to $Y(v)$. Over $\overline{R}$ we have an isomorphism $pr_1^* G = pr_2^* G$ [Str10, 3.1.5], and thus we have over $R(v)$ an isomorphism

$$G/pr_1^* H_\star^n \xrightarrow{\sim} G/pr_2^* H_\star^n.$$

As $\overline{R}$ is normal (because smooth) and R is dense in $\overline{R}$, by [Str10, Théorème 1.1.2] (due to Faltings and Chai) there is an isomorphism over $\overline{R}$

$$G/pr_1^* H_\star^n \xrightarrow{\sim} G/pr_2^* H_\star^n.$$

Thus, by faithfully flat descent, $H_\tau^n \subset H_{\sigma\tau}^n$ exist on $X(v)$. In particular, $K_m = H_{\sigma\tau}^m + H_\tau^{2m}$ also exists on $X(v)$.

A priori, this filtration does not extend to a formal model of $X(v)$, but as $X/\operatorname{Spec}(\mathcal{O})$ is a normal scheme, we will be able to use the following proposition.

DEFINITION 5.10. For $K/\mathbb{Q}_p$ an extension, define the category $\mathfrak{Adm}$ of admissible $\mathcal{O}_K$-algebra, i.e. flat quotient of power series ring $\mathcal{O}_K\langle\langle X_1, \ldots, X_r\rangle\rangle$ for some $r \in \mathbb{N}$. Define $\mathfrak{NAdm}$ the sub-category of normal admissible $\mathcal{O}_K$-algebra.

Proposition 5.11. *Let m be an integer, $S = \mathrm{Spf}\, R$ a normal formal scheme over $\mathcal{O}$, and $G \longrightarrow S$ a truncated p-divisible $\mathcal{O}$-module of level $2m+1$ and signature $(p_\tau = 1, p_{\sigma\tau} = 2)$. Suppose that for all $x \in S^{\mathrm{rig}}$, $\mathrm{ha}_\tau(x) < 1/4p^{2m-1}$. Then the subgroup $K_m := H_\tau^{2m} + H_{\sigma\tau}^m \subset G[p^{2m}]$ of S^{rig} extends to S.*

Proof. As we know that K_m coincides with the kernel of some Frobenius morphism on points, this is exactly as [AIP15, Proposition 4.1.3]. □

6. Construction of torsors

6.1 Hodge–Tate map and image sheaves

Let p a prime, inert in E, and $\mathcal{O} = \mathcal{O}_{E,p}$, a degree-2 unramified extension of $\mathbb{Z}_p$. Let K be a valued extension of E_p. Let $m \in \mathbb{N}^\times$ and $v < 1/4p^{2m-1}$. Let $S = \mathrm{Spec}(R)$ where R is an object of $\mathfrak{NAdm}/\mathcal{O}_K$, and $G \longrightarrow S$ a truncated p-divisible $\mathcal{O}$-module of level $2m$ and signature,

$$\begin{cases} p_\tau = 1, & q_\tau = 2, \\ p_{\sigma\tau} = 2, & q_{\sigma\tau} = 1, \end{cases}$$

where $\tau : \mathcal{O} \longrightarrow \mathcal{O}_S$ is the fixed embedding. Suppose moreover that for all $x \in S^{\mathrm{rig}}$, $\mathrm{ha}_\tau(x) \leqslant v$. According to the previous section, there exists on S^{rig} a filtration of $G[p^{2m}]$ by finite flat $\mathcal{O}$-modules,

$$0 \subset H_\tau^{2m} \subset H_{\sigma\tau}^{2m} \subset G[p^{2m}],$$

of $\mathcal{O}$-heights $2m$ and $4m$ respectively. Moreover, we have on S a subgroup $K_m \subset G[p^{2m}]$, finite flat of $\mathcal{O}$-height $3m$, étale-locally isomorphic (on S^{rig}) to $\mathcal{O}/p^{2m}\mathcal{O} \oplus \mathcal{O}/p^m\mathcal{O}$, and on S^{rig}, $K_m = H_\tau^{2m} + H_{\sigma\tau}^{2m}[p^m]$.

Proposition 6.1. *Let $w_\tau, w_{\sigma\tau} \in v(\mathcal{O}_K)$ such that $w_{\sigma\tau} < m - ((p^{2m}-1)/(p^2-1))v$ and $w_\tau < 2m - ((p^{4m}-1)/(p^2-1))v$. Then, the morphism of sheaves on S $\pi : \omega_G \longrightarrow \omega_{K_m}$, induce by the inclusion $K_m \subset G$, induces isomorphisms,*

$$\pi_\tau : \omega_{G,\tau,w_\tau} \xrightarrow{\sim} \omega_{K_m,\tau,w_\tau} \quad \text{and} \quad \pi_{\sigma\tau} : \omega_{G,\sigma\tau,w_{\sigma\tau}} \xrightarrow{\sim} \omega_{K_m,\sigma\tau,w_{\sigma\tau}}.$$

Proof. If $G/\mathrm{Spec}(\mathcal{O}_C)$ (C a complete algebraically closed extension of $\mathbb{Q}_p$), the degrees of the canonical filtration of G assure that

$$\deg_{\sigma\tau}(G[p^m]/H_{\sigma\tau}^m) \geqslant \frac{p^{2m}-1}{p^2-1}v \quad \text{and} \quad \deg_\tau(G[p^{2m}]/H_\tau^{2m}) \geqslant \frac{p^{4m}-1}{p^2-1}v,$$

and there is thus an isomorphism,

$$\omega_{G[p^m],\tau,w_\tau} \xrightarrow{\sim} \omega_{H_\tau^n,\tau,w_\tau},$$

and also for $\sigma\tau$ and $G[p^{2m}]$. But there are inclusions $H_{\sigma\tau}^m = H_{\sigma\tau}^{2m}[p^m] \subset K_m \subset G$ and $H_\tau^{2m} \subset K_m \subset G$ such that the composite,

$$\omega_{G,\sigma\tau,w_{\sigma\tau}} \longrightarrow \omega_{K_m,\sigma\tau,w_{\sigma\tau}} \longrightarrow \omega_{H_{\sigma\tau}^m,\sigma\tau,w_{\sigma\tau}},$$

is an isomorphism, which implies that the first one is. The same reasoning applies for τ. We can thus conclude the following for general S as in [AIP15, Proposition 4.2.1]. Up to reducing R we can suppose ω_G is a free R/p^{2m+1}-module, and look at the surjection $\alpha_{\sigma\tau} : R^2 \twoheadrightarrow \omega_{G,\sigma\tau} \twoheadrightarrow \omega_{K_m,\sigma\tau,w_{\sigma\tau}}$; it is enough to prove that for any (x_1, x_2) in $\ker \alpha_{\sigma\tau}$ we have $x_i \in p^{w_\tau}R$, but, as R is normal, it suffices to do it for $R_\mathfrak{p}$, and even for $\widehat{R_\mathfrak{p}}$, for all codimension-1 prime ideals $\mathfrak{p}$ that contain (p). But now $\widehat{R_\mathfrak{p}}$ is a complete, discrete valuation ring of mixed characteristic, and this reduces to the preceding assertion. □

PROPOSITION 6.2. *Suppose there is an isomorphism* $K_m^D(R) \simeq \mathcal{O}/p^m\mathcal{O} \oplus \mathcal{O}/p^{2m}\mathcal{O}$. *Then the cokernel of the* $\sigma\tau$*-Hodge–Tate map,*

$$\mathrm{HT}_{K_m^D,\sigma\tau} \otimes 1 : K_m^D(R)[p^m] \otimes_{\mathcal{O}} R \longrightarrow \omega_{K_m,\sigma\tau},$$

is killed by $p^{(p+v)/(p^2-1)}$, *and the cokernel of the* τ*-Hodge–Tate map,*

$$\mathrm{HT}_{K_m^D,\tau} \otimes 1 : K_m^D(R)[p^m] \otimes_{\mathcal{O}} R \longrightarrow \omega_{K_m,\tau},$$

is killed by $p^{v/(p^2-1)}$.

Proof. This is true for $G/\operatorname{Spec}(\mathcal{O}_C)$ by [Her19, Théorème 6.10(2)] with the previous proposition (because $(p+v)/(p^2-1) < 1-v$, already for $m=1$). For a general normal R, we can reduce to previous case (see also [AIP15, Proposition 4.2.2]): up to reducing $\operatorname{Spec}(R)$, we have a diagram,

$$\begin{tikzcd} R^2 \arrow[r, "\gamma"] \arrow[d] & R^2 \arrow[d] \\ K_n^D(R)[p^n] \otimes_{\mathcal{O}} R \arrow[r, "\mathrm{HT}_{\sigma\tau} \otimes 1"] & \omega_{K_n,\sigma\tau} \end{tikzcd}$$

and $\mathrm{Fitt}^1(\gamma)$ (which is just a determinant here) annihilates the cokernel of γ, and it suffices to prove that $p^{(p+v)/(p^2-1)} \in \mathrm{Fitt}^1(\gamma)$. But as R is normal, it suffices to prove that $p^{(p+v)/(p^2-1)} \in \mathrm{Fitt}^1(\gamma)R_{\mathfrak{p}}$ for every codimension-1 prime ideal $\mathfrak{p}$ that contains (p). But by the previous case, we can conclude. The same works for τ. □

PROPOSITION 6.3. *Suppose we have an isomorphism* $K_m^D(R) \simeq \mathcal{O}/p^m\mathcal{O} \oplus \mathcal{O}/p^{2m}\mathcal{O}$. *Then there exist on* $S = \operatorname{Spec} R$ *locally free subsheaves* $\mathcal{F}_{\sigma\tau}, \mathcal{F}_\tau$ *of* $\omega_{G,\sigma\tau}$ *and* $\omega_{G,\tau}$ *respectively, of ranks 2 and 1, which contain* $p^{(p+v)/(p^2-1)}\omega_{G,\sigma\tau}$ *and* $p^{v/(p^2-1)}\omega_{G,\tau}$, *and which are equipped, for all* $w_{\sigma\tau} < m - ((p^{2m}-1)/(p^2-1))v$ *and* $w_\tau < 2m - ((p^{4m}-1)/(p^2-1))v$, *with maps,*

$$\mathrm{HT}_{\sigma\tau,w_{\sigma\tau}} : K_m^D(R) \longrightarrow \mathcal{F}_{\sigma\tau} \otimes_R R_{w_{\sigma\tau}}, \quad \text{and} \quad \mathrm{HT}_{\tau,w_\tau} : K_m^D(R) \longrightarrow \mathcal{F}_\tau \otimes_R R_{w_\tau},$$

which are surjective after tensoring $K_m^D(R)$ *with* R *over* $\mathcal{O}$.

More precisely, via the projection,

$$K_m^D(R) \twoheadrightarrow (H_\tau^{2m})^D(R_K),$$

we have induced isomorphisms,

$$\mathrm{HT}_{\sigma\tau,w_{\sigma\tau}} : K_m^D(R_K) \otimes_{\mathcal{O}} R_{w_{\sigma\tau}} \longrightarrow \mathcal{F}_{\sigma\tau} \otimes_R R_{w_{\sigma\tau}},$$

and

$$\mathrm{HT}_{\tau,w_\tau} : (H_\tau^{2m})^D(R_K) \otimes_{\mathcal{O}} R_{w_\tau} \longrightarrow \mathcal{F}_\tau \otimes_R R_{w_\tau}.$$

Proof. This is the same construction as [AIP15, proposition 4.3.1]. To check the assertion about the isomorphism with H_τ^{2m}, it suffices to show that the map $\mathrm{HT}_{\tau,w_\tau}$ factors, but it is true over $R_{\mathfrak{p}}$ (as the canonical filtration is given by kernels of Hodge–Tate maps) for every codimension-1 ideal $\mathfrak{p}$, and it is moreover surjective, so it globally factors and is globally surjective, but the two free R_{w_τ}-modules are free of the same rank 1, so it is an isomorphism. □

Moreover the construction of the sheaves $\mathcal{F}$ is functorial in the following sense.

Proposition 6.4. *Suppose given G, G' two truncated p-divisible $\mathcal{O}$-module such that for all $x \in S^{\mathrm{rig}}$*

$$\mathrm{ha}_\tau(G_x), \mathrm{ha}_\tau(G'_x) < v,$$

and an $\mathcal{O}_E$-isogeny,

$$\phi : G \longrightarrow G'.$$

Assume moreover that we are given trivializations of the points of $K_m^D(G)$ and $K_m^D(G')$. Then ϕ^* induces maps

$$\phi_\tau^* : \mathcal{F}'_\tau \longrightarrow \mathcal{F}'_\tau \quad \text{and} \quad \phi_{\sigma\tau}^* : \mathcal{F}'_{\sigma\tau} \longrightarrow \mathcal{F}_{\sigma\tau},$$

that are compatible with inclusion in ω, reduction modulo p^w and the Hodge–Tate maps of K_m^D.

Proof. Once we know that ϕ will send $K_m^D(G)$ inside $K_m^D(G')$ this is straightforward as $\mathcal{F}_?$ corresponds to sections of $\omega_{G,?}$ that are modulo $p^{w_?}$ generated by the image of $\mathrm{HT}_?$. But K_m is generated by the subgroup $H_{\sigma\tau}^m$ and H_τ^{2m} each being a breakpoint of the Harder–Narasimhan filtration $\mathrm{HN}_{\sigma\tau}(G[p^m])$ and $\mathrm{HN}_\tau(G[p^{2m}])$ respectively, and thus by functoriality of these filtrations, ϕ sends each subgroup for G inside the one for G' and thus sends $K_m(G)$ inside $K_m(G')$. □

Remark 6.5. Strictly speaking, we cannot apply this section for $S^{\mathrm{rig}} \subset X(v)$ an affinoid. The reason is that even if $H_{\sigma\tau}^n, H_\tau^n$ descend to $X(v)$, it is not the case of $\widetilde{G}[p^n]$. But we can apply the results of this section with $\widetilde{G}[p^n]$ for S^{rig} an affinoid of $\overline{U}$. As $\omega_{\widetilde{G}} \simeq \omega \times_{X(v)} \overline{U}(v)$, it is enough to check Propositions 6.1 and 6.2 over $\overline{U}/\operatorname{Spec}(\mathcal{O})$. Thus Propositions 6.3 and 6.4 remain true over S^{rig} and affinoid of $X(v)$ (for Proposition 6.4 we will restrict anyway to the open Picard variety, but it would remain true looking at an isogeny of semi-abelian schemes).

6.2 The torsors

To simplify the notations, fix $w = w_\tau = w_{\sigma\tau} < m - ((p^{2m}-1)/(p^2-1))v$ to use the previous propositions. Let $R \in \mathcal{O}_K - \mathfrak{NAdm}$ and $S = \mathrm{Spf}(R)$. In rigid fiber, we have a subgroup of $K_m[p^m]/S^{\mathrm{rig}}$, $H_\tau^m \subset K_m[p^m]$ which induces a filtration,

$$0 \subset (H_{\sigma\tau}^m/H_\tau^m)^D(R_K) \subset K_m^D(R),$$

of cokernel isomorphic to $(H_\tau^m)^D(R_K)$.

Suppose we are given a trivialization,

$$\psi : \mathcal{O}/p^m\mathcal{O} \oplus \mathcal{O}/p^{2m}\mathcal{O} \simeq K_m^D(R),$$

which induces trivializations (first coordinate and quotient),

$$\psi_{\sigma\tau} : (H_{\sigma\tau}^m/H_\tau^m)^D(R_K) \simeq \mathcal{O}/p^m\mathcal{O} \quad \text{and} \quad \psi_\tau : (H_\tau^{2m})^D(R_K) \simeq \mathcal{O}/p^{2m}\mathcal{O}.$$

Let $\mathcal{G}r_{\sigma\tau} \longrightarrow S$ be the Grassmanian of locally direct factor sheaves of rank 1, $\mathrm{Fil}^1 \mathcal{F}_{\sigma\tau} \subset \mathcal{F}_{\sigma\tau}$. Let $\mathcal{G}r_{\sigma\tau}^+ \longrightarrow \mathcal{G}r_{\sigma\tau}$ be the $\mathbb{G}_m^2$-torsor of trivializations of $\mathrm{Fil}^1 \mathcal{F}_{\sigma\tau}$ and $\mathcal{F}_{\sigma\tau}/\mathrm{Fil}^1\mathcal{F}_{\sigma\tau}$. Also let $\mathcal{G}r_\tau^+ \longrightarrow S$ be the $\mathbb{G}_m$-torsor of trivializations of $\mathcal{F}_\tau$.

Definition 6.6. We say that a A-point of $\mathcal{G}r_{\sigma\tau}$ (respectively $\mathcal{G}r_{\sigma\tau}^+$ or $\mathcal{G}r_\tau^+$)

$$\mathrm{Fil}^1(\mathcal{F}_{\sigma\tau} \otimes A) \quad (\text{respectively } (\mathrm{Fil}^1(\mathcal{F}_{\sigma\tau} \otimes A), P_1^{\sigma\tau}, P_2^{\sigma\tau}) \text{ or } P^\tau)$$

is w-compatible with $\psi_\tau, \psi_{\sigma\tau}$ if, respectively:

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(i) $\mathrm{Fil}^1(\mathcal{F}_{\sigma\tau} \otimes A) \otimes_R R_w = \mathrm{HT}_{\sigma\tau,w}((H^m_{\sigma\tau}/H^m_\tau)^D(R_K) \otimes_{\mathcal{O}} R_w) \otimes_R A$;
(ii) $P_1^{\sigma\tau} \otimes_R R_w = \mathrm{HT}_{\sigma\tau,w} \circ (\psi_{\sigma\tau} \otimes_{\mathcal{O}} R_w) \otimes_R A$;
(iii) $P_2^{\sigma\tau} \otimes_R R_w = \mathrm{HT}_{\sigma\tau,w} \circ (\psi_\tau \otimes_{\mathcal{O}} R_w) \otimes_R A$;
(iv) $P^\tau \otimes_R R_w = \mathrm{HT}_{\tau,w} \circ (\psi_\tau \otimes_{\mathcal{O}} R_w) \otimes_R A$.

We can define the functors,

$$\mathfrak{IW}_{\sigma\tau,w} : \begin{array}{ccc} R-\mathfrak{Adm} & \longrightarrow & SET \\ A & \longmapsto & \{w-\text{compatible}\,\mathrm{Fil}^1(\mathcal{F}_{\sigma\tau} \otimes_R A) \in \mathcal{G}r_{\sigma\tau}(A)\}, \end{array}$$

$$\mathfrak{IW}^+_{\sigma\tau,w} : \begin{array}{ccc} R-\mathfrak{Adm} & \longrightarrow & SET \\ A & \longmapsto & \{w-\text{compatible}(\mathrm{Fil}^1(\mathcal{F}_{\sigma\tau} \otimes_R A), P_1^\tau, P_2^\tau) \in \mathcal{G}r^+_{\sigma\tau}(A)\}, \end{array}$$

$$\mathfrak{IW}^+_{\tau,w} : \begin{array}{ccc} R-\mathfrak{Adm} & \longrightarrow & SET \\ A & \longmapsto & \{w-\text{compatible}P^\tau \in \mathcal{G}r^+_\tau(A)\}. \end{array}$$

The previous functors are representable by formal schemes, affine over $S = \mathrm{Spf}(R)$, and locally isomorphic to

$$\begin{pmatrix} 1 & \\ p^w\mathfrak{B}(0,1) & 1 \end{pmatrix} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(R) \quad \text{for } \mathfrak{IW}_{\sigma\tau,w}, \quad 1 + p^w\mathfrak{B}(0,1) \quad \text{for } \mathfrak{IW}^+_{\tau,w}$$

where $\mathfrak{B}(0,1) = \mathrm{Spf}(\mathcal{O}_K\langle T\rangle)$ is the one-dimensional formal unit ball, and

$$\begin{pmatrix} 1+p^w\mathfrak{B}(0,1) & \\ p^w\mathfrak{B}(0,1) & 1+p^w\mathfrak{B}(0,1) \end{pmatrix} \times_{\mathrm{Spf}(O_K)} \mathrm{Spf}(R) \quad \text{for } \mathfrak{IW}^+_{\sigma\tau,w}.$$

We also define $\mathfrak{IW}^+_w = \mathfrak{IW}^+_{\tau,w} \times_S \mathfrak{IW}^+_{\sigma\tau,w}$. The previous constructions are independent of $n = 2m$ (because $\mathcal{F}_\tau, \mathcal{F}_{\sigma\tau}$ are).

Let $T^1 = \mathrm{Res}_{\mathcal{O}/\mathbb{Z}_p} \mathbb{G}_m \times U(1)$ the torus of $U(2,1)$ over $\mathbb{Z}_p$ whose $\mathbb{Z}_p$-points are $\mathcal{O}^\times \times \mathcal{O}^1$. Its scalar extension $T = T^1 \otimes_{\mathbb{Z}_p} \mathcal{O}$ is isomorphic to $\mathbb{G}_m^3$, and $\mathcal{G}r^+ = \mathcal{G}r^+_\tau \times \mathcal{G}r^+_{\sigma\tau} \longrightarrow \mathcal{G}r = \mathcal{G}r_\tau$ is a T-torsor. Denote by $\mathfrak{T} \longrightarrow \mathrm{Spf}(\mathcal{O})$ the formal completion of T along its special fiber, and $\mathfrak{T}_w$ the torus defined by

$$\mathfrak{T}_w(A) = \mathrm{Ker}(\mathfrak{T}(A)) \longrightarrow \mathfrak{T}(A/p^wA).$$

Then $\mathfrak{IW}^+_w \longrightarrow \mathfrak{IW}_{\sigma\tau,w}$ is a $\mathfrak{T}_w$-torsor.

Denote by $\mathcal{IW}_{\sigma\tau,w}, \mathcal{IW}^+_{\tau,w}, \mathcal{IW}^+_{\sigma\tau,w}, \mathcal{IW}^+_w, \mathcal{T}$ the generic fibers of the previous formal schemes.

7. The Picard surface and overconvergent automorphic sheaves

7.1 Constructing automorphic sheaves

Let us consider the datum $(E, V, \psi, \mathcal{O}_E, \Lambda = \mathcal{O}_E^3, h)$, the PEL datum introduced in § 2. Let p be a prime, inert in E and G the reductive group associated over $\mathbb{Z}_p$. We fix K^p a compact open subgroup of $G(\mathbb{A}_f^p)$ sufficiently small and $\mathfrak{C} = G(\mathbb{Z}_p)$ an hyperspecial subgroup at p. Let $X = X_{K^p\mathfrak{C}}/\mathrm{Spec}(\mathcal{O})$ the (integral) Picard variety associated to the previous datum (cf. [Kot92, Lan13, LR92]).

Let $K/\mathcal{O}[1/p]$ be a finite extension (that we will choose sufficiently large) and still write $X = X_{\mathcal{O}_K} = X \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}(\mathcal{O}_K)$.

Denote by A the universal semi-abelian scheme, X^{rig} the rigid fiber of X, X^{ord} the ordinary locus and for $v \in v(K)$, $X(v)$ the rigid-analytic open $\{x \in X^{\mathrm{rig}} : \mathrm{ha}_\tau(x) < v\}$. Denote also $\mathfrak{X} \longrightarrow \mathrm{Spf}(\mathcal{O}_K)$ the formal completion of X along its special fiber, $\widetilde{\mathfrak{X}(v)}$ the admissible blow up of $\mathfrak{X}$ along the ideal $(\widetilde{\mathrm{ha}_\tau}, p^v)$ and $\mathfrak{X}(v)$ its open subscheme where $(\widetilde{\mathrm{ha}_\tau}, p^v)$ is generated by $\widetilde{\mathrm{ha}_\tau}$.

Lemma 7.1. *The formal scheme $\mathfrak{X}(v)$ is normal.*

Proof. As $X(v)$ is smooth, thus normal, and $\widetilde{\mathrm{ha}_\tau}$ is reduced, this follows from the next lemma.

Lemma 7.2. *Let $A \in \mathcal{O}_K - \mathfrak{Adm}$ such that A_K is normal and A/π_K is reduced. Then A is normal.*

Proof. For R a (noetherian) ring denote by R^{norm} its integral closure in its total ring of fractions. Write, for all $x \in A_K$,

$$v_A(x) = \sup\{n \in \mathbb{Z} : \pi_K^{-n} x \in A\} \quad \text{and} \quad |x|_A = \pi_K^{-v_A(x)}.$$

Then we can check that $|f+g|_A \leqslant \sup(|f|_A, |g|_A)$ and that $|f^n|_A = |f|_A^n$. Indeed, $\pi_K^{-v_A(f)} f \in A \backslash \pi_K A$. Thus, as $A/\pi_K A$ is reduced, $(\pi_K^{-v_A(f)} f)^n \in A \backslash \pi_K A$ and thus $v_A(f^n) = n v_A(f)$. For this norm, we have that

$$A = \{x \in A_K : |x|_A \leqslant 1\}.$$

Now let us verify that A is normal. Let $x \in A^{\mathrm{norm}}$, in particular, $x \in A_K^{\mathrm{norm}}$. However, as A_K is normal, $x \in A_K$. Now write, for $a_i \in A, 0 \leqslant i \leqslant n$ and $a_n = 1$,

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x = -a_0.$$

Then $\sup_{1 \leqslant i \leqslant n} |x^i a_{n-i}|_A = |a_0|_A \leqslant 1$, thus $|x^n|_A = |x|_A^n \leqslant 1$, and $|x|_A \leqslant 1$. Thus $x \in A$. □

By the previous sections, we have on $X(v)$ a filtration of G by finite flat $\mathcal{O}$-modules,

$$0 \subset H_\tau^{2m} \subset H_{\sigma\tau}^{2m} \subset G[p^{2m}],$$

locally isomorphic to $\mathcal{O}/p^{2m}\mathcal{O}$ and $(\mathcal{O}/p^{2m}\mathcal{O})^2$. Moreover, the subgroup $K_m = H_\tau^{2m} + H_{\sigma\tau}^{2m}[p^m]$ extends to $\mathfrak{X}(v)$ by Proposition 5.11, and over $X(v)$ is locally isomorphic to

$$\mathcal{O}/p^{2m}\mathcal{O} \oplus \mathcal{O}/p^m\mathcal{O}.$$

Definition 7.3. We write

$$X_1(p^{2m})(v) = \mathrm{Isom}_{X(v),\mathrm{pol}}(K_m^D, \mathcal{O}/p^m\mathcal{O} \oplus \mathcal{O}/p^{2m}\mathcal{O}),$$

where the condition *pol* means that we are looking at isomorphisms $\psi = (\psi_1, \psi_2)$ which induce an isomorphism 'in first coordinate',

$$\psi_{1,1} = (\psi_1)_{|(H_{\sigma\tau}^m/H_\tau^m)^D} : (H_{\sigma\tau}^m/H_\tau^m)^D \simeq \mathcal{O}/p^m\mathcal{O},$$

such that $(\psi_{1,1})^D = ((\psi_{1,1})^{(\sigma)})^{-1}$, and such that the quotient morphism,

$$\psi_{2,1} = \psi_1/(\psi_1)_{|(H_{\sigma\tau}^m/H_\tau^m)^D} : (H_\tau^{2m})^D \longrightarrow \mathcal{O}/p^m\mathcal{O},$$

is zero.

Remark 7.4. The map $\psi_{1,1}$ is automatically an isomorphism. Moreover,

$$(\psi_{1,1})^D : \mathcal{O}/p^m\mathcal{O} \longrightarrow H_{\sigma\tau}^m/H_\tau^m \overset{\lambda}{\simeq} (H_{\sigma\tau}^m/H_\tau^m)^{D,(\sigma)},$$

where the last morphism is induced by λ, the polarization of A.

Denote by B_n the subgroup of $\mathrm{GL}(\mathcal{O}/p^m\mathcal{O} \oplus \mathcal{O}/p^{2m}\mathcal{O})$ of matrices,

$$\begin{pmatrix} a & p^m b \\ 0 & d \end{pmatrix}$$

such that $a^{-1} = a^{(\sigma)}$ i.e. $a \in (\mathcal{O}/p^m\mathcal{O})^1$. We can map $\mathcal{O}^\times \times \mathcal{O}^1$ (diagonally) to B_n.

$$B_n \simeq \begin{pmatrix} (\mathcal{O}/p^m\mathcal{O})^1 & \mathcal{O}/p^m\mathcal{O} \\ & (\mathcal{O}/p^{2m}\mathcal{O})^\times \end{pmatrix}.$$

Write also

$$B_\infty(\mathbb{Z}_p) = \begin{pmatrix} \mathcal{O}^1 & \mathcal{O} \\ 0 & \mathcal{O}^\times \end{pmatrix}$$

which surjects to B_n and that we can embed into $\mathrm{GL}_2 \times \mathrm{GL}_1$ (even in its upper triangular Borel subgroup) via

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \longmapsto \begin{pmatrix} \sigma\tau(a) & \sigma\tau(b) & \\ & \sigma\tau(d) & \\ & & \tau(d) \end{pmatrix}.$$

We denote by ψ_τ and $\psi_{\sigma\tau}$ the inverses of the induced isomorphisms,

$$\psi_{\sigma\tau} : \mathcal{O}/p^m\mathcal{O} \simeq (H^m_{\sigma\tau}/H^m_\tau)^D,$$

and the quotient,

$$\psi_\tau = \psi^{-1}/\psi_{\sigma\tau} : \mathcal{O}/p^{2n}\mathcal{O} \simeq (H^{2m}_\tau)^D.$$

We also denote by $\mathfrak{X}_1(p^{2m})(v)$ the normalization of $\mathfrak{X}(v)$ in $X_1(p^{2m})(v)$. Over $\mathfrak{X}_1(p^{2m})(v)$, we have, by the previous section, locally free subsheaves of $\mathcal{O}_{\mathfrak{X}_1(p^{2m})(v)}$-modules $\mathcal{F}_\tau, \mathcal{F}_{\sigma\tau}$ of $\omega_{G,\tau}$ and $\omega_{G,\sigma\tau}$ together with morphisms,

$$\mathrm{HT}_{\tau,w} \circ \psi_\tau[p^m] : (\mathcal{O}/p^m\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{X}_1(p^{2m})(v)} \xrightarrow{\sim} \mathcal{F}_\tau \otimes_{\mathcal{O}_K} \mathcal{O}_K/p^w,$$

$$\mathrm{HT}_{\sigma\tau,w} \circ \psi_{\sigma\tau} : (\mathcal{O}/p^m\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{X}_1(p^{2m})(v)} \hookrightarrow \mathcal{F}_{\sigma\tau} \otimes_{\mathcal{O}_K} \mathcal{O}_K/p^w,$$

and denote by $\mathcal{F}^{\mathrm{can}}_{\sigma\tau,w}$ the image of the second morphism. It is a locally direct factor of $\mathcal{F}_{\sigma\tau} \otimes \mathcal{O}_K/p^w$, and, passing through the quotient, we get a map,

$$\overline{\mathrm{HT}_{\sigma\tau,w}} \circ \psi_\tau[p^m] : (\mathcal{O}/p^m\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{X}_1(p^m)(v)} \xrightarrow{\sim} (\mathcal{F}_{\sigma\tau} \otimes_{\mathcal{O}_K} \mathcal{O}_K/p^w)/(\mathcal{F}^{\mathrm{can}}_{\sigma\tau,w}).$$

Using the construction of torsors of the previous section, we get a chain of maps,

$$\mathfrak{IW}^+_w \xrightarrow{\pi_1} \mathfrak{IW}_w \xrightarrow{\pi_2} \mathfrak{X}_1(p^{2m})(v) \xrightarrow{\pi_3} \mathfrak{X}(v).$$

Moreover, π_1 is a torsor over the formal torus $\mathfrak{T}_w$, π_2 is affine, and we have an action of $\mathcal{O}^\times \times \mathcal{O}^1$ and B_n on $\mathfrak{X}_1(p^{2m})$ over $\mathfrak{X}(v)$. Denote by B the Borel subgroup of $\mathrm{GL}_2 \times \mathrm{GL}_1$, $\mathfrak{B}$ its formal completion along its special fiber, and $\mathfrak{B}_w$,

$$\mathfrak{B}_w(A) = \mathrm{Ker}(\mathfrak{B}(A) \longrightarrow \mathfrak{B}(A/p^w A)).$$

We can embed $\mathfrak{T}$ in $\mathfrak{B}$ (which induce an embedding $\mathfrak{T}_w \subset \mathfrak{B}_w$) and $\mathcal{O}^\times \times \mathcal{O}^1$ in $\mathfrak{T}$, via

$$(a,b) \in \mathcal{O}^\times \times \mathcal{O}^1 \longmapsto \begin{pmatrix} \sigma\tau(b) & & \\ & \sigma\tau(a) & \\ & & \tau(a) \end{pmatrix} \in \mathfrak{T},$$

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such that the action of $\mathcal{O}^\times \times \mathcal{O}^1$ on $X_1(p^{2m})(v)$ and via $\mathfrak{T}$ on $\mathcal{G}r^+$ preserves $\mathfrak{IW}_w^+$ (over $\mathfrak{X}(v)$). More generally, the action of $B_\infty(\mathbb{Z}_p)$ on $X_1(p^{2m})(v)$ (and thus $\mathfrak{X}_1(p^{2m})(v)$) and via $\mathfrak{B}$ on $\mathcal{G}r^+$ preserves $\mathfrak{IW}_w^+$.

Let $\kappa \in \mathcal{W}_w(L)$. The character $\kappa : \mathcal{O}^\times \times \mathcal{O}^1 \longrightarrow \mathcal{O}_L^\times$ extends to a character,

$$\kappa : (\mathcal{O}^\times \times \mathcal{O}^1)\mathfrak{T}_w \longrightarrow \widehat{\mathbb{G}_m},$$

which can be extended as a character of

$$\kappa : (\mathcal{O}^\times \times \mathcal{O}^1)\mathfrak{B}_w \longrightarrow \widehat{\mathbb{G}_m},$$

where $\mathfrak{U}_w \subset \mathfrak{B}_w$ acts trivially, and even as a character,

$$\kappa : B(\mathbb{Z}_p)\mathfrak{B}_w \longrightarrow \widehat{\mathbb{G}_m},$$

where $U(\mathbb{Z}_p)\mathfrak{U}_w$ acts trivially. Let us write $\pi = \pi_3 \circ \pi_2 \circ \pi_1$.

Proposition 7.5. *The sheaf $\pi_*\mathcal{O}_{\mathfrak{IW}_w^+}[\kappa]$ is a formal Banach sheaf, in the sense of* [AIP15, Definition A.1.2.1].

Proof. We can use the same dévissage as presented in [AIP15]: denote by κ^0 the restriction of κ to $\mathfrak{T}_w$. Then $(\pi_1)_*\mathcal{O}_{\mathfrak{IW}_w^+}[\kappa^0]$ is an invertible sheaf on $\mathfrak{IW}_w$. Its pushforward via π_2 is then a formal Banach sheaf because π_2 is affine, and pushing through π_3 and taking invariants over $B_\infty(\mathbb{Z}_p)/p^n = B_n, \pi_*\mathcal{O}_{\mathfrak{IW}_w^+}[\kappa]$ is a formal Banach sheaf. □

Definition 7.6. We call $\mathfrak{w}_w^{\kappa\dagger} := \pi_*\mathcal{O}_{\mathfrak{IW}_w^+}[\kappa]$ the sheaf of v-overconvergent w-analytic modular forms of weight κ. The space of integral v-overconvergent, w-analytic modular forms of weight κ and level (outside p) K^p, for the group G is

$$M_w^{\kappa\dagger}(\mathfrak{X}(v)) = H^0(\mathfrak{X}(v), \mathfrak{w}_w^{\kappa\dagger}).$$

Remark 7.7. Unfortunately it does not seem clear how to define an involution $\kappa \mapsto \kappa'$ on all $\mathcal{W}$ which extends the one on classical weights, and thus we only get that classical modular forms of (classical, integral) weight κ embed in overconvergent forms of weight $\kappa' \ldots$.

7.2 Changing the analytic radius

Let $m - ((p^{2m}-1)/(p^2-1))v > w' > w$ and $\kappa \in \mathfrak{W}_w(L)$, and thus $\kappa \in \mathfrak{W}_{w'}(L)$. There is a natural inclusion,

$$\mathfrak{IW}_{w'}^+ \hookrightarrow \mathfrak{IW}_w^+,$$

compatible with the action of $(\mathcal{O}^\times \times \mathcal{O}^1)\mathfrak{B}_w$. This induces a map $\mathfrak{w}_w^{\kappa\dagger} \longrightarrow \mathfrak{w}_{w'}^{\kappa\dagger}$ and thus a map,

$$M_w^{\dagger\kappa}(\mathfrak{X}(v)) \longrightarrow M_{w'}^{\dagger\kappa}(\mathfrak{X}(v)).$$

Definition 7.8. The space of integral overconvergent locally analytic Picard modular forms of weight κ, and level (outside p) K^p, is

$$M_\kappa^\dagger(\mathfrak{X}) = \varinjlim_{v\to 0, w\to\infty} M_w^{\kappa\dagger}(\mathfrak{X}(v)).$$

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7.3 Classical and overconvergent forms in rigid fiber

Denote by $\mathfrak{X}_{Iw^+(p^{2m})}(v)$ the quotient of $\mathfrak{X}_1(p^{2m})(v)$ by $\tilde{U_m} \subset B_m$, which is isomorphic to

$$\begin{pmatrix} 1 & \mathcal{O}/p^m\mathcal{O} \\ & 1+p^m\mathcal{O}/p^{2m}\mathcal{O}\end{pmatrix} \subset \begin{pmatrix} (\mathcal{O}/p^m\mathcal{O})^\times & \mathcal{O}/p^m\mathcal{O} \\ & (\mathcal{O}/p^{2m}\mathcal{O})^\times\end{pmatrix}.$$

Let us also denote by $X_{Iw^+(p^{2m})}(v)$ the corresponding rigid space. Over the scheme X, we have the locally free sheaf $\omega_A = \omega_{A,\tau} \oplus \omega_{A,\sigma\tau}$, which is locally isomorphic to $\mathcal{O}_X \oplus \mathcal{O}_X^2$, with the corresponding action of $\mathcal{O}$. Denote by $\mathcal{T}$ the scheme $\mathrm{Hom}_{X,\mathcal{O}}(\mathcal{O}_X^2 \oplus \mathcal{O}_X, \omega_G)$ of trivialization of ω_G as a $\mathcal{O}_X \otimes_{\mathbb{Z}_p} \mathcal{O} = \mathcal{O}_X \oplus \mathcal{O}_X$-sheaf, and denote by $\mathcal{T}^\times$ its subsheaf of isomorphisms; it is a $\mathrm{GL}_2 \times \mathrm{GL}_1$-torsor over X, where $g \in \mathrm{GL}_2 \times \mathrm{GL}_1$ acts on $\mathcal{T}^\times$ by $g \cdot \phi = \phi \circ g^{-1}$. For $\kappa \in X^+(T)$ a classical weight, denote by ω^κ the sheaf $\pi_*\mathcal{O}_{\mathcal{T}^\times}[\kappa']$, where $\pi : \mathcal{T}^\times \longrightarrow X$ is the projection and $\kappa \longrightarrow \kappa'$ the involution on classical weights. In down-to-earth terms, $\kappa = (k_1, k_2, l)$ where $k_1 \geqslant k_2$ and

$$\omega^\kappa = \mathrm{Sym}^{k_1-k_2} \omega_{G,\sigma\tau} \otimes (\det \omega_{G,\sigma\tau})^{\otimes k_2} \otimes (\det \omega_{G,\tau})^{\otimes l}.$$

We have defined $X(v)$, which is the rigid fiber of $\mathfrak{X}(v)$. Denote by $\mathcal{T}_{\mathrm{an}}, \mathcal{T}_{\mathrm{an}}^\times, (\mathrm{GL}_2 \times \mathrm{GL}_1)_{\mathrm{an}}$ the analytification of the schemes $\mathcal{T}, \mathcal{T}^\times, \mathrm{GL}_g$, and $\mathcal{T}_{\mathrm{rig}}, \mathcal{T}_{\mathrm{rig}}^\times, (\mathrm{GL}_2 \times \mathrm{GL}_1)_{\mathrm{rig}}$ Raynaud's rigid fiber of the completion along the special fibers of the same schemes. As $\mathcal{T}^\times/B$ is complete, $\mathcal{T}_{\mathrm{an}}^\times/B_{\mathrm{an}} = \mathcal{T}_{\mathrm{rig}}^\times/B_{\mathrm{rig}}$, over which there is the diagram,

$$\begin{tikzcd} \mathcal{T}_{\mathrm{rig}}^\times/U_{\mathrm{rig}} \arrow[rr] \arrow[dr, "f"'] & & \mathcal{T}_{\mathrm{an}}^\times/U_{\mathrm{an}} \arrow[dl, "g"] \\ & \mathcal{T}_{\mathrm{rig}}^\times/B_{\mathrm{rig}} & \end{tikzcd}$$

where f is a torsor over $U_{\mathrm{rig}}/B_{\mathrm{rig}} = T_{\mathrm{rig}}$ (the torus, not to be mistaken with $\mathcal{T}_{\mathrm{rig}}$) and g a torsor over T_{an} (same remark).

Definition 7.9. Let $\kappa \in \mathcal{W}_w(K)$. We denote by $\omega_w^{\kappa\dagger}$ the rigid fiber of $\mathfrak{w}_w^{\kappa\dagger}$ on $X(v)$. It exists by [AIP15, Proposition A.2.2.4]. It is called the sheaf of w-analytic overconvergent modular forms of weight κ. The space of v-overconvergent, w-analytic modular forms of weight κ is the space,

$$H^0(X(v), \omega_w^{\kappa\dagger}).$$

The space of locally analytic overconvergent Picard modular forms of weight κ (and level K^p) is the space,

$$M_\kappa^\dagger(X) = \varinjlim_{v\to 0, w\to\infty} H^0(X(v), \omega_w^{\kappa\dagger}).$$

The injection of $\mathcal{O}_{\mathfrak{X}(v)}$-modules $\mathcal{F}_\tau \oplus \mathcal{F}_{\sigma\tau} \subset \omega_A = \omega_{A,\tau} \oplus \omega_{A,\sigma\tau}$ is an isomorphism in generic fiber, and this induces an open immersion,

$$\mathcal{IW}_w \hookrightarrow \mathcal{T}_{\mathrm{rig}}^\times/B_{\mathrm{rig}} \times_{X(v)} X_1(p^{2m})(v).$$

We also have an open immersion,

$$\mathcal{IW}_w^+ \hookrightarrow \mathcal{T}_{\mathrm{an}}^\times/U_{\mathrm{an}} \times_{X(v)} X_1(p^{2m})(v).$$

The action of B_n on $\mathfrak{X}_1(p^{2m})(v)$ (or $X_1(p^{2m})(v)$) lifts to an action on $\mathfrak{IW}_w$ (or $\mathcal{IW}_w$) because being w-compatible for $\mathrm{Fil}^1 \mathcal{F}_\tau$ it only depends on the trivialization of K_n^D modulo B_n. Similarly the action of $\tilde{U_n}$ lifts to $\mathfrak{IW}_w^+$ and $\mathcal{IW}_w^+$. We can thus define $\mathcal{IW}_w^0$ and $\mathcal{IW}_w^{+,0}$ the respective quotients of $\mathcal{IW}_w$ and $\mathcal{IW}_w^+$ by B_n and $\tilde{U_n}$, which induces open immersions,

$$\mathcal{IW}_w^0 \hookrightarrow \mathcal{T}_{\mathrm{rig}}^\times/B_{\mathrm{rig}} \times_{X(v)} X(v) \quad \text{and} \quad \mathcal{IW}_w^{+,0} \hookrightarrow \mathcal{T}_{\mathrm{an}}^\times/U_{\mathrm{an}} \times_{X(v)} X_{Iw^+(p^{2m})}(v).$$

Proposition 7.10. *Suppose $w > m-1+(p+v)/(p^2-1)$. Then there are embeddings*

$$\mathcal{IW}_w^0 \subset (\mathcal{T}^{\mathrm{an}}/B)_{X(v)} \quad \text{and} \quad h : \mathcal{IW}_w^{0,+} \subset (\mathcal{T}^{\mathrm{an}}/U)_{X(v)}.$$

Proof. Let S be a set of representatives in $I_\infty \simeq \begin{pmatrix} \mathcal{O}^1 & \mathcal{O} \\ p\mathcal{O} & \mathcal{O}^\times \end{pmatrix}$ of $I_n/\widetilde{U_n}$ which we can suppose of the form,

$$\begin{pmatrix} [b] & \\ p[c] & [a] \end{pmatrix}, \quad a \in (\mathcal{O}/p^m)^\times, b \in (\mathcal{O}^1/p^m), c \in \mathcal{O}/p^{m-1}.$$

Here, $[.]$ denotes any lift. Then h is locally (over $X(v)$) isomorphic to

$$h : \coprod_{\gamma \in S} \begin{pmatrix} 1+p^wB(0,1) & & \\ p^wB(0,1) & 1+p^wB(0,1) & \\ & & 1+p^wB(0,1) \end{pmatrix} M\widetilde{\gamma} \longrightarrow (\mathrm{GL}_2 \times \mathrm{GL}_1/U)_{\mathrm{an}},$$

where M is the matrix which is locally given by the Hodge–Tate map, and thus corresponds to the inclusion $\mathcal{F}_\tau \oplus \mathcal{F}_{\sigma\tau} \subset \omega_\tau \oplus \omega_{\sigma\tau}$, and if $\gamma \in S$, then $\widetilde{\gamma}$ is given by

$$\widetilde{\gamma} = \begin{pmatrix} \sigma\tau(b) & & \\ p\sigma\tau(c) & \sigma\tau(a) & \\ & & \tau(a) \end{pmatrix} \quad \text{if} \quad \gamma = \begin{pmatrix} b & \\ pc & a \end{pmatrix}.$$

But there exists M' with integral coefficients such that

$$M'M = \begin{pmatrix} p^{(p+v)/(p^2-1)}I_2 & \\ & p^{v/(p^2-1)} \end{pmatrix},$$

and it is easily checked that $M' \circ h$ is then injective if $w > m-1+(p+v)/(p^2-1)$. The proof for the other embedding is similar (and easier). $\square$

Remark 7.11. Of course not every w satisfies $m-1+(p+v)/(p^2-1) < w < m - ((p^{2m}-1)/(p^2-1))v$ for some m. But every w-analytic character κ is w'-analytic for all $w' > w$, and in particular we can choose a w' which satisfies the previous inequalities, *a priori* for another m (for each choice of w' there is a unique such m).

From now on, we suppose that we have fixed m, and we suppose that $m-1+(p+v)/(p^2-1) < w < m - ((p^{2m}-1)/(p^2-1))v$.

We could have defined $\omega_w^{\kappa\dagger}$ directly, by $g_*\mathcal{O}_{\mathcal{IW}_w^{0,+}}[\kappa]$ where g is the composite,

$$\mathcal{IW}_w^{0,+} \longrightarrow \mathcal{IW}_w^0 \longrightarrow X(v)$$

as shown by the next proposition. Remark that $X(v) \subset X_{Iw(p)}(v)$ via the canonical filtration of level 1.

Proposition 7.12. *The sheaf $\omega_w^{\kappa\dagger}$ (defined as the rigid fiber of $\mathfrak{w}_w^{\kappa\dagger}$) is isomorphic to $g_*\mathcal{O}_{\mathcal{IW}_w^{0,+}}[\kappa]$.*

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Proof. In the rigid setting, we did a quotient by $\widetilde{U_n}$ to get $\mathcal{IW}_w^{0,+}$. But $\omega_w^{\kappa\dagger}$ is constructed as $((\pi_2 \circ \pi_1)_* \mathcal{O}_{\mathcal{IW}^+}[\kappa^0])(-\kappa)^{B_n}$, and the action of $\widetilde{U_n}$ on $(\pi_2 \circ \pi_1)_* \mathcal{O}_{\mathcal{IW}^+}[\kappa^0]$ is trivial and it thus descends to $X_{Iw^+(p^{2m})}(v)$ and is isomorphic to the κ^0-variant vectors in the pushforward of $\mathcal{O}_{\mathcal{IW}_w^{0,+}}$. □

Proposition 7.13. *For $\kappa \in X_+(T^0)$ and $\omega > 0$, there is a restriction map,*

$$\omega^\kappa_{X(v)} \hookrightarrow \omega_w^{\kappa'\dagger},$$

induced by the inclusion $\mathcal{IW}_w^{0,+} \subset (\mathcal{T}^{\mathrm{an}}/U)$. Moreover, locally for the étale topology, this inclusion is isomorphic to the following composition,

$$V_{\kappa'} \hookrightarrow V_{\kappa'}^{w\text{-an}} \xrightarrow{\mathrm{res}_0} V_{0,\kappa'}^{w\text{-an}}.$$

Proof. Locally for the étale topology, ω^κ is identified with algebraic functions on $\mathrm{GL}_2 \times \mathrm{GL}_1$ which are invariant by U and varies as κ' under the action of T, i.e. to $V_{\kappa'}$. But a function $f \in \omega_w^{\kappa'\dagger}$ is locally identified with a function,

$$f : \left\{ \begin{pmatrix} \tau(a)(1+p^wB(0,1)) & & \\ p^wB(0,1) & \tau(b)(1+p^wB(0,1)) & \\ & & \sigma\tau(b)(1+p^wB(0,1)) \end{pmatrix}, a \in \mathcal{O}^1, b \in \mathcal{O}^\times \right\} \longrightarrow L,$$

which varies as κ' under the action on the right of $T(\mathbb{Z}_p)\mathfrak{T}_w$. As $\kappa' = (k_1, k_2, k_3) \in X_+(T)$ we can extend f to a κ'-varying function on

$$I_{p^w} = \left\{ \begin{pmatrix} \mathbb{G}_m & B(0,1) & \\ p^wB(0,1) & \mathbb{G}_m & \\ & & \mathbb{G}_m \end{pmatrix} \right\},$$

extending it ‘trivially’; i.e.

$$\begin{aligned} f\left(\begin{pmatrix} x & u & \\ p^wz & y & \\ & & t \end{pmatrix}\right) &= f\left(\begin{pmatrix} 1 & 0 & \\ p^wzx^{-1} & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} x & u & \\ 0 & y-p^wzu & \\ & & t \end{pmatrix}\right) \\ &= x^{k_1}(y-p^wzu)^{k_2}t^{k_3} f\left(\begin{pmatrix} 1 & 0 & \\ p^wzx^{-1} & 1 & \\ & & 1 \end{pmatrix}\right). \end{aligned}$$

Under this identification, locally for the étale topology $\omega_w^{\kappa'\dagger}$ is identified with $V_{0,\kappa'}^{w\text{-an}}$. □

8. Hecke operators, classicity

As explained in [AIP15] and [Bra16], it is not possible to find a toroidal compactification for more general PEL Shimura varieties (already for GSp_4) that is preserved with all the Hecke correspondences, but this can be overcome by looking at *bounded* sections on the open variety. For the Picard modular variety, there is only one choice of a toroidal compactification, and thus this problem does not appear, but we will keep the general strategy (and thus we will not have to check that the correspondences extend to the boundary). Thus, instead we will define Hecke operators on the open Picard Variety $\mathcal{Y}_{Iw(p)}$ of Iwahori level, and, as bounded sections on the open variety extend automatically to the compactification (see [AIP15, Theorem 5.5.1, Proposition 5.5.2], which follows from a theorem of Lutkebohmert), we show that Hecke operators send bounded functions to bounded functions, and thus induce operators on overconvergent locally analytic modular forms.

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8.1 Hecke operators outside p

These operators have been defined already in [Bra16, § 4]. We explain their definition quickly, and refer to [Bra16] (see also [AIP15, § 6.1]) for the details. Let $\ell \neq p$ be an integer, and suppose $\ell \nmid N$, the set of places where K_v is not maximal. Let $\gamma \in G(\mathbb{Q}_\ell) \cap \mathrm{End}\,\mathcal{O}_{E,\ell}(\mathcal{O}_{E,\ell}^3) \times \mathbb{Q}_\ell^\times$, and consider

$$C_\gamma \rightrightarrows \mathcal{Y}_{Iw(p)},$$

the moduli space of isogeny $f : A_1 \longrightarrow A_2$ such that:

(i) f is $\mathcal{O}_E$-linear, and of degree a power of ℓ;
(ii) f is compatible with polarizations, i.e. $f^*\lambda_2$ is a multiple of λ_1;
(iii) f is compatible with the K^p-level structure (at places that divides N) (we remark that f is an isomorphism on $T_q(A_i)$ when $q \neq \ell$ is a prime);
(iv) f is compatible with the filtration given by the Iwahori structure at p;
(v) the type of f is given by the double class $G(\mathbb{Z}_\ell)\gamma G(\mathbb{Z}_\ell)$.

Remark 8.1. The space C_γ does not depend on γ, only on the double class $G(\mathbb{Z}_\ell)\gamma G(\mathbb{Z}_\ell)$.

We could similarly define C_γ without Iwahori level at p (i.e. for $\mathcal{Y}_{G(\mathbb{Z}_p)K^p}$) without the condition that f is compatible with the filtration given by the Iwahori structure at p. In our case, this Iwahori structure at p will always be the canonical one, and thus f is automatically compatible as it sends the canonical filtration of A_1 in the one of A_2.

Denote by $p_i : C_\gamma \longrightarrow \mathcal{Y}$ the two (finite) maps that sends f to A_i. Denote by $C_\gamma(p^n)$ the fiber product with p_1 of C_γ with $\mathcal{Y}_1(p^n)(v) \longrightarrow \mathcal{Y}(v) \subset_s \mathcal{Y}_{Iw(p)}(v)$, where s is the canonical filtration of $A[p]$. Denote by f the universal isogeny over $G_\gamma(p^n)$. It induces an $\mathcal{O}$-linear isomorphism,

$$p_2^*(\mathcal{F}_\tau \oplus \mathcal{F}_{\sigma\tau}) \longrightarrow p_1^*(\mathcal{F}_\tau \oplus \mathcal{F}_{\sigma\tau}).$$

In particular, we get a $B(\mathbb{Z}_p)\mathfrak{B}_w$-equivariant isomorphism,

$$f^* : p_2^*\widetilde{\mathcal{IW}}^+_{w,|\mathcal{Y}_1(p^n)(v)} \xrightarrow{\sim} p_1^*\widetilde{\mathcal{IW}}^+_{w,|\mathcal{Y}_1(p^n)(v)}.$$

We can thus form the composite morphism,

$$H^0(\mathcal{Y}_1(p^n)(v), \mathcal{O}_{\widetilde{\mathcal{IW}}_w^+}) \xrightarrow{p_2^*} H^0(C_\gamma(p^n)(v), p_2^*\mathcal{O}_{\widetilde{\mathcal{IW}}_w^+}) \xrightarrow{(f^*)^{-1}} H^0(C_\gamma(p^n)(v), p_1^*\mathcal{O}_{\widetilde{\mathcal{IW}}_w^+})$$
$$\xrightarrow{Tr(p_1)} H^0(\mathcal{Y}_1(p^n)(v), \mathcal{O}_{\widetilde{\mathcal{IW}}_w^+}).$$

As f^* is an isomorphism, it sends bounded functions to bounded functions, and we can make the following definition.

DEFINITION 8.2. Let $\kappa \in \mathcal{W}_w(K)$ a weight. We define the Hecke operator,

$$T_\gamma : M_{v,w}^{\kappa\dagger} \longrightarrow M_{v,w}^{\kappa\dagger},$$

as the restriction of the previous operator to the bounded, κ-equivariant sections under the action of $B(\mathbb{Z}_p)\mathfrak{B}_w$. It induces an operator,

$$T_\gamma : M^{\kappa\dagger} \longrightarrow M^{\kappa\dagger}.$$

DEFINITION 8.3. Define $\mathcal{H}$ to be the commutative $\mathbb{Z}$ -algebra generated by all operators T_γ for all $\ell \nmid Np$ and all double classes γ. These operators commute on overconvergent forms, and thus $\mathcal{H}$ acts on them.

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8.2 Hecke operator at p

Here we stress that, as in the previous sections, the prime p is inert in E. We will define a first Hecke operator at p, U_p. Define C the moduli space over K which parametrizes data (A,λ,i,η,L) where $(A,\lambda,i,\eta)\in X_K(v)$ and $L\subset A[p^2]$ is a totally isotropic $\mathcal{O}$-module for λ (i.e. $L\subset L^\perp := (G[p^2]/L)^D \subset G[p^2]^D \overset{\lambda}{\simeq} G[p^2])$ of $\mathcal{O}$-height p^3 (and thus $L=L^\perp$) such that

$$L[p]\oplus H^1_\tau = A[p] \quad \text{and} \quad pL\oplus H^1_{\sigma\tau}=A[p].$$

As remarked by Bijakowski in [Bij16], the second condition is implied by the first one and the isotropy condition. We then define two projections,

$$p_1,p_2 : C \longrightarrow X_K,$$

where p_1 is the forgetful map which sends (A,λ,i,η,L) to (A,λ,i,η,L) and p_2 sends (A,λ,i,η,L) to $(A/L,\lambda',i',\eta')$. To compare the correspondence with the canonical filtration we will need the following lemma.

Lemma 8.4. *Let $p>2$ and G be a p-divisible $\mathcal{O}$-module of unitary type and signature $(2,1)$. Let H be a sub-$\mathcal{O}$-module of p-torsion and of $\mathcal{O}$-height 1. Then the two following assertions are equivalent:*

(i) $\mathrm{Deg}_\tau(H) > 1+p-\frac{1}{2}$;
(ii) $\mathrm{ha}_\tau(G)<\frac{1}{2}$ *and H is the canonical subgroup of $G[p]$ associated to τ.*

Let H be a sub-$\mathcal{O}$-module of p-torsion and of $\mathcal{O}$-height 2. Then the two following assertions are equivalent:

(i) $\mathrm{Deg}_{\sigma\tau}(H) > p+2-\frac{1}{2}$;
(ii) $\mathrm{ha}_\tau(G)<\frac{1}{2}$ *and H is the canonical subgroup of $G[p]$ associated to $\sigma\tau$.*

In both cases we can be more precise: if $v=1+p-\mathrm{Deg}_\tau(H)$ (respectively $2+p-\mathrm{Deg}_{\sigma\tau}(H)$), then $\mathrm{ha}_\tau(G)\leqslant v$.

Proof. In both cases we only need to prove that the first assumption implies the second, by the existence theorem of the canonical filtration ($\mathrm{ha}_\tau(G)=\mathrm{ha}_{\sigma\tau}(G)$). Moreover, we only have to prove that $\mathrm{ha}_\tau(G)<\frac{1}{2}$, because then $G[p]$ will have a canonical filtration, and in both cases, H will be a group of this filtration because it will correspond to a break-point of the Harder–Narasimhan filtration (for the classical degree, as we only care about filtration in $G[p]$). Let us do the second case, as it is the most difficult one (the first case can be treated similarly, even using only technics introduced in [Far11]). Let $v=2+p-\mathrm{Deg}_{\sigma\tau}(H)$. We can check that $\deg_{\sigma\tau}(H)>2-v$, thus $\deg_{\sigma\tau}(H^D)<v$, and thus, for all $\varepsilon>1-v$, if $E=G[p]/H$, then

$$\omega_{G^D,\sigma\tau,\varepsilon}\simeq\omega_{E^D,\sigma\tau,\varepsilon}.$$

But then the cokernel of $\alpha_{E,\sigma\tau,\varepsilon}\otimes 1$ is of degree $(1/(p^2-1))\,\mathrm{Deg}_{\sigma\tau}(E)$ (E is a Raynaud subgroup of type $(p\ldots p)$), and the following square is commutative,

$$\begin{array}{ccc} G[p](\mathcal{O}_K) & \longrightarrow & E(\mathcal{O}_K) \\ \big\downarrow{\scriptstyle \alpha_{G,\sigma\tau,\varepsilon}} & & \big\downarrow{\scriptstyle \alpha_{E,\sigma\tau,\varepsilon}} \\ \omega_{G[p]^D,\sigma\tau,\varepsilon} & \xrightarrow{\sim} & \omega_{E^D,\sigma\tau,\varepsilon} \end{array}$$

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and thus, in particular, $\deg \operatorname{Coker}(\alpha_{E,\sigma\tau,\varepsilon} \otimes 1) = \deg \operatorname{Coker}(\alpha_{G[p],\sigma\tau,\varepsilon} \otimes 1)$. But according to [Her19, Proposition 5.25], we can check that the image of $\alpha_{G[p],\sigma\tau}$ is always included inside $up^{\mathrm{ha}_{\sigma\tau}(G)/(p^2-1)}\mathbb{F}_{p^2} + p^{1/p^2}\mathcal{O}_C/p \subset \omega_{G[p]^D,\sigma\tau} \simeq \mathcal{O}_C/p$ for some $u \in \mathcal{O}_C^\times$. Rewriting the inequality with $\operatorname{Deg}_{\sigma\tau}(E) = \operatorname{Deg}_{\sigma\tau}(G[p]) - \operatorname{Deg}_{\sigma\tau}(H)$ we get

$$\min\left(\mathrm{ha}_\tau(G), \frac{p^2-1}{p^2}\right) \leqslant 2 + p - \operatorname{Deg}_{\sigma\tau}(H) = v,$$

but as $v < \frac{1}{2} < 1 - 1/p^2$ we get $\mathrm{ha}_\tau(G) \leqslant v$. □

Thus, we can deduce the following,

LEMMA 8.5. *Let (A, λ, i, η, L) as before with corresponding $(A, \lambda, i, \eta) \in \mathfrak{X}(v)$ and $v < 1/2(p^2+1)$. Then $A/L \in \mathfrak{X}(v)$, and $A[p^2]/L$ coincides with the group $K_1(A/L)$.*

Proof. By hypothesis on L, the map,

$$H_\tau^1 \longrightarrow A[p]/L[p],$$

is an isomorphism on generic fiber, thus $\operatorname{Deg}_\tau(A[p]/L[p]) \geqslant \operatorname{Deg}_\tau(H_\tau^1) > 1 + p - v$. Thus by the previous lemma, we get that, $\mathrm{ha}_\tau(A/L) \leqslant v$ and moreover $A[p]/L[p]$ coincide with the first canonical subgroup associated to τ. Moreover, we deduce that $\deg A[p]/L[p] \geqslant 2 - v$. Now consider the composite map,

$$H_{\sigma\tau}^2 \longrightarrow A[p^2]/L \longrightarrow (A[p^2]/L)/(A[p]/L[p]) = Q.$$

Because $H_{\sigma\tau}^1$ is sent inside $A[p]/L[p]$, we get the factorization,

$$H_{\sigma\tau}^2/H_{\sigma\tau}^1 \longrightarrow Q.$$

This is a generic isomorphism by the second hypothesis on L, and thus $\operatorname{Deg}_{\sigma\tau}(Q) \geqslant \operatorname{Deg}_{\sigma\tau}(H_{\sigma\tau}^2/H_{\sigma\tau}^1)$. But by construction, $H_{\sigma\tau}^2/H_{\sigma\tau}^1$ is the canonical subgroup (for $\sigma\tau$) of $A/H_{\sigma\tau}^1$ and thus $\operatorname{Deg}_{\sigma\tau}(H_{\sigma\tau}^2/H_{\sigma\tau}^1) \geqslant p + 2 - \mathrm{ha}_{\sigma\tau}(A/H_{\sigma\tau}^1)$, and $\mathrm{ha}_{\sigma\tau}(A/H_{\sigma\tau}^1) \leqslant p^2 \,\mathrm{ha}_{\sigma\tau}(A)$ (this is [Her19, Proposition 8.1]), and this implies that $\deg Q > 3 - p^2 v$. Using the exact sequence,

$$0 \longrightarrow A[p]/L[p] \longrightarrow A[p^2]/L \longrightarrow Q \longrightarrow 0,$$

we get that $\deg A[p^2]/L > 5 - p^2 v - v$. A similar argument also shows that $\deg K_1(A/L) \geqslant 5 - (p^2+1)\,\mathrm{ha}_\tau(A/L)$. But using Bijakowski's proposition recalled in [Her19, Proposition A.2], we get that if $2(p^2+1)v \leqslant 1$, then $A[p^2]/L = K_1(A/L)$. □

LEMMA 8.6. *Suppose $v < 1/2p^4$. Let $G/\operatorname{Spec}(\mathcal{O}_C)$ be a p-divisible group such that $\mathrm{ha}_\tau(G) < v$, then $K_1 \subset G[p^2]$ coincides over $\mathcal{O}_C/p^{1/2p^2}$ with $\operatorname{Ker} F^2 \subset G[p^2]$. In particular,*

$$\mathrm{ha}_\tau(G/K_1) = p^2\,\mathrm{ha}_\tau(G).$$

Proof. This is Appendix B. □

PROPOSITION 8.7. *Let $v < 1/2p^4$. The Hecke correspondence U_p defined by the two previous maps preserves $X(v)$. More precisely, if $y \in p_2(p_1^{-1}(\{x\}))$ where $x \in X(v)$, then $y \in X(v/p^2)$.*

Proof. The proof follows from the two previous lemmas as $(A/L)/(A[p^2]/L) = A$. □

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Denote the universal isogeny over C by

$$\pi : \mathcal{A} \longrightarrow \mathcal{A}/L,$$

which induces maps $\omega_{A/L,\tau} \xrightarrow{\pi^\star_\tau} \omega_{A,\tau}$ and $\omega_{A/L,\sigma\tau} \xrightarrow{\pi^\star_{\sigma\tau}} \omega_{A,\sigma\tau}$. We define

$$\widetilde{\pi^\star} : p_2^*\mathcal{T}_{\mathrm{an}}^\times \longrightarrow p_1^*\mathcal{T}_{\mathrm{an}}^\times$$

with $\widetilde{\pi^\star} = \widetilde{\pi^\star}_\tau \oplus \widetilde{\pi^\star}_{\sigma\tau}$ by $\widetilde{\pi^\star}_\tau = \pi^\star{}_\tau$, and $\widetilde{\pi^\star}_{\sigma\tau}$ sends a basis (e_1, e_2) of $\omega_{A/L,\sigma\tau}$ to $((1/p)\pi^\star e_1, \pi^\star e_2)$. This is an isomorphism.

We will need to slightly change the objects as in [AIP15, Proposition 6.2.2.2].

Definition 8.8. Denote by $w_0 = m - ((p^{2m}-1)/(p^2-1))v$ and for $\underline{w} = (w_{1,1}, w_{2,1}, w_{2,2}, w_\sigma)$, define $\mathcal{IW}_{\underline{w}}^{0,+}$ as the subspace of $\mathcal{T}^\times/U_{\mathrm{an}}$ (over $X_1(p^n)(v)$) of points for a finite extension L of K consisting of $(A, \psi_N, \mathrm{Fil}_{\sigma\tau}, P_1^{\sigma\tau}, P_2^{\sigma\tau}, P^\tau)$ such that there exists a polarized trivialization ψ of K_m^D satisfying:

(i) $\mathrm{Fil}_{\sigma\tau}$ is (w_0, ψ)-compatible with H_τ^m;
(ii) $P_1^{\sigma\tau} = a_{1,1}\,\mathrm{HT}_{\sigma\tau,w_0}(\psi(e_1)) + a_{2,1}\,\mathrm{HT}_{\sigma\tau,w_0}(\psi_{e_2}) \pmod{p^{w_0}\mathcal{F}_{\sigma\tau}}$;
(iii) $P_2^{\sigma\tau} = a_{2,2}\,\mathrm{HT}_{\sigma\tau,w_0}(\psi(e_2)) \pmod{p^{w_0}\mathcal{F}_{\sigma\tau} + \mathrm{Fil}_{\sigma\tau}}$;
(iv) $P_\tau = t\,\mathrm{HT}_{\tau,w_0}(\psi(e_2)) \pmod{p^{w_0}\mathcal{F}_\tau}$,

where $a_{1,1} \in B(1, p^{w_{1,1}}), a_{2,2} \in B(1, p^{w_{2,2}}), t \in B(1, p^{w_\sigma}), a_{2,1} \in B(0, p^{w_{2,1}})$.

Let $\underline{w}$ as before, with $w_{2,1}, w_{2,2} < m - 1 - ((p^{2m}-1)/(p-1))v$. Write $\underline{w}' = (w_{1,1}, w_{2,1}+1, w_{2,2}, w_\sigma)$.

Proposition 8.9. *The quotient map,*

$$\widetilde{\pi^\star}^{-1} : p_1^*\mathcal{T}_{\mathrm{an}}^\times/U_{\mathrm{an}} \longrightarrow p_2^*\mathcal{T}_{\mathrm{an}}^\times/U_{\mathrm{an}}$$

sends $p_1^*\mathcal{IW}_{\underline{w}}^{0,+}$ to $p_2^*\mathcal{IW}_{\underline{w}'}^{0,+}$ (i.e. improves the analycity radius).

Proof. Let $x = (A, \psi_N, L)$ be a point of C. Let (e_1, e_2) be a basis of K_m^D $(p^m\mathcal{O}/p^{2m}\mathcal{O}e_1 \oplus \mathcal{O}/p^{2m}\mathcal{O}e_2 = K_n^D)$ and denote by (e_1', e_2') a similar basis for A/L such that if x_1, x_2 and x_1', x_2' denote the dual basis then $\pi^D : K_m^{\prime,D} \longrightarrow K_m^D$ in these basis is given by

$$\begin{pmatrix} p & \\ & 1 \end{pmatrix}.$$

Let $(\mathrm{Fil}', w') \in p_2^*\mathcal{T}_{\mathrm{an}}^\times/U_{\mathrm{an}}$. As π^D is a generic isomorphism on the multiplicative part, it is enough to check the proposition on $\mathcal{F}_{\sigma\tau}$. Suppose $\widetilde{\pi}^*(\mathrm{Fil}', w') = (\mathrm{Fil}, w) \in p_1^*\mathcal{IW}_{\underline{w}}^{0,+}$, which means (on the $\sigma\tau$-factor),

$$\frac{1}{p}\pi^* w_1' \in a_{1,1}\,\mathrm{HT}_{\sigma\tau,w}(e_1) + a_{2,1}\,\mathrm{HT}_{\sigma\tau,w}(e_2) + p^{w_0}\mathcal{F}_{\sigma\tau},$$

$$\pi^* w_2' \in a_{2,2}\,\mathrm{HT}_{\sigma\tau,w}(e_2) + p^{w_0}\mathcal{F}_{\sigma\tau} + \mathrm{Fil}^1.$$

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But then,

$$\begin{aligned}\pi^* w'_1 &\in pa_{1,1}\,\mathrm{HT}_{\sigma\tau,w}(e_1) + pa_{2,1}\,\mathrm{HT}_{\sigma\tau,w}(e_2) + p^{w_0+1}\mathcal{F}_{\sigma\tau}\\ &= a_{1,1}\,\mathrm{HT}_{\sigma\tau,w}(\pi^D e'_1) + pa_{2,1}\,\mathrm{HT}_{\sigma\tau,w}(\pi^D e'_2) + p^{w_0+1}\mathcal{F}_{\sigma\tau},\end{aligned}$$

and thus, as $p\mathcal{F} \subset \pi^*\mathcal{F}'$,

$$w'_1 \in a_{1,1}\,\mathrm{HT}_{w,\sigma\tau}(e'_1) + pa_{2,2}\,\mathrm{HT}_{w,\sigma\tau}(e'_2) + p^{w_0}\mathcal{F}',$$

$$w'_2 \in a_{2,2}\,\mathrm{HT}_{w,\sigma\tau}(e'_2) + \mathrm{Fil}' + p^{w_0-1}\mathcal{F}'.$$

As $w_{2,2} \leqslant w_0 - 1$ we get the result. □

Suppose $v < 1/2(p^2+1)$, and define $\omega^{\kappa\dagger}_{\underline{w}'} = g_*\mathcal{IW}^{0,+}_{\underline{w}'}[\kappa]$. Suppose $w < m - 1 - (((p^{2m}-1))/((p^2-1)))v$. We can then look at the following composition,

$$H^0(\mathcal{Y}(v/p^2), \omega^{\kappa\dagger}_{\underline{w}'}) \xrightarrow{p_2^\star} H^0(C, p_2^*\omega^{\kappa\dagger}_w) \xrightarrow{(\widetilde{\pi}^\star)^{-1}} H^0(C, p_1^*\omega^{\kappa\dagger}_w) \xrightarrow{(1/p^3)\,\mathrm{Tr}_{p_1}} H^0(\mathcal{Y}(v), \omega^{\kappa\dagger}_w),$$

where $w'_{2,1} = w+1$ (we remark that if κ is $(w+1)$-analytic, there is an isomorphism between $g_*\mathcal{IW}^{0,+}_{\underline{w}'}[\kappa]$ and $\omega^{\kappa\dagger}_{w_{2,1}}$).

Remark 8.10. The normalization of the Trace map is the same as in [Bij17], the normalization of $\pi^\star$ giving a factor p^{-k_2} on algebraic weights.

DEFINITION 8.11. Suppose $v < 1/2(p^2+1)$. The operator U_p is defined as the previous composition on bounded functions precomposed by $H^0(\mathcal{Y}(v), \omega^{\kappa\dagger}_w) \hookrightarrow H^0(\mathcal{Y}(v/p^2), \omega^{\kappa\dagger}_{\underline{w}'})$. In particular it is compact as $H^0(\mathcal{Y}(v), \omega^{\kappa\dagger}_w) \hookrightarrow H^0(\mathcal{Y}(v/p^2), \omega^{\kappa\dagger}_{\underline{w}'})$ is.

PROPOSITION 8.12. *Let L be a finite extension of K, and $x, y \in X(v)(L)$ such that $y \in p_2(p_1^{-1}(x))$, and let κ be a ω-analytic character. Then U_p is identified with δ, i.e. there is a commutative diagram as follows.*

$$\begin{array}{ccc} (\omega^{\kappa'\dagger}_w)_y & \xrightarrow{(\widetilde{\pi}^\star)^{-1}} & (\omega^{\kappa'\dagger}_w)_x \\ \big\downarrow & & \big\downarrow \\ V^{w\text{-an}}_{0,\kappa',L} & \xrightarrow{\delta} & V^{w\text{-an}}_{0,\kappa',L} \end{array}$$

We can also define an operator S_p, by considering the two maps,

$$p_1, p_2 : X_{Iw(p)} \longrightarrow X_{Iw(p)}$$

defined by $p_1 = \mathrm{id}$ and $p_2(A, \mathrm{Fil}(A[p])) = (A/A[p], p^{-1}\,\mathrm{Fil}(A[p])/A[p])$. The map p_2 corresponds to multiplication by p on ω and the universal map $\pi : A \longrightarrow A/A[p]$ over $X_{Iw(p)}$ induces a map,

$$\pi^* = p_2^*\mathcal{T}^\times_{\mathrm{an}} \longrightarrow \mathcal{T}^\times_{\mathrm{an}}.$$

Define $\widetilde{\pi^*} = (1/p)\pi^*$ and consider the operator,

$$H^0(X_{Iw(p)}, \omega^{\kappa\dagger}_w) \xrightarrow{p_2^*} H^0(X_{Iw(p)}, p_2^*\omega^{\kappa\dagger}_w) \xrightarrow{\widetilde{\pi^*}^{-1}} H^0(X_{Iw(p)}, \omega^{\kappa\dagger}_w).$$

The map π^* preserves the Hasse invariant (as $A[p^\infty]/A[p] \simeq A[p^\infty]$) and sends the canonical filtration, if it exists, to itself (if $v < 1/4p$). In concrete terms, on the classical sheaf $\omega^{\kappa'} \subset \omega^{\kappa\dagger}$, if we write $\kappa' = (k_1, k_2, k_3) \in \mathbb{Z}^3$ (and thus $\kappa = (-k_2, -k_1, -k_3)$) the previous composition corresponds to a normalization by $p^{-k_1-k_2-k_3}$ of the map that sends $f(A, dz_i) \mapsto f(A, pdz_i)$.

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DEFINITION 8.13. Define also the Hecke operator S_p to be the previous composition. The operator S_p is invertible as p is invertible in $\mathcal{T}^\times$.

We define the Atkin–Lehner algebra at p as $\mathcal{A}(p) = \mathbb{Z}[1/p][U_p, S_p^{\pm 1}]$. It acts on the space of (classical) modular forms too.

Classically it is also possible to define geometrically operators U_p and S_p at p on classical modular forms of Iwahori level at p, and they obviously coincide with ours through the inclusion of classical forms to overconvergent ones. It is actually proven in [Bij16] that these Hecke operators preserve a strict neighborhood of the canonical-μ-ordinary locus of X_I, given in terms of the degree.

Remark 8.14. Because of the normalization, the definition of the Hecke operator slightly differs with the one by convolution on automorphic forms. The reason is that the Hodge–Tate or automorphic weights do not vary continuously in families. This is already the case in other constructions. Let us be more specific. Let $f \in H^0(X_{Iw,K}, \omega^\kappa)$ a classical automorphic form of weight $\kappa = (k_1, k_2, k_3)$ and Iwahori level at p. To f, as explained in Proposition 2.6 is associated a (non-scalar) automorphic form Φ_f (and a scalar one φ_f whose Hecke eigenvalues are the same as the one of Φ_f). The Hecke action on f and Φ_f is equivariant for the classical (i.e. non-renormalized) action at p, more precisely at p if we denote S_p and U_p the previous (normalized operators) the classical ones are $S_p^{\mathrm{class}} = p^{|k|} S_p$ and $U_p^{\mathrm{class}} = p^{k_2} U_p$. The operators S_p^{class} and U_p^{class} correspond to the two matrices,

$$\begin{pmatrix} p & & \\ & p & \\ & & p \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p^2 & & \\ & p & \\ & & 1 \end{pmatrix} \in GU(2,1)(\mathbb{Q}_p).$$

Their similitude factor is in both cases $p^2 = N(p)$. Let $f \in H^0(X_{Iw,K}, \omega^\kappa)$ be a classical eigenform that is proper for the Hecke operator U_p and S_p, of respective eigenvalues μ, λ, then φ_f has eigenvalues for the corresponding (non-normalized) Hecke operators at p, $p^{k_2}\mu$ and $p^{k_1+k_2+k_3}\lambda$.

8.3 Remarks on the operators on the split case

When p splits in E, the eigenvariety for $U(2,1)_E$ is a particular case of Brasca's construction (see [Bra16]). Unfortunately as noted by Brasca, there is a slight issue with the normalization of the Hecke operators at p constructed in [Bra16, § 4.2.2], where there should be a normalization in families that depends on the weights, as in [Bij16, § 2.3.1] for classical sheaves (without this normalization Hecke operators do not vary in family). More explicitly on the split Picard case, we have four Hecke operators at p (Bijakowski only consider two of them, which are relevant for classicity), written $U_i, i = 0, \dots, 3$, following [Bij16, § 2.3.1] (allowing $i = 0$ and $i = 3$). The normalization are the following on classical weights,

$$U_0 = \frac{1}{p^{k_3}} U_0^c, \quad U_1 = U_1^c, \quad U_2 = \frac{1}{p^{k_2}} U_2^c, \quad U_3 = \frac{1}{p^{k_1+k_2}} U_3^c,$$

where U_i^c denotes the classical Hecke operators, and we choose a splitting of the universal p-divisible group $A[p^\infty] = A[v^\infty] \times A[\overline{v}^\infty]$ and $A[\overline{v}^\infty] = A[v^\infty]^D$, where v coincide with τ_∞ through the fixed isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}_p}$. Thus $G = A[v^\infty]$ has height 3 and dimension 1, and modular forms of weight $\kappa = (k_1 \geqslant k_2, k_3) \in \mathbb{Z}^3_{\mathrm{dom}}$ are sections of

$$\mathrm{Sym}^{k_1-k_2} \omega_{G^D} \otimes (\det \omega_{G^D})^{k_2} \otimes \omega_G^{\otimes k_3}.$$

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8.4 Classicity results

In this section, we will prove a classicity result. As in [AIP15], this is realized in two steps. First show that a section in $M_\kappa^\dagger$ is actually a section of $\omega^{\kappa'}$ over $X(v)$ (this is called a result of classicity at the level of sheaves), then we show that this section extends to all X_{Iw}, but this is done in [Bij16].

If n is big enough, there is an action of $I_{p^n} \subset \mathrm{GL}_2 \times \mathrm{GL}_1$ on $\mathcal{IW}_w^{0,+}$ which can be derived as an action of $U(\mathfrak{g})$ on $\mathcal{O}_{\mathcal{IW}_w^{0,+}}$ denoted by $\star$. As in § 4, let $\kappa = (k_1, k_2, r)$ be a classical weight, and we denote by d_κ the map,

$$f \in \mathcal{O}_{\mathcal{IW}_w^{0,+}} \mapsto X^{k_1-k_2+1} \star f,$$

which sends $\omega_w^{\kappa\dagger}$ to $\omega_w^{(k_2-1,k_1+1,r)\dagger}$.

PROPOSITION 8.15. *Let $\kappa = (k_1, k_2, r)$ be a classical weight. There is an exact sequence of sheaves on $X(v)$,*

$$0 \longrightarrow \omega^{\kappa'} \longrightarrow \omega_w^{\kappa\dagger} \xrightarrow{d_\kappa} \omega_w^{(k_2-1,k_1+1,r)\dagger}.$$

Proof. This is exactly as in [AIP15, Proposition 7.2.1] (we do not need assumption on w as $V_{\kappa,L}^{0,w\text{-an}}$ is isomorphic to analytic functions on one ball only, and Jones' theorem applies [Jon11]). □

PROPOSITION 8.16. *On $\omega_w^{\kappa\dagger}$ we have the following commutativity,*

$$U_p \circ d_\kappa = p^{-k_1+k_2-1} d_\kappa \circ U_p.$$

In particular, if $H^0(X(v), \omega_w^{\kappa\dagger})^{<k_1-k_2+1}$ denotes the union of generalized eigenspaces for eigenvalues of slope smaller than $k_1 - k_2 + 1$, and $f \in H^0(X(v), \omega_w^{\kappa\dagger})^{<k_1-k_2+1}$, then $f \in H^0(X(v), \omega^{\kappa'})$.

Proof. We can work étale-locally, in which case by the previous results on $\omega_w^{\kappa\dagger}$ locally the first part reduces to § 4. Now, if f is a generalized eigenvector for U_p of eigenvalue λ of slope (strictly) smaller than $k_1 - k_2 + 1$, then $d_\kappa f$ is generalized eigenvector of slope $\lambda p^{-k_1+k_2-1}$ which is of negative valuation, but this is impossible as U_p (and étale-locally δ) is of norm strictly less than 1. Thus $d_\kappa f = 0$ and f is a section of ω^κ. □

The previous result is sometimes referred to as a classicity at the level of sheaves. Moreover, we have the following classicity result of Bijakowski [Bij16].

THEOREM 8.17 (Bijakowski). *Let f be an overconvergent section of the sheaf ω^κ, $\kappa = (k_1 \geqslant k_2, k_3)$, which is proper for U_p of eigenvalue α. Then if*

$$3 + v(\alpha) < k_2 + k_3,$$

then f is a classical form of weight κ and level K^pI.

9. Constructing the eigenvariety

In this section we will construct the eigenvariety associated to the algebra $\mathcal{H} \otimes \mathcal{A}(p)$ and the spaces of overconvergent modular forms $M_\kappa^\dagger$. In order to do this, we will use Buzzard's construction of eigenvarieties, and we need to show that the spaces $M_\kappa^\dagger$ (and a bit more) are projective. The method of proof follows closely the lines of [AIP15], but as this case is simpler (because the toroidal compactification is) we chose to write the argument in details.

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9.1 Projection to the minimal compactification

Definition 9.1. Let X^* be the minimal compactification of Y as a (projective) scheme over $\mathrm{Spec}(\mathcal{O})$. There is a map

$$\eta : X \longrightarrow X^*,$$

from the toroidal to the minimal compactification. Denote by X^*_{rig} the rigid fiber and $X^*(v)$ the image of $X(v)$ in X^*_{rig}. If $v \in \mathbb{Q}$ this is an affinoid as X^*_{ord} is ($\det \omega$ is ample on the minimal compactification). Denote also by D the boundary in the toroidal compactification X, and by abuse of notation in $X_1(p^{2m})$ and $X_1(p^{2m})(v)$.

The idea to check that our spaces of cuspidal overconvergent modular forms are projective is to push the sheaves to $X^*(v)$, which is affinoid, and use the dévissage of [AIP15, Proposition A.1.2.2]. But we need to show that the pushforward of the family of sheaves $\mathfrak{w}_w^{\kappa^{0,\mathrm{un}}}(-D)$ is a small Banach sheaf. In order to do this, we will do as in [AIP15] and prove that the pushforward of the trivial sheaf has no higher cohomology, and we will need to calculate this locally.

9.1.1 *Description of the toroidal compactification.* Let $V = \mathcal{O}_E^3$ with the hermitian form $\langle\,,\rangle$ chosen in the datum. For all totally isotropic factor V', we denote by $C(V/(V')^\perp)$ the cone of symmetric hermitian semi-definite positive forms on $(V/(V')^\perp) \otimes_{\mathbb{Z}} \mathbb{R}$ with rational radical. Denote by $\mathfrak{C}$ the set of such V', and

$$\mathcal{C} = \coprod_{V' \in \mathfrak{C} \text{ non-zero}} C(V/(V')^\perp).$$

Remark 9.2. The subspaces V' are of dimension 1 (if non-zero), and $C(V/(V')^\perp) \simeq \mathbb{R}_+$.

Fix ψ_N-level N structure,

$$\psi_N : (\mathcal{O}_E/N\mathcal{O}_E)^3 \simeq V/NV,$$

and ψ of level p^{2m},

$$\psi : \mathcal{O}_E/p^{2m}e_1 \oplus p^m\mathcal{O}_E/p^{2m}e_2 \subset V/p^{2m}V.$$

Let $\Gamma \subset G(\mathbb{Z})$ be the congruence subgroup fixing the level outside p, and $\Gamma_1(p^{2m})$ fixing ψ_N and ψ. Suppose that N is big enough so that Γ is neat. Fix $\mathcal{S}$ a polyhedral decomposition of $\mathcal{C}$ which is Γ-admissible: on each $C(V/(V')^\perp) = \mathbb{R}_+$ there is a unique polyhedral decomposition and thus there is a unique decomposition $\mathcal{S}$ and it is automatically Γ (or $\Gamma_1(p^{2m})$)-admissible, smooth, and projective.

Recall the local charts of the toroidal compactification X. For each $V' \in \mathfrak{C}$ non-zero, we have a diagram,

$$\begin{tikzcd}
\mathcal{M}_{V'} \arrow[rr] \arrow[dr] & & \mathcal{M}_{V',\sigma} \arrow[dl] \\
& \mathcal{B}_{V'} \arrow[d] & \\
& Y_E &
\end{tikzcd}$$

where Y_E is the moduli space of elliptic curves with complex multiplication by $\mathcal{O}_E$ of principal level N structure. Denote by $\mathcal{E}$ the universal elliptic curve. Then $\mathcal{B}_{V'} = Ext^1(\mathcal{E}, \mathbb{G}_m \otimes O_E)$ is isogenous to ${}^t\mathcal{E}$, and is a $\mathbb{G}_m$-torsor, $\mathcal{M}_{V'} \longrightarrow \mathcal{B}_{V'}$ is a $\mathbb{G}_m$-torsor; it is the moduli space of principally polarized 1-motives, with ψ_N-level structure, and $\mathcal{M}_{V'} \longrightarrow \mathcal{M}_{V',\sigma}$ is an affine toroidal embedding associated to the cone decomposition of $C(V/(V')^\perp)$, locally isomorphic over $\mathcal{B}_{V'}$ to $\mathbb{G}_m \subset \mathbb{G}_a$.

Over $\mathcal{B}_{V'}$ we have a semi-abelian scheme of constant toric rank,

$$0 \longrightarrow \mathbb{G}_m \otimes_{\mathbb{Z}} \mathcal{O}_E \longrightarrow \widetilde{G}_{V'} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Denote by $Z_{V'}$ the closed stratum of $\mathcal{M}_{V',\sigma}$.

Recall that X is the toroidal compactification of our moduli space Y (it is unique as the polyhedral decomposition $\mathcal{S}$ is), as defined in [Lar92] or [Lan13] in full generality, and X^* is the minimal compactification. The toroidal compactification X is proper and smooth, and X^* is proper. Moreover, we have a (proper) map,

$$\eta : X \longrightarrow X^*.$$

Moreover, η is the identity on Y. As sets, X^* is a union of Y to which we glue points corresponding to elliptic curves with complex multiplication, one for each component of D, the boundary of X, and, over each $x \in X^*\backslash Y$, $\eta^{-1}(x)$ is a CM elliptic curve.

Denote by $\widehat{\mathcal{M}_{V',\sigma}}$ the completion of $\mathcal{M}_{V',\sigma}$ along the closed stratum $Z_{V'}$. On X there is a stratification indexed by $\mathfrak{C}/\Gamma$ (the open subset Y corresponds to $V' = \{0\}$). For all non-zero V', the completion of X along the V' stratum is isomorphic to $\widehat{\mathcal{M}_{V',\sigma}}$, as $\Gamma_{V'}$, the stabilizer of V', is trivial: $V' \simeq \mathcal{O}_E$ so $\Gamma_{V'} \subset \mathcal{O}_E^\times$, which is finite as E is quadratic imaginary, and thus because Γ is neat, $\Gamma_{V'} = \{1\}$.

As the Hasse invariant on the special fiber of X is defined as the one of the abelian part of the semi-abelian scheme, we can identify it with the same one on $\mathcal{M}_{V',\sigma}$, which comes from the special fiber of $Y_{V'} \simeq Y_E$. Denote by $\mathcal{Y}$, $\mathcal{X}$ the formal completions of Y, X along the special fiber. We have defined $\mathfrak{X}(v) \longrightarrow \mathfrak{X}$ as an open subset of a blow up. We will describe its boundary locally. Denote by $\mathfrak{Y}(v)$ the inverse image of $\mathfrak{Y}$. Denote by:

- $\mathfrak{Y}_E$ the formal completion along p of $Y_{V'}$;
- $\mathfrak{Y}_E(v)$ the open subset of $\mathcal{Y}_E(v)$ along $I = (p^v, \mathrm{Ha}_\tau)$ where I is generated by Ha_τ, but as every CM elliptic curve is μ-ordinary, $\mathfrak{Y}_E(v) = \mathfrak{Y}_E$;
- $\mathfrak{B}_{V'}$ the formal completion of $\mathcal{B}_{V'}$;
- and define $\mathfrak{M}_{V'}, \mathfrak{M}_{V',\sigma}, \mathfrak{Z}_{V'}$ similarly.

PROPOSITION 9.3. *The formal scheme $\mathfrak{X}(v)$ has a stratification indexed by $\mathfrak{C}/\Gamma$, and the stratum corresponding to V' is isomorphic to $\mathfrak{Z}_{V'}$ if V' is non-zero, and $\mathfrak{Y}(v)$ if $V' = \{0\}$. For all non-zero $V' \in \mathfrak{C}$ the completion of $\mathfrak{X}(v)$ along the V' stratum is isomorphic to $\widehat{\mathfrak{M}_{V',\sigma}}$ (completion along $\mathfrak{Z}_{V'}$).*

Proof. We complete and pull back the stratification of X. The analogous result on $\mathfrak{X}$ is true since we can invert the completion along p and the stratum. If $V' \neq 0$, it is simply that the boundary of $\mathfrak{X}$ is inside the μ-ordinary locus. For $V' = 0$ the stratum is the pull back of $\mathfrak{Y}$ inside $\mathfrak{X}(v)$, i.e. $\mathfrak{Y}(v)$. $\square$

We used the space $\mathfrak{X}_1(p^{2m})$ in the previous sections, and we would like to describe its boundary.

Let $\mathfrak{C}'$ be the subset of $V' \in \mathfrak{C}$ such that $\mathrm{Im}(\psi) \subset (V')^\perp/p^{2m}(V')^\perp$. The (unique) polyhedral decomposition previously considered induces also a (unique) polyhedral decomposition on

$$\coprod_{V' \in \mathfrak{C}' \text{ non-zero}} C(V/(V')^\perp),$$

which is $\Gamma_1(p^{2m})$ admissible.

For $V' \in \mathfrak{C}'$ non-zero, decompose

$$0 \longrightarrow V'/p^{2m} \longrightarrow (V')^\perp/p^{2m} \longrightarrow (V')^\perp/(V' + p^{2m}(V')^\perp) \longrightarrow 0,$$

and denote by W the image in $(V')^\perp/(V'+p^{2m}(V')^\perp)$ of $\psi(\mathcal{O}/p^{2m} \oplus p^m\mathcal{O}/p^m)$. This is isomorphic to $\mathcal{O}/p^m$. Indeed, as $(V')^\perp$ contains $e_1, p^m e_2$ modulo p^{2m}, $(V')^\perp/p^{2m} = \mathcal{O}/p^{2m}(e_1, e_2)$. Then $\overline{V'} = V'/p^{2m}$ is totally isotropic inside, i.e. generated by $ae_1 + be_2$ where $p^m|b$ (totally isotropic) and $a \in \mathcal{O}^\times$ (direct factor). Thus the image of ψ is generated in $(V')^\perp/(V' + p^{2m}(V')^\perp)$ by the image of $e_1 = a^{-1}be_2$ which is p^m-torsion.

We specify the following.

(i) Denote by $\mathcal{Y}_{V'}$ the rigid fiber of $\mathfrak{Y}_{V'}$.

(ii) Denote by $H_{m,V'}$ the canonical subgroup of level m of the universal elliptic scheme $\mathfrak{E}_{V'}$ over $\mathfrak{Y}_{V'}$.

(iii) Denote by $\mathcal{Y}_1(p^m)_{V'}$ the torsor $\mathrm{Isom}_{\mathcal{Y}_{V'}}((H_{m,V'})^D, W^\vee)$, and $\psi_{V'}$ the universal isomorphism.

(iv) Denote by $\mathfrak{Y}_1(p^m)$ the normalization of $\mathfrak{Y}_{V'}$ in $\mathcal{Y}_1(p^m)_{V'}$.

(v) There is an isogeny $i : \mathfrak{B}_{V'} \longrightarrow \mathfrak{E}_{V'}$, and if we write $i' : \mathfrak{E}_{V'} \longrightarrow \mathfrak{E}_{V'}/H_{m,V'}$, set

$$\mathfrak{B}_1(p^m)_{V'}(v) = \mathfrak{B}_{V'} \times_{i,\mathfrak{E},i'} \mathfrak{E}_{V'}/H_{m,V'}.$$

(vi) Denote by $\mathfrak{M}_1(p^m)_{V'}, \mathfrak{M}_1(p^m)_{V',\sigma}, \mathfrak{Z}_1(p^m)_{V',\sigma}$ the fibered products of the corresponding formal schemes with $\mathfrak{B}_1(p^m)_{V'}$ over $\mathfrak{B}_{V'}$.

PROPOSITION 9.4. *The formal scheme $\mathfrak{X}_1(p^{2m})(v)$ has a stratification indexed by $\mathfrak{C}'/\Gamma_1(p^{2m})$, for all non-zero V', the completion of $\mathfrak{X}_1(p^{2m})(v)$ along the V'-stratum is isomorphic to the completion $\widehat{\mathfrak{M}_1}(p^{2m})_{V',\sigma}$ along $\mathfrak{Z}_1(p^{2m})_{V'}$.*

Proof. This is known in rigid fiber, with the same construction, but the previous local charts are normal, and thus coincide with the normalization in their rigid fiber of the level Γ-charts. Thus $\mathfrak{M}_1(p^{2m})_{V',\sigma}$ is the normalization of $\mathfrak{M}_{V',\sigma}$ in $\mathcal{M}_1(p^{2m})_{V',\sigma}$. But the completion of $\mathfrak{M}_1(p^{2m})_{V',\sigma}$ along $\mathfrak{Z}_1(p^{2m})_{V',\sigma}$ coincides with the normalization of $\widehat{\mathfrak{M}_{V',\sigma}}^{V'} = \widehat{\mathfrak{X}(v)}^{V'}$ inside $\mathcal{M}_1(p^{2m})_{V',\sigma} = \widehat{X_1(p^n)(v)}^{V'}$. ◻

9.2 Minimal compactification

Let X^* be the minimal compactification of Y of level Γ. As a topological space, it corresponds to adding a finite set of points to Y, corresponding to CM elliptic curves. X^* is also stratified by $\mathfrak{C}/\Gamma$. Let $\overline{x} \in X^* \backslash Y$ a geometric point of the boundary, it corresponds to a point $x \in Y_E$.

Using the previous description of X, we can describe the local rings of X^*. Let $V' \in \mathfrak{C}$ be non-zero. Over $\mathcal{B}_{V'}$, $\mathcal{M}_{V'}$ is an affine $\mathbb{G}_m$-torsor, and we can thus write

$$\mathcal{M}_{V'} = \mathrm{Spec}_{B_V} \mathcal{L},$$

where $\mathcal{L}$ is a quasi-coherent $\mathcal{O}_{\mathcal{B}_{V'}}$-algebra endowed with an action of $\mathbb{G}_m$, that can be decomposed,

$$\mathcal{L} = \bigoplus_{k\in\mathbb{Z}} \mathcal{L}(k).$$

For all k, $\mathcal{L}(k)$ is locally free of rank 1 over $\mathcal{B}_{V'}$. Denote by $\widehat{B_{V';\overline{x}}}$ the completion of $\mathcal{B}_{V'}$ along the fiber over $\overline{x}$. We have the following proposition.

Proposition 9.5. *The scheme X^* is stratified by $\mathfrak{C}/\Gamma$ and $\eta : X \longrightarrow X^*$ is compatible with stratifications. Moreover, for all V', $X^*_{V'}$ is isomorphic to $Y_{V'}$ and for all $\overline{x} \in X^\star_{V'}$ a geometric point,*

$$\widehat{\mathcal{O}_{X^*,\overline{x}}} = \prod_{k\in\mathbb{Z}} H^0(\widehat{B_{V',\overline{x}}}, \mathcal{L}(k)),$$

where $\widehat{\mathcal{O}_{X^*,\overline{x}}}$ is the completion of the strict henselisation of $\mathcal{O}_{X^*}$ at $\overline{x}$.

Proof. This is Serre's theorem on global sections of the structure sheaf on proper schemes (as η is proper and X^* is normal), the theorem of formal functions and the previous description of X. □

The Hasse invariant ha_τ descends to the special fiber of X^*, and we can thus define $\mathfrak{X}^*$ the formal completion of X^* along its special fiber and $\mathfrak{X}^*(v)$ the normalization of the open subspace of the blow up of X^* along (p^v, ha_τ) where this ideal is generated by ha_τ.

Proposition 9.6. *For all $V' \in \mathfrak{C}$ the V'-strata of $\mathfrak{X}^*(v)$ is $\mathfrak{Y}_{V'}(v)$ (and Y_E if V' is non-zero).*

Proof. This is known before the blow up, and thus for V' non-zero as the boundary is contained in the μ-ordinary locus. But for $V' = 0$ this is tautologous. □

9.3 Higher cohomology and projectivity of the space of overconvergent automorphic forms

We will look at the following diagram.

$$\begin{tikzcd}
\mathfrak{X}_1(p^n)(v) \arrow[rr, "\pi_4"] \arrow[dr, "\eta"'] & & \mathfrak{X}(v) \arrow[dl, "\pi"] \\
& \mathfrak{X}^*(v) &
\end{tikzcd}$$

Proposition 9.7. *Let D be the boundary of $\mathfrak{X}_1(p^{2m})(v)$. Then for all $q > 1$*

$$R^q\eta_*\mathcal{O}_{\mathfrak{X}_1(p^{2m})(v)}(-D) = 0.$$

Proof. It is enough to work locally at $\overline{x}$ a geometric point of the boundary of $\mathfrak{X}^*(v)$, and by the theorem of formal functions,

$$(\eta_*\mathcal{O}_{\widehat{\mathfrak{X}_1(p^{2m})(v)}}(-D))_{\overline{x}} = H^q(\mathfrak{X}_1\widehat{(p^{2m})}(v)^{\eta^{-1}(x)}, \mathcal{O}_{\mathfrak{X}_1\widehat{(p^{2m})}(v)^{\eta^{-1}(x)}}(-D)).$$

We will thus show that the right-hand side is zero. But the completion $\mathfrak{X}_1\widehat{(p^{2m})}(v)^{\eta^{-1}(x)}$ is isomorphic to a finite disjoint union of spaces of the form $\mathfrak{M}_1\widehat{(p^{2m})}_{V',\sigma}^{\overline{y}}$ for $\overline{y}$ a geometric point in Y_E. Denote by M_σ this completed space. As

$$M_\sigma = \mathrm{Spf}_{\mathfrak{B}_1\widehat{(p^m)}_{V'}}\Big(\widehat{\bigoplus_{k\geqslant 0}}\mathcal{L}(k)\Big),$$

and thus the morphism,

$$M_\sigma \longrightarrow \widehat{\mathfrak{B}_1(p^m)}_{V'},$$

is affine, we have the equality,

$$H^q(M_\sigma, \mathcal{O}(-D)) = \prod_{k>0} H^q(\widehat{\mathfrak{B}_1(p^m)}_{V'}, \mathcal{L}(k))$$

(the product is over $k > 0$ as we take the cohomology in $\mathcal{O}(-D)$). But for $k > 0$, $\mathcal{L}(k)$ is very ample on the elliptic curve $\mathfrak{B}_1(p^m)_{V'}$, and thus $H^q(\widehat{\mathfrak{B}_1(p^m)}_{V'}, \mathcal{L}(k)) = 0$ for all $q > 0$. □

THEOREM 9.8. *For $m \geqslant l$ two integers, we have the following commutative diagram,*

$$\begin{tikzcd} X_1(p^n)(v)_l \arrow[r, "i"] \arrow[d, "\eta_l"] & X_1(p^n)(v)_m \arrow[d, "\eta_m"] \\ X_*(v)_l \arrow[r, "i'"] & X_*(v)_m \end{tikzcd}$$

and the following base change property is satisfied,

$$i'^*((\eta_m)_*\mathfrak{w}^{\kappa^0\dagger}_{w,m}(-D)) = (\eta_l)_*\mathfrak{w}^{\kappa^0\dagger}_{w,l}(-D).$$

In particular, $\eta_*\mathfrak{w}^{\kappa^0\dagger}_w(-D)$ is a small Banach sheaf on $\mathfrak{X}^*(v)$. The same result is true over $\mathfrak{X}^*(v) \times \mathfrak{W}(w)^0$ for

$$(\eta \times 1)_*\mathfrak{w}^{\kappa^{0,\mathrm{un}}\dagger}_w(-D).$$

Proof. We can just restrict to $l = m-1$, but as inductive limit and direct image commute, and as the kernel $\mathfrak{w}^{\kappa^{0,\mathrm{un}}\dagger}_{w,m} \longrightarrow \mathfrak{w}^{\kappa^{0,\mathrm{un}}\dagger}_{w,m-1}$ is isomorphic to $\mathfrak{w}^{\kappa^{0,\mathrm{un}}\dagger}_{w,1}$ which is itself a direct limit of sheaves with graded pieces isomorphic to $\mathcal{O}_{X_1(p^m)}/\pi$ (see Corollary C.5) and thus by the previous proposition we have the announced equality. We can thus use [AIP15, Proposition A.1.3.1] which proves that $(\eta \times 1)_*\mathfrak{w}^{\kappa^{0,\mathrm{un}}\dagger}_w$ is a small formal Banach sheaf (recall that η is proper). □

PROPOSITION 9.9. *Let $w > 0$. Write $\mathfrak{W}(w)^0 = \mathrm{Spf}(A)$. Then*

$$M^{\kappa^{0,\mathrm{un}}\dagger,\mathrm{cusp}}_{v,w} = H^0(X^*(v) \times \mathcal{W}(w)^0, (\eta \times 1)_*\omega^{\kappa^{0,\mathrm{un}}\dagger}_w(-D))$$

is a projective $A[1/p]$-Banach module. Moreover the specialization map, for $\kappa \in \mathcal{W}(w)^0$,

$$M^{\kappa^{0,\mathrm{un}}\dagger,\mathrm{cusp}}_{v,w} \longrightarrow H^0(X^*(v), \eta_*\omega^{\kappa\dagger}_w(-D)),$$

is surjective.

Proof. This is proved exactly as in [AIP15, Corollary 8.2.3.2]. Let us sketch the ideas. Fix $(\mathfrak{U}_i)_{1\geqslant i\geqslant r}$ a (finite) affine covering of $\mathfrak{X}^*(v)$, and for $\underline{i} = (i_1, \dots, i_s) \in \{1, \dots, r\}^s$ denote by $\mathfrak{U}_{\underline{i}}$ the intersection $\mathfrak{U}_{i_1} \cap \dots \mathfrak{U}_{i_s}$. Then

$$M_{\underline{i},\infty} = H^0(\mathfrak{U}_{\underline{i}} \times \mathfrak{W}(w)^0, (\eta \times 1)_*\mathfrak{w}^{\kappa^{0,\mathrm{un}}\dagger}_w(-D)),$$

is isomorphic to the p-adic completion of a free A-module (i.e. is orthonormalizable). This is essentially Corollary C.5 and topological Nakayama's lemma. But then, as $X^*(v)$ is affinoid, the

Cech complex after inverting p is exact and thus [AIP15, Theorem A.1.2.2] provides a resolution of $M_{v,w}^{\kappa^{0,\mathrm{un}}\dagger,\mathrm{cusp}}$ by the $M_{\underline{i},\infty}[1/p]$, and thus $M_{v,w}^{\kappa^{0,\mathrm{un}}\dagger,\mathrm{cusp}}$ is projective. For the surjectivity assertion, fix $\mathfrak{p}_\kappa$ the maximal ideal of $A[1/p]$ corresponding to κ and consider the Koszul resolution of $A[1/p]/\mathfrak{p}_\kappa$. Tensoring this for each $\underline{i}$ with $\eta_*\mathfrak{w}^{\kappa^{0,\mathrm{un}}\dagger}(-D)(\mathfrak{U}_{\underline{i}})$ gives a resolution of $\eta_*\mathfrak{w}_w^{\kappa\dagger}(-D)(\mathfrak{U}_{\underline{i}})$. This gives a double complex where each column (for a fixed index of the Koszul complex) is exact. But each line (for a fixed $\underline{i}$ non-trivial) is also exact by the previous acyclicity, and thus we have the following bottom right square,

$$\begin{tikzcd}
\prod_{1\geqslant i\geqslant r}\omega^{\kappa^{0,un}\dagger}(\mathcal{U}_i) \arrow[r] \arrow[d] & \prod_{1\geqslant i\geqslant r}\omega^{\kappa\dagger}(\mathcal{U}_i) \arrow[r] \arrow[d] & 0 \\
\omega^{\kappa^{0,un}\dagger}(X^*(v) \arrow[r, "\pi_\kappa"] \arrow[d] & \omega^{\kappa\dagger}(X^*(v)) \arrow[d] & \\
0 & 0 &
\end{tikzcd}$$

which proves that π_κ is surjective. $\square$

Proposition 9.10. *Write* $\mathrm{Spm}(B)=\mathcal{W}(w)$. *Then the* B*-module*

$$H^0(\mathcal{X}(v)\times\mathcal{W}(w),\omega_w^{\kappa^{\mathrm{un}}\dagger}(-D))$$

is projective. Moreover, for every $\kappa\in\mathcal{W}(w)$, *the specialization map,*

$$H^0(\mathcal{X}(v)\times\mathcal{W}(w),\omega_w^{\kappa^{\mathrm{un}}\dagger}(-D))\longrightarrow H^0(\mathcal{X}(v),\omega_w^{\kappa\dagger}(-D)),$$

is surjective.

Proof. We can identify the B-module,

$$M'=H^0(\mathcal{X}(v)\times\mathcal{W}(w),\omega_w^{\kappa^{\mathrm{un}}\dagger}(-D)),$$

with $(M_{v,w}^{\kappa^{0,\mathrm{un}}\dagger,\mathrm{cusp}}\otimes_{A[1/p]}B(-\kappa^{\mathrm{un}}))^{B_n}$. But now B_n is a finite group, and B is of characteristic zero, thus M' is a direct factor in a projective B-module, and is thus surjective. Moreover, as B_n is finite, the (higher) group cohomology vanishes, and the specialization map remains surjective. $\square$

9.4 Types

Let K_f^p be a compact open subgroup of $G(\mathbb{A}_f^p)$. Let $K_f=K_f^pI$ where $I\subset G(\mathbb{Q}_p)$ is the Iwahori subgroup. Fix (J,V_J) a complex continuous irreducible representation of K_f, trivial at p and outside a level N, it is of finite dimension and finite image, and thus defined over a number field. Denote by $K^0\subset K_f$ its Kernel.

Definition 9.11. The space of Picard modular forms of weight κ, v-overconvergent, w-analytic, of type (K_f,J) is

$$\mathrm{Hom}_{K_f}(J,H^0(\mathcal{X}(v),\omega_w^{\kappa\dagger})).$$

The space of overconvergent locally analytic Picard modular forms of weight κ and type (K_f,J) is then

$$M_\kappa^{\dagger,(K_f,J)}=\mathrm{Hom}_{K_f}\Bigl(J,\varinjlim_{v\to0,w\to\infty}\Bigr)H^0(\mathcal{X}(v),\omega_w^{\kappa\dagger}).$$

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Remark 9.12. In the beginning of this section we made the assumption that the level Γ, outside p, is big enough ('neat'). But using the previous definition we can get rid of this assumption by taking K_f^p big enough to have the neatness assumption, and take J the trivial representation to descend our families for any level outside p, as the following proposition shows.

Proposition 9.13. *The space*

$$M_{\mathrm{cusp},v,w}^{(K,J),\kappa^{\mathrm{un}}\dagger} := \mathrm{Hom}_{K_f}(J, H^0(\mathcal{X}(v)\times\mathcal{W}(w), \omega_w^{\kappa^{\mathrm{un}}\dagger}(-D))),$$

is a projective $\mathcal{O}_{\mathcal{W}(w)}$-module, and the specialization map is surjective.

Proof. Suppose $K(N) = K \subset K^0 = \mathrm{Ker}(J)$ is neat (outside p, up to enlarging it). Then we have shown that, in level K,

$$H^0(\mathcal{X}(v)\times\mathcal{W}(w), \omega_w^{\kappa^{\mathrm{un}}\dagger}(-D)),$$

is projective, and that the corresponding specialization map is surjective. We can thus twist the K_f/K action by V_J^* and take the invariants over J; as K_f/K is finite, the space is a direct factor inside $H^0(\mathcal{X}(v)\times\mathcal{W}(w), \omega_w^{\kappa^{\mathrm{un}}\dagger}(-D))\otimes V_J^*$ and higher cohomology vanishes. $\square$

Remark 9.14. The same argument applies when p splits in E, for the spaces of overconvergent modular forms defined in [Bra16, AIP15]. In particular we can construct families of Picard modular forms with fixed type when p is unramified.

9.5 Eigenvarieties

Theorem 9.15. *Let p be a prime number, unramified in E. Let $\mathcal{W}$ be the p-adic weight space of $U(2,1)$. When p is inert, it is defined in § 3, and when p splits it is a disjoint union of three-dimensional balls over $\mathbb{Q}_p$. Fix (K_J, J) a type outside p, $K \subset \mathrm{Ker}\, J$ a neat level outside p, and let S be the set of places where K is not compact maximal or p. There exists an equidimensional, of dimension 3, eigenvariety $\mathcal{E}$ and a locally finite map,*

$$w : \mathcal{E} \longrightarrow \mathcal{W},$$

such that for any $\kappa \in \mathcal{W}$, $w^{-1}(\kappa)$ is in bijection with eigensystems for $\mathbb{T}^S \otimes_{\mathbb{Z}} \mathcal{A}(p)$ acting on the space of overconvergent, locally analytic, modular forms of weight κ and type-level (K_J, J) (and Iwahori level at p), finite slope for U_p.

Proof. If p is split this is a particular case of the main result of [Bra16] (taking into account the previous remark and the normalization of the Hecke operators). If p is inert, this is a consequence of Buzzard and Coleman's machinery [Buz07] using for all compatible v, w $(\mathcal{W}(w), M_{\mathrm{cusp},v,w}^{(K,J),\kappa^{\mathrm{un}}\dagger}$, $U_p, \mathcal{H}^{Np}\otimes\mathcal{A}(p))$ and gluing along v, w. $\square$

9.6 Convention on weights

As in [BC04], we set as convention the Hodge–Tate weight of the cyclotomic character to be -1. Fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}_p}$ compatible with the inclusions $\overline{E} \subset \mathbb{C}$ that extend τ_∞ and denote by τ, σ_τ the p-adic places at p inert corresponding to $\tau_\infty, c\tau_\infty$. If p is split, we will instead call $v, \overline{v}$ the places corresponding to $\tau_\infty, c\tau_\infty$, but in this section we focus on p inert, even if a similar result hold with $v, \overline{v}$.

Let us recall the different parameters that are associated to an algebraic automorphic representation π of $GU(2,1)$ that we will need, following partly [Ski12]. There is $\lambda = ((\lambda_1, \lambda_2, \lambda_3), \lambda_0)$, the Harish-Chandra parameter, there is $c = ((c_1, c_2, c_3), c_0)$, the highest weight of the

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algebraic representation which has the same infinitesimal character as π_∞ in the discrete series case and (c_0, c_0') is the parameter at infinity of ω_π^c the conjugate of the central character of π. There is $\kappa = (k_1, k_2, k_3)$ the classical weight such that π_f appears in $H^0(X_K, \omega^\kappa)$ (if it exists) and there are the Hodge–Tate weights $((h_1^\tau, h_2^\tau, h_3^\tau), (h_1^{\sigma\tau}, h_2^{\sigma\tau}, h_3^{\sigma\tau}))$ of the Galois representation of G_E associated to π by Blasius and Rogawski [BR92] or Skinner [Ski12]. Let us explain how they are related in the case of discrete series.

First denote by ρ_n the half-sum of the positive non-compact roots and ρ_c the half-sum of the positive compact roots (see [Gol14, § 5.3]). We have then, for $i \geqslant 1, \lambda_i = (c+\rho_n+\rho_c)_i$, and (c_0, c_0') is the infinite weight of the dual of the central character. The calculation of Harris and Goldring gives $(-k_3, k_1, k_2) = \lambda + \rho_n - \rho_c$ (forgetting the λ_0 factor here, it is because we only considered three parameters in the weight space). The Hodge–Tate weights of the Galois representation associated to π depends of course on the normalization of the correspondence, but take the one of Skinner [Ski12, § 4.2, 4.3 and after Theorem 10], we get

$$\begin{aligned}&((h_1^\tau, h_2^\tau, h_3^\tau), (h_1^{\sigma\tau}, h_2^{\sigma\tau}, h_3^{\sigma\tau}))\\ &\quad = ((-c_0 - c_1, -c_0 - c_2 + 1, -c_0 - c_3 + 2), (-c_0' + c_3, -c_0' + c_2 + 1, -c_0' + c_1 + 2)).\end{aligned}$$

Remark 9.16. Let $f \in H^0(X, \omega^\kappa)$ be a classical form. To f is associated φ_f an automorphic form, with equivariant Hecke action, cf. Proposition 2.6, and thus an autormorphic representation π_f.

Before going further, let us remark that a (algebraic) representation π of $GU(2,1)$ is equivalent to a pair (π^0, ψ) of π^0 a (algebraic) automorphic representation of $U(2,1)$ (the restriction of π) and a (algebraic) Hecke character of $GU(1) = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ (the central character of π) which extend the central character of π^0 (see § 10.4). To an algebraic (nice) π is associated a (non-necessarily polarized) Galois representation ρ_π, but also a pair π^0, ψ_π, and to π^0 is associated a polarized Galois representation, which is what we will need. Thus from Skinner's normalization, removing the central character of π, we get the following proposition (we could also directly use [CH13]).

PROPOSITION 9.17. *Let $\kappa = (k_1, k_2, k_3) \in \mathbb{Z}^3$ and $f = H^0(X, \omega^\kappa)$ which is an eigenvector for the Hecke operators outside p. Write $|k| = k_1 + k_2 + k_3$. Let π be the automorphic representation corresponding to f (i.e. a irreducible factor in the representation generated by Φ_f of § 2.2).*

Suppose π_∞ is a (regular) discrete series of Harish-Chandra parameter λ, then

$$\lambda = ((k_1, k_2 - 1, 1 - k_3), |k|)$$

(see [Gol14, § 5]) with $k_1 \geqslant k_2 > 2 - k_3$. Denote by $\rho_{\pi,\mathrm{Ski}}$ the p-adic Galois representation associated to π by Skinner [Ski12]. Then $\rho_{\pi,\mathrm{Ski}}$ satisfies the following essentially self-polarization,

$$\rho_{\pi,\mathrm{Ski}}^c \simeq \rho_{\pi,\mathrm{Ski}}^\vee \otimes \varepsilon_{\mathrm{cycl}}^{-|k|-2} \otimes \rho_\psi,$$

where ε is the cyclotomic character, ψ is a finite Hecke character, and if ω_π denotes the central character of π, $\omega_\pi\omega_\pi^c$ is of the form $N^{-|k|}\psi$.

Then, the τ-Hodge–Tate weights of $\rho_{\pi,\mathrm{Ski}}$ are

$$(k_2 + 1, 2 + k_1, k_3 + k_1 + k_2),$$

and the $\sigma\tau$-Hodge–Tate weights are

$$(2, k_2 + k_3, k_1 + k_3 + 1).$$

To $\pi^0 = \pi_{|U(2,1)}$ is associated a polarized continuous Galois representation ρ_π verifying

$$\rho_\pi^c \simeq \rho_\pi^\vee,$$

of τ-Hodge–Tate weights $(1-k_3, k_2-1, k_1)$ and (thus) $\sigma\tau$-Hodge–Tate weights $(-k_1, 1-k_2, k_3-1)$.

Proof. The calculation of λ in terms of κ is exactly [Gol14, Theorem 5.5.1]. Remark also that we can calculate in terms of κ which are the discrete series by Harish-Chandra theorem [Kna16, Theorem 6.6], and we find $k_1 \geqslant k_2 > 2 - k_3$. Thus, the calculation of the Hodge–Tate weights of the Galois representations associated to π are [Ski12, Theorem 10], with the previous calculation of c in terms of λ. The representation ρ_π is given by $\rho_{\pi,\mathrm{Ski}}\rho_{\overline{\omega_\pi}}^{-1}(1)$. □

Remark 9.18. In terms of the τ-Hodge–Tate weights of ρ_π, discrete series correspond to $h_1 < h_2 < h_3$. All of this is coherent with the Bernstein–Gelfand–Gelfand (BGG) decomposition, see for example [Lan12, (2.3)], for $\underline{c} = \mu = (a,b,c)$ a highest weight representation of G,

$$H^2_{\mathrm{dR}}(Y, V_\mu^\vee) = H^2(X, \omega^{(-b,-a,c)}) \oplus H^1(X, \omega^{(1-c,-a,b+1)}) \oplus H^0(X, \omega^{(1-c,1-b,a+2)}).$$

Denote by $\mathcal{Z} \subset \mathcal{E}$ the set of characters corresponding to regular (i.e. $w_2(z) \leqslant w_1(z) < -2 - w_3(z) \in \mathbb{Z}^3$) classical modular forms (recall that if f is classical of weight (k_1,k_2,k_3), $w(f) = (-k_2, -k_1, -k_3)$). For each $z \in \mathcal{Z}$, there exists f a classical form, which determines Π an automorphic representation of $GU(2,1)$ (generated by Φ_f defined in § 2.2). Such a Π corresponds to a packet, to which by the work of Blasius and Rogawski [BR92, Theorem 1.9.1] (see also for generalization to higher-dimension unitary groups and local global compatibilities the work of many authors, in particular [Bel06a, CH13, Ski12, BGHT11]) is associated a number field E_z, and compatible system of Galois representations,

$$\rho_{z,\lambda} : G_E \longrightarrow \mathrm{GL}_3(E_{z,\lambda}), \quad \forall \lambda \in \mathrm{Spm}(\mathcal{O}_{E_z}),$$

satisfying local global compatibilities (see for example [Ski12], where the association is normalized by the previous proposition (the Hodge–Tate weight of the cyclotomic character being -1) and the previous proposition for a normalization suitable to our needs). In particular, denote S the set of prime of E where $\mathrm{Ker}(J)I$ is not hyperspecial, and if ℓ a prime under λ, denote by S_ℓ the set of places of E dividing λ. Then $\rho_{z,\lambda}$, is unramified outside SS_ℓ.

We have the classical proposition, which is one reason why eigenvarieties are so useful (see for example [BC09] proposition 7.5.4),

Proposition 9.19. *Let p be unramified in E. To each $z \in \mathcal{Z}$, denote by ρ_z the (p-adic) polarized representation associated to z by Proposition 9.17. There exists a unique continuous pseudocharacter*

$$T : G_{E,S} \longrightarrow \mathcal{O}_{\mathcal{E}},$$

such that for all $z \in \mathcal{Z}$, $T_z = \mathrm{tr}(\rho_z)$. Moreover the pseudocharacter T satisfies $T^\perp = T$, where $T^\perp(g) = T((\tau g \tau)^{-1})$ for all $g \in G_E$.

Proof. $\mathcal{Z}$ is dense in $\mathcal{E}$ by density of very regular weights in $\mathcal{W}$ and the two classicity results (8.16 and 8.17). We only need [Che04, Proposition 7.1] to finish, the hypothesis (H) there being verified by the Frobenius classes in S. The polarization assumption follows from the case of $z \in \mathcal{Z}$ by density. □

10. Application to a conjecture of Bloch and Kato

Let E be a quadratic imaginary field, and fix an algebraic Hecke character,

$$\chi : \mathbb{A}_E^\times / E^\times \longrightarrow \mathbb{C}^\times,$$

such that, for all $z \in \mathbb{C}^\times, \chi_\infty(z) = z^a \overline{z}^b$, for some $a, b \in \mathbb{Z}$. Call $w = -a - b$ the motivic weight of χ. Denote by $\chi_p : G_E \longrightarrow K$, where K is a finite extension of $\mathbb{Q}_p$, the p-adic realization of χ.

We are interested in the Selmer group $H^1_f(E, \chi_p)$, which parameterizes extensions U,

$$0 \longrightarrow \chi_p \longrightarrow U \longrightarrow 1 \longrightarrow 0,$$

which have good reduction everywhere ([BK90, FP94], [BC09, ch. 5] and the introduction of this article).

Associated to χ there is also an L-function $L(\chi, s)$, where s is a complex variable, which is a meromorphic function on $\mathbb{C}$, which satisfies a functional equation,

$$\Lambda(\chi, s) = \varepsilon(\chi, s)\Lambda(\chi^*(1), -s),$$

where χ^* is the contragredient representation, $\Lambda(V, s)$ is the completed L-function of V, a product of $L(V, s)$ by a finite number of Γ-factors.

The conjecture of Bloch and Kato (more precisely a particular case of) in this case is the following equality,

$$\dim H^1_f(E, \chi_p) - \dim(\chi_p)^{G_E} = \operatorname{ord}_{s=0} L(\chi^*(1), s).$$

The conjecture is more generally for a Galois representation ρ of the Galois group G_F of a number field, but in the previous case we have a special case by the theorem of Rubin on Iwasawa's main conjecture for CM elliptic curves,

Conjecture 10.1. Suppose that χ is polarized and of weight -1, i.e.

$$\chi^\perp = \chi|.|^{-1},$$

where $\chi^\perp(z) = \chi^{-1}(czc)$, and $c \in G_\mathbb{Q}$ induces the complex conjugation in E. Then

$$\operatorname{ord}_{s=0} L(\chi, s) \neq 0 \Rightarrow \dim H^1_f(E, \chi_p) \geqslant 1.$$

Remark 10.2. Under the previous polarization assumption, we have $L(\chi^*(1), s) = L(\chi, s)$. The previous conjecture is mainly known by the work of Rubin [Rub91] if $p \nmid |\mathcal{O}_E^\times|$.

In the rest of the article, we will prove the following theorem.

Theorem 10.3. *Suppose that χ is polarized and of weight -1, i.e.*

$$\chi^\perp = \chi|.|^{-1},$$

where $\chi^\perp(z) = \chi^{-1}(czc)$, and $c \in G_\mathbb{Q}$ induces the complex conjugation in E. Suppose that p is unramified in E and that $p \nmid \operatorname{Cond}(\chi)$. If p is inert suppose, moreover, that $p \neq 2$. Then

$$\operatorname{ord}_{s=0} L(\chi, s) \neq 0 \quad \textit{and} \quad \operatorname{ord}_{s=0} L(\chi, s) \textit{ is even } \Rightarrow \dim H^1_f(E, \chi_p) \geqslant 1.$$

Remark 10.4. The same result but for $\operatorname{ord}_{s=0} L(\chi, s)$ odd (and strictly speaking for p split) is proved in [BC04], and using the same method with the eigenvariety for the group $U(3)$.

Definition 10.5. To stick with notations of [BC04], denote by k the positive odd integer such that $\chi_\infty(z) = z^{(k+1)/2}\overline{z}^{(1-k)/2}$ (i.e. $k = 2a - 1 = 1 - 2b$). We suppose $k \geqslant 1$, i.e. $a \geqslant 1$ (which we can always suppose up to changing χ by χ^c, which does not change either the L-function nor the dimension of the Selmer group).

10.1 Endoscopic transfer, after Rogawski

Let $\chi_0 = \chi|.|^{-1/2}$ the unitary character as in [BC04]. We will define following Rogawski [Rog92] an automorphic representation of $U(2,1)$, by constructing it at each place.

10.1.1 *If ℓ is split in E.* Write $\ell = v\overline{v}$ and the choice of say v induces an isomorphism $U(2,1)(\mathbb{Q}_\ell) \overset{i_v}{\simeq} \mathrm{GL}_3(\mathbb{Q}_\ell)$. Let $P = MN$ be the standard parabolic of $\mathrm{GL}_3(\mathbb{Q}_p)$ with Levi $M = \mathrm{GL}_2 \times \mathrm{GL}_1$. Define

$$\widetilde{\chi_{0,\ell}} : \begin{pmatrix} A & \\ & b \end{pmatrix} \in \mathrm{GL}_2 \times \mathrm{GL}_1 \longmapsto \chi_{0,\ell}(\det A),$$

trivially extended to P, and denote by $\mathrm{ind}-n_P^G(\widetilde{\chi_{0,\ell}})$ the normalized induction of $\widetilde{\chi_{0,\ell}}$. Then set

$$\pi_\ell^n(\chi) = i_v^* \,\mathrm{ind}-n_P^G(\widetilde{\chi_{0,\ell}}).$$

If χ_ℓ is unramified, then so is $\pi_\ell^n(\chi)$. Fix in this case K_ℓ a maximal compact subgroup of $U(2,1)(\mathbb{Q}_\ell)$.

10.1.2 *If ℓ is inert or ramified in E.* In this case write $T = \mathcal{O}_{E_\ell}^\times \times \mathcal{O}_{E_\ell}^1$ the torus of $U(2,1)(\mathbb{Q}_\ell)$, and consider the following character of T,

$$\widetilde{\chi}_\ell : \begin{pmatrix} a & & \\ & b & \\ & & \overline{a}^{-1} \end{pmatrix} \longmapsto \chi_\ell(a),$$

trivially extended to the Borel subgroup B of $U(2,1)(\mathbb{Q}_\ell)$. Then the normalized induction $\mathrm{ind}-n_B^{U(2,1)(\mathbb{Q}_\ell)}(\widetilde{\chi}_\ell)$ has two Jordan–Holder factors, one which is non-tempered that we denote by $\pi_\ell^n(\chi)$ and the other one, which is square integrable, that we denote by $\pi_\ell^2(\chi)$, see [Rog92].

If ℓ is inert and $\chi_{0,\ell}$ is unramified, $\pi_\ell^n(\chi)$ is also unramified (Satake) and we can choose K_ℓ a maximal compact for which $\pi_\ell^n(\chi)$ has a non-zero fixed vector.

If ℓ is ramified and $\chi_{0,\ell}$ is unramified, there are two conjugacy classes of maximal compact subgroup, but only one of them, denoted K_ℓ (called very special), verifies that $\pi_\ell^n(\chi)$ has a non-zero fixed vector under K_ℓ whereas $\pi_\ell^2(\chi)$ has none.

10.1.3 *Construction at infinity.* As in the inert case, let $\pi_\infty^n(\chi)$ be the non-tempered Jordan–Holder factor of $\mathrm{ind}-n_B^{U(2,1)(\mathbb{R})}(\widetilde{\chi_\infty})$.

Then we have the following proposition, following Rogawski.

Proposition 10.6 (Rogawski). *Suppose $a \geqslant 1$. Recall that $\mathrm{ord}_{s=0} L(\chi,s)$ is even. Then the representation,*

$$\pi^n(\chi) = \bigotimes_\ell{}' \pi_\ell^n(\chi) \otimes \pi_\infty^n(\chi),$$

is an automorphic representation of $U(2,1)$. If moreover $L(\chi,0) = 0$, it is a cuspidal representation. Its Galois representation (associated by the work of [LR92] *or see also* [BC04, § 3.2.3 and Proposition 4.1]*) $\rho_{\pi^n(\chi),p} : G_E \longrightarrow \mathrm{GL}_3(\overline{\mathbb{Q}_p})$ is*

$$\rho_{\pi^n(\chi),p} = (1 \oplus \chi_p \oplus \chi_p^\perp).$$

Moreover, its τ-Hodge–Tate weights are $(-(k+1)/2, -(k-1)/2, 0) = (-a, 1-a, 0)$.

10.2 Accessible refinement (at p) for $\pi^n(\chi)$

In order to construct a p-adic family of modular forms passing through $\pi^n(\chi)$, we need to construct inside $\pi_p^n(\chi)^I$ a form which is proper for the operator U_p previously defined. Strictly speaking, U_p is defined for $GU(2,1)$, and when p is inert, U_p is associated to the operator of the double Iwahori class IU_p^cI where

$$U_p^c = \begin{pmatrix} p^2 & & \\ & p & \\ & & 1 \end{pmatrix}.$$

This class is not in $U(2,1)(\mathbb{Q}_p)$, but $p^{-1}U_p^c$ is. Fix $T \subset B \subset U(2,1)(\mathbb{Q}_p)$. As p is unramified, we have as a representation of T/T^0 (T^0 a maximal compact in T), for π a representation of $U(2,1)(\mathbb{Q}_p)$ an isomorphism, see [BC04],

$$\pi^I \simeq (\pi_N)^{T^0} \otimes \delta_B^{-1}.$$

Thus to understand how the double coset operator U_p^c in the Iwahori–Hecke algebra acts, we only need to determine the Jacquet functor $(\pi_p^n(\chi))_N$ as a representation of T. If p splits, this is computed in [BC04] (and [BC09] in greater generality), so suppose that p is inert.

Proposition 10.7. *Let $\widetilde{\chi}$ be the (unramified) character of the torus T of $U(2,1)(\mathbb{Q}_p)$ defined by*

$$\begin{pmatrix} a & & \\ & e & \\ & & \overline{a}^{-1} \end{pmatrix} \longmapsto \chi_p(a).$$

Denote by $w \in W_{U(2,1)(\mathbb{Q}_p)} \simeq \mathbb{Z}/2\mathbb{Z}$ the non-trivial element and $\widetilde{\chi}^w$ the corresponding character of T ($\widetilde{\chi}^w = \chi(w \cdot w)$). Then the unique admissible refinement of $\pi_p^n(\chi)$ is given by $\widetilde{\chi}^w$, i.e. $\pi_p^n(\chi)_N = \widetilde{\chi}^w \delta_B^{1/2}$.

Proof. Write for the proof $G = U(2,1)(\mathbb{Q}_p)$. According to Rogawski we have $\mathrm{ind}-n_B^G(\widetilde{\chi})^{ss} = \{\pi_p^n, \pi_p^2\}$ and $(\mathrm{ind}-n_B^G(\widetilde{\chi}))_N^{ss} = \{\widetilde{\chi}\delta_B^{1/2}, \widetilde{\chi}^w\delta_B^{1/2}\}$ by Bernstein and Zelevinski's geometric lemma. Following [BC09], denote for $\sigma \in W_G$ $S(\widetilde{\chi}^\sigma)$ the unique subrepresentation of $\mathrm{ind}-n_B^G(\widetilde{\chi}^\sigma)$ (this induction is non-split by [Key84] for example). It is also the Jordan–Hölder factor that contains $\widetilde{\chi}^\sigma\delta_B^{1/2}$ inside its semi-simplified Jacquet functor. Thus $S(\widetilde{\chi}) = \pi_p^2$ or $S(\widetilde{\chi}) = \pi_p^n$. Also, as changing $\widetilde{\chi}$ by $\widetilde{\chi}^w$ exchanges the subrepresentation and the quotient in the induced representation, $S(\widetilde{\chi}) \neq S(\widetilde{\chi}^w)$. So the proposition is equivalent to $\pi_p^2 = S(\widetilde{\chi})$. Let us remark that it is announced in [Rog92], as π_p^n is said to be the Langlands quotient, but let us give an argument for that fact. We can use Casselman's criterion for π_p^2 [Cas95, Theorem 4.4.6]. For $A = T^{\mathrm{split}} = \mathbb{G}_m \subset B$,

$$A^-\backslash A(\mathcal{O})A_\delta = \{\mathrm{Diag}(x, 1, x^{-1}) : x \in \mathbb{Z}_p\backslash\mathbb{Z}_p^\times = p\mathbb{Z}_p\},$$

and thus

$$\forall x \in p\mathbb{Z}_p, |\widetilde{\chi}\delta_B^{1/2}(\mathrm{Diag}(x,1,x^{-1}))| = |\chi(x)| = |\chi_0(x)||x|^{1/2} < 1,$$

as χ_0 is unitary, and thus $\widetilde{\chi}\delta_B^{1/2}$ is an exponent of $r_B^G(\pi^2)$ and $\pi_p^2 \subset \mathrm{ind}-n_B^G(\widetilde{\chi})$ i.e.

$$(\pi_p^n)_N = \widetilde{\chi}^w\delta_B^{1/2}. \qquad \square$$

When p is split, the calculation is done in [BC04] and we get the following up to identifying an unramified character of $T(\mathbb{Q}_p) \simeq (\mathbb{Q}_p^\times)^3$,

$$\psi : \begin{array}{ccc} T(\mathbb{Q}_p) & \longrightarrow & \mathbb{C} \\ \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} & \longmapsto & \psi_1(x_1)\psi_2(x_2)\psi_3(x_3) \end{array}$$

with the triple $(\psi_1(p), \psi_2(p), \psi_3(p))$.

PROPOSITION 10.8 (Bellaïche–Chenevier [BC04, BC09]). *If $p = \overline{v}v$, the accessible refinements of $\pi_p^n(\chi)$ are given (with identification with $\mathrm{GL}_3(\mathbb{Q}_p)$ using v) by:*

- $\sigma = 1, (1, \chi_v^\perp(p), \chi_v(p))$;
- $\sigma = (3,2), (\chi_v^\perp(p), 1, \chi_v(p))$;
- $\sigma = (3,2,1), (\chi_v^\perp(p), \chi_v(p), 1)$.

Proof. Indeed, the Langlands class associated to $\pi_p^n(\chi)$ is $(\chi_v^\perp(p), 1, \chi_v(p))$ which corresponds, up to a twist of the central character by $(\chi_v^\perp)^{-1}$, to the class $(1, (\chi_v^\perp(p))^{-1}, |p|)$ which in turn is associated by Satake (up to twist by $\mu^{-1}|.|^{1/2}$) to the unramified induction studied in [BC09, Lemma 8.2.1], $n = 1, m = 3$ and $\pi = \chi_0^c = \chi_0^{-1}$ (which satisfies the hypothesis of [BC09] 6.9.1). Thus $L(\pi_p|.|^{1/2}) = (\chi^\perp(p))^{-1}$. The refinements are then given by [BC09, Lemma 8.2.1]. □

10.3 Coherent cohomology

In order to associate to the automorphic representation $\pi^n(\chi)$ a point in the eigenvariety constructed in § 8, we need also to show that $\pi^n(\chi)$ appears in the global sections (over X) of a coherent automorphic sheaf. The full calculation is made in Appendix D. Here we give an alternative proof in the case $a > 1$, which corresponds to a regular weight, as the case $a = 1$ will correspond to a singular weight (and for $a = 1$, π^2 is a non-holomorphic limit of discrete series). Thus suppose $a > 1$. According to Rogawski [Rog90, Proposition 15.2.1], the (regular) parameter $\varphi = \varphi(a,b,c) = \varphi(a, a-1, 0)$ (see [Rog90, p. 176], χ corresponding to χ_φ^-) we already know that

$$H^i(\mathfrak{g}, K, \pi_\infty^n(\chi) \otimes \mathcal{F}_\varphi^\vee) = \begin{cases} \mathbb{C} & \text{if } i = 1, 3, \\ 0 & \text{otherwise,} \end{cases}$$

for $\mathcal{F}_\varphi$ the representation of $U(2,1)(\mathbb{R})$ of highest weight $(a-1, a-1, 1)$, and

$$H^i(\mathfrak{g}, K, \pi_\infty^2(\chi) \otimes \mathcal{F}_\varphi^\vee) = \begin{cases} \mathbb{C} & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular the system of Hecke Eigenvalues of $\pi^n(\chi)$ appears in the first intersection cohomology group of a local system associated to $\mathcal{F}_\varphi^\vee$ (thus coming from a representation of the group G), $IH^1(X^{\mathrm{BS}}, \mathcal{F}_\varphi^\vee)$, where X^{BS} is the Borel–Serre compactification, or also by a theorem of Borel in $H^1_{\mathrm{dR},c}(Y, (V_\varphi, \nabla))$ the de Rham cohomology with compact support (where $V_\varphi = \mathcal{F}_\varphi^\vee \otimes \mathcal{O}_X$ is the associated vector bundle with connection, see [Har90a, 1.4]). If our Picard surface were compact, then $IH^1(X^{\mathrm{BS}}, \mathcal{F}_\varphi^\vee) = H^1_{\text{ét}}(X, \mathcal{F}_\varphi^\vee)$ is just étale cohomology and as $\mathcal{F}_\phi^\vee$ comes from a representation of our group, using the Hodge-decomposition for $H^1_{\text{ét}}(X, \mathcal{F}_\varphi^\vee)$ or for the de Rham cohomology (see [Fal83] or [MM63]), we know that there exists a coherent automorphic sheaf $V_\varphi = \mathcal{F}_\varphi^\vee \otimes \mathcal{O}_X$ such that

$$H^1_{\text{ét}}(X, \mathcal{F}_\varphi^\vee) = H^1(X, \mathcal{F}_\varphi^\vee \otimes \mathcal{O}_X) \oplus H^0(X, \mathcal{F}_\varphi^\vee \otimes \Omega_X^1).$$

More generally, if X is not compact (which is the case here) we still have a BGG-decomposition for the de Rham cohomology with compact support, see [Lan12, (2.4)],

$$H^1_{\mathrm{dR},c}(Y,(V_\varphi,\nabla)) = H^1(X,\omega^{(1-a,1-a,1)}(-D)) \oplus H^0(X,\omega^{(0,1-a,a)}(-D)).$$

We thus need to show that the system of Hecke eigenvalues appears in the last factor. But, denote by $\overline{I}$ the opposite induced representation $\mathrm{ind}-n_B^{U(2,1)(\mathbb{R})}(\chi_\infty^w)$, so that π_∞^n is the subrepresentation of $\overline{I}$ and π^2 its quotient. Writing the long exact sequence of $(\mathfrak{g},K)$-cohomology associated to

$$0 \longrightarrow \pi_\infty^n \longrightarrow \overline{I} \longrightarrow \pi_\infty^2 \longrightarrow 0,$$

we get that $H^1(\mathfrak{g},K,\pi_\infty^n\otimes\mathcal{F}_\varphi^\vee) = H^1(\mathfrak{g},K,\overline{I}\otimes\mathcal{F}_\varphi^\vee)$. Using Hodge decomposition for this, we get that

$$\mathrm{Hom}_K(\mathfrak{p}^+\otimes\mathcal{F}_\varphi,\pi_\infty^n) = \mathrm{Hom}_K(\mathfrak{p}^+\otimes\mathcal{F}_\varphi,\overline{I}),$$

and using Frobenius reciprocity we can calculate the last term as

$$\mathrm{Hom}_{T\cap K}(\mathfrak{p}^+\otimes\mathcal{F}_\varphi,(\chi_\infty^w)_{T\cap K}),$$

and, as we know $\mathcal{F}_\varphi$, we can calculate its restriction to $T\cap K$, we get

$$(\mathcal{F}_\varphi)_{T\cap K} = t^a e^{a-1}\oplus\cdots\oplus t^{2a-2}e,$$

where

$$t^ke^l : \begin{pmatrix} t & & \\ & e & \\ & & t\end{pmatrix} \in T\cap K = U(1)\times U(1) \longmapsto t^ke^l.$$

We can also explicitly calculate that, by conjugacy, $T\cap K$ acts on $\mathfrak{p}^+$ by $1\oplus te^{-1}$ and on $\mathfrak{p}^-$ by $1\oplus t^{-1}e$.

Remark 10.9. This is because of our choice of h. If we change h by its conjugate, then the action on $\mathfrak{p}^+,\mathfrak{p}^-$ would have been exchanged, and $\pi^n(\chi)$ would be anti-holomorphic (but we could have used χ^c instead of χ in this case, $\pi^n(\chi^c)$ would have been holomorphic).

As $\chi^w_{T\cap K} = t^{2a-1}$, we get that

$$\mathrm{Hom}_K(\mathfrak{p}^+\otimes\mathcal{F}_\varphi,\overline{I}) = \mathbb{C} \quad\text{and}\quad \mathrm{Hom}_K(\mathfrak{p}^-\otimes\mathcal{F}_\varphi,\overline{I}) = \{0\}.$$

Remark 10.10. Changing χ by χ^c inverts the previous result, as predicted by the Hodge structure, so we could have argued without explicitly calculating these spaces.

Proposition 10.11. *If $a>1$, the Hecke eigensystem corresponding to $\pi^n(\chi)$ appears in $H^0(X, F_{s\cdot(\varphi^\vee)}\otimes\mathcal{O}_X(-D)) = H^0(X,\omega^{(0,1-a,a)}(-D))$. More generally, for $a\geqslant 1$, the Hecke eigenvalues of $\pi^n(\chi)$ appear in the global section of the coherent sheaf $\omega^{(0,1-a,a)}(-D)$.*

Proof. When $a>1$ we know that the Hecke eigensystem corresponding to $\pi^n(\chi)$ appears in the de Rham cohomology associated to φ. If Y were compact, the proposition would just be Matsushima's formula and the Hodge decomposition [Yos08, Theorem 4.7] of the first $(\mathfrak{g},K)$-cohomology group we calculated above. More generally, the previous calculation still shows that the class corresponding to $\pi^n(\chi)$ cannot be represented by classes in $H^1(X,\omega^{(1-a,1-a,1)})$ (whose cohomology is calculated by $(\mathfrak{p}^-,K)$-cohomology), and thus, by the BGG resolution, it appears in $H^0(X,\omega^{(0,1-a,a)}(-D))$.

For the general case, this is Appendix D (which proves that the Hecke eigensystem appears in the 0th coherent cohomology group). ◻

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10.4 Transfer to $GU(2,1)$

From now on, denote by $G = GU(2,1)$ the algebraic group over $\mathbb{Q}$ of unitary similitudes (relatively to (E^3, J)). It is endowed with a morphism ν, and there is an exact sequence,

$$0 \longrightarrow G_1 \longrightarrow G \xrightarrow{\nu} \mathbb{G}_m,$$

where $G_1 = U(2,1)$ is the unitary group of (E^3, J).

Let $T = \mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_m$ be the center of G, $Nm : T \longrightarrow \mathbb{G}_m$ the norm morphism, and T^1 its Kernel; the center of G_1. We have the exact sequence,

$$1 \longrightarrow T^1 \longrightarrow T \times G_1 \longrightarrow G,$$

where the first map is given by $\lambda \mapsto (\lambda, \lambda^{-1})$.

Let π_1 be an automorphic (respectively a smooth admissible local) representation of $G_1(\mathbb{A}_\mathbb{Q})$ (respectively of $G_1(\mathbb{Q}_p)$ or $G_1(\mathbb{R})$), of central character χ_1 of T^1. Let χ be a character of T (local or global) that extends χ_1; we can thus look at the representation,

$$(z, g) \in T \times G_1 \longmapsto \chi(z)\pi_1(g),$$

of $T \times G_1$. We can check that it factors through the action of T^1 and gives a representation of a subgroup of G.

PROPOSITION 10.12. *The automorphic representation π^n of $U(2,1)$ given in Proposition 10.6 has central character ω equal to the restriction of χ to E^1. We can extend ω as an algebraic Hecke character $\widetilde{\omega}$ of T by the algebraic character $\widetilde{\omega} = N^{-1}\chi$, where N is the norm of E. Thus, there exists an automorphic representation $\widetilde{\pi^n}$ of G such that for ℓ a prime, unramified for χ_0, $(\pi_\ell^n)^{K_\ell} = (\widetilde{\pi_\ell^n})^{K_\ell}$ (where $K_\ell \subset G(\mathbb{Q}_\ell)$ is the hyperspecial, respectively special if ℓ ramifies in E – subgroup) and the Galois representation associated to $\widetilde{\pi}^n$ by* [BR92, Theorem 1.9.1] *(or* [Ski12]*) is (with the normalization of* [Ski12]*)*

$$(1 \oplus \chi \oplus \chi^\perp)\overline{\chi}(-2) = (\overline{\chi} \oplus \omega_{\mathrm{cycl}} \oplus 1)(-2).$$

Moreover $(\pi_p^n)^I = (\widetilde{\pi_p^n})^I$.

Proof. To calculate ω, we only need to look at π_p^n for every place p, and we can use that $\pi_p^n =$ ind $-n_P^{\mathrm{GL}_3(\mathbb{Q}_p)}(\chi_0)$ for split p, and $\pi_p^n \subset$ ind $-n_B^{U(3)(\mathbb{Q}_p)}(\widetilde{\chi}^w)$ for $p = \infty$ and inert or ramified p.

The character $\widetilde{\omega}$ extends ω. Once we have extended the central character of π^n, the existence of a $\widetilde{\pi}^n$ is unique and assured by [CHT08, Proposition 1.1.4] (as $2+1 = 3$ is odd). More precisely,

$$\widetilde{\pi}^n(zg_\mathbb{Q}g_1) = \widetilde{\omega}(z)\pi^n(g_1),$$

where $g = zg_\mathbb{Q}g_1$ is written following the decomposition $G(\mathbb{A}) = T(\mathbb{A})G(\mathbb{Q})G_1(\mathbb{A})$.

Denote by V_p the space of π_p^n (and thus of $\widetilde{\pi}_p^n$). Also, $I \subset GU(2,1)(\mathbb{Q}_p)$ the Iwahori subgroup, and I_1 is its intersection with $U(2,1)(\mathbb{Q}_p)$. Because if $M \in I$, then $M \equiv B(\mathrm{mod}\ p)$, up to multiplying by an element of $T \in T(\mathcal{O})$, suppose that $TM \equiv U \pmod p$. In this case $c(TM) \equiv 1 \pmod p$; thus, as p is unramified in E, there exist $T' \in T(\mathbb{Z}_p)$ such that $c(T'TM) = 1$ and hence $M = T^{-1}(T'TM) \in T(\mathbb{Z}_p)I_1$. Thus, as we can write

$$\begin{pmatrix} a & & \\ & e & \\ & & N(e)\overline{a}^{-1} \end{pmatrix} = \begin{pmatrix} ae^{-1} & & \\ & 1 & \\ & & \overline{ea}^{-1} \end{pmatrix} \begin{pmatrix} e & & \\ & e & \\ & & e \end{pmatrix} \in I_1T(\mathbb{Z}_p).$$

We get that

$$V_p^I = \{z \in V_p^{I_1} : \forall \lambda \in T(\mathbb{Q}_p) \cap I, \widetilde{\omega}_p(\lambda) z = z\},$$

but as $T(\mathbb{Q}_p) \cap I = \mathcal{O}_{E_p^\times} \mathrm{Id}$ and $\widetilde{\omega}_p$ is unramified, $V_p^I = V_p^{I_1}$. The assertion for K_ℓ follows the same lines and is easier. □

Remark 10.13. We could have lifted the central character of π^n simply by χ, in which case the resulting representation would have been a twist of the previous one, but as we only used three variables on the weight space, which means that we do not allow families which are twists by power of the norm of the central character, only one choice of the lift of the central character gives a point in our eigenvariety. We can check that the Hecke eigenvalues of $\widetilde{\pi^n}$ appears in $H^0(X, \omega^\kappa)$, with

$$\kappa = (0, 1-a, a).$$

How can we find the power of the norm and the coherent weight? First, as Hodge–Tate and coherent weights vary continuously on $\mathcal{E}$, and $\widetilde{\pi^n(\chi)}$ appears as a classical form of $\mathcal{E}$ (Proposition D.2), according to Propositions 9.17 and 9.19, the *polarized* Galois representation associated to $\widetilde{\pi^n}(\chi)$ is

$$1 \oplus \chi \oplus \chi^\perp,$$

and thus $(k_1, k_2 - 1, 1 - k_3) = (-a, 0, 1-a)$ up to order. This leaves us six possibilities for κ:

(i) $(0, 1-a, a)$;
(ii) $(0, 2-a, a+1)$;
(iii) $(1-a, 1-a, 1)$;
(iv) $(-a, 1-a, a)$;
(v) $(-a, 2-a, 1+a)$;
(vi) $(1-a, 1, 1+a)$,

but as for classical points (as $\widetilde{\pi^n}(\chi)$) $k_1 \geqslant k_2$, and $a \geqslant 1$, this eliminates the three last possibilities (and the second when $a = 1$). But then we know that the lowest K_∞-type for $\pi^n(\chi)$ is of dimension a by restriction to $U(2)$ and the calculation of Appendix D, Proposition D.2, which makes only the first coherent weight possible when $a > 1$. When $a = 1$, the first and third weights are the same. Another possibility is also to find the infinitesimal character of $\pi^n(\chi)$ (using for example [Kna16, Proposition 8.22]), and that $\eta = (-k_3, k_1, k_2)$ is the highest weight character of $V^\vee_{\lambda+\rho_n-\rho_c}$ in the notations of [Gol14] (paying attention to the dual). Finally, if $a > 1$, then the BGG decomposition tells us which weight κ has to be. Then to find the corresponding power of the norm, note that $|\kappa|$ must be equal to the opposite of the power of the norm of the central character of $\widetilde{\pi^n(\chi)}$ by the calculation before Proposition 2.6 and conventions on weights (see § 9.6 about (c_0, c_0')).

10.5 Refinement of representations of $GU(2,1)(\mathbb{Q}_p)$

Let $G = GU(2,1)(\mathbb{Q}_p)$. Consider in $C_c(I \backslash G / I, \mathbb{Z}[1/p])$ the double classes,

$$U_p^c = \begin{pmatrix} p^2 & & \\ & p & \\ & & 1 \end{pmatrix} \quad \text{and} \quad S_p^c = \begin{pmatrix} p & & \\ & p & \\ & & p \end{pmatrix}.$$

The characteristic functions of S_p^c and U_p^c are invertible in $C_c(I \backslash G / I, \mathbb{Z}[1/p])$ and denote by $\mathcal{A}(p)$ the sub-algebra generated by the characteristic functions of U_p^c, S_p^c and their inverses.

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Proposition 10.14. *For π a smooth complex representation of G, we have a natural $\mathbb{C}[\mathcal{A}(p)]$-isomorphism,*

$$\pi^I \longrightarrow (\pi_N)^{T^0} \otimes \delta_B^{-1}.$$

Let π be a smooth admissible representation of G, such that π is a subquotient of the (normalized) induction of an unramified character ψ of the torus T of G. For example, this is the case if π is unramified, or if $\pi^I \neq \{0\}$ (by the previous equality and adjunction between Jacquet functor and induction for example).

Definition 10.15. Following [BC09], an accessible refinement of π is a $\sigma \in W$ such that $\psi^\sigma \delta_B^{1/2}$ is a subrepresentation of $\pi_N^{T^0}$ (equivalently if $\psi^\sigma \delta_B^{-1/2}$ appears in π^I).

Another way to see it is that a refinement is an ordering of the eigenvalues of the Frobenius class of $LL(\pi)$, the Weil representation associated by local Langlands to π, and it is accessible if it appears in the previous sense in π^I (or $\pi_N^{T^0}$).

For $GU(2,1)$ when p is split, $GU(2,1)(\mathbb{Q}_p) \simeq \mathrm{GL}_3(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$ and ψ is an unramified character of $\mathbb{Q}_p^4$. The local Langlands representation associated to $\pi = \pi_1 \otimes \psi_4$ in this case is $LL(\pi_1) \otimes \overline{\psi_4}$ which has eigenvalues $(\psi_1(p)\overline{\psi}_4(p), \psi_2(p)\overline{\psi}_4(p), \psi_3(p)\overline{\psi}_4(p))$ and an ordering of these eigenvalues is given by an element of $\mathfrak{S}_3 = W_{\mathrm{GL}_3} = W_{\mathrm{GL}_3 \times \mathrm{GL}_1}$. Of course, *a priori*, not all refinements are accessible (π_p^n will be an example).

When p is inert, $W_G \simeq \mathbb{Z}/2\mathbb{Z}$, and a character of $T = (\mathrm{Res}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p} \mathbb{G}_m)^2$ is given by two characters (χ_1, χ_2), by

$$\begin{pmatrix} a & & \\ & e & \\ & & N(e)\overline{a}^{-1} \end{pmatrix}, a, e \in \mathbb{Q}_{p^2}^\times \longmapsto \chi_1(a)\chi_2(e).$$

The non-trivial element $w \in W_G$ acts on the character by $w \cdot (\chi_1, \chi_2) = (\chi_1^\perp, \chi_2(\chi_1 \circ N))$. Thus a refinement in this case is simply given by 1 or w.

Remark 10.16. In terms of Galois representation, the base change morphism from $GU(2,1)$ to $\mathrm{GL}_3 \times \mathrm{GL}_1$ sends the (unramified) Satake parameter χ_1, χ_2 (if χ_2 is unramified, it is trivial on E^1) to the parameter $((\chi_1, 1, \overline{\chi}_1^{-1}), \chi_1\chi_2)$ (see [BR92, Theorem 1.9.1] or [Ski12, § 2]), whose semi-simple class in GL_3 associated by local Langlands has Frobenius class given by

$$\begin{pmatrix} \chi_2(p) & & \\ & \chi_1(p)\chi_2(p) & \\ & & \chi_2(p)\chi_1^2(p) \end{pmatrix}.$$

In the inert case, say $\sigma \in W_G$ is a refinement, then the action of U_p on the σ-part of $\pi_N^{T^0}$ is given by $\chi_1^\sigma(p)^2\chi_2^\sigma(p)$, and the action of s is given by $\chi_1^\sigma(p)\chi_2^\sigma(p)$. In particular, the action of $\mathcal{A}(p)$ (through T/T^0), and actually of U_p or $u_1 = U_p S_p^{-1}$, on π^I determines the refinement.

This is also true (and easier) if p splits.

As we normalized our Galois representation ρ_π so that they are polarized, i.e. forgetting the central character, the previous class does not directly relate to the Frobenius eigenvalues of ρ_π but rather of the one of $\rho_{\pi,\mathrm{Ski}}$. But as the link between both only differ through the central character of π, it is straightforward that the Frobenius eigenvalues of (a crystalline) ρ_π are given by $(\psi_1, 1, \psi_1^\perp)$, when p is inert, and ψ_1 is given (if unramified) by the action of the Iwahori–Hecke double class

$$\begin{pmatrix} p & & \\ & 1 & \\ & & p^{-1} \end{pmatrix}$$

which corresponds to $U_pS_p^{-1}$ (see next subsection). In the split case, an unramified character of the torus of $\mathrm{GL}_3 \times \mathrm{GL}_1$ gives Frobenius eigenvalues $(p\psi_1(p)\psi_4(p), p\psi_2(p)\psi_4(p), p\psi_3(p)\psi_4(p))$ for (crystalline) $\rho_{\pi,\mathrm{Ski}}$ and $(\psi_1(p), \psi_2(p), \psi_3(p))$ for ρ_π, which relates to operators U_{i-1}/U_3 (see next subsection).

Thus, using the previous definition of refinement, local global compatibility at p, we can associate to $\Pi = \pi_\infty \otimes \bigotimes_\ell \pi_\ell$ an algebraic regular cuspidal automorphic representation of $GU(2,1)$ of level $K^{Np}I$ a representation $\rho_{\pi,p}$ together with an (accessible) ordering of its crystalline-Frobenius eigenvalues for each choice of a character in π_p^I under $\mathcal{A}(p)$, such that the following proposition holds.

Proposition 10.17. *The automorphic representation $\widetilde{\pi}^n(\chi)$ of $GU(2,1)$ constructed by Proposition 10.12 has only one accessible refinement at p if p is inert, it is given by*

$$\omega \neq 1 \in W_G,$$

which correspond to the ordering $((\chi^\perp(p), 1, \chi(p)), \chi(p))$ or $(1, \chi(p), |p|)$. If $p = \overline{v}v$ is split, there are three accessible refinements, given by:

- $\sigma = 1, ((1, \chi_v^\perp(p), \chi_v(p)), \chi_v(p))$ *which corresponds to* $(\chi_v(p), 1, |p|)$*;*
- $\sigma = (3,2), ((\chi_v^\perp(p), 1, \chi_v(p)), \chi_v(p))$ *which corresponds to* $(1, \chi_v(p), |p|)$*;*
- $\sigma = (3,2,1), ((\chi_v^\perp(p), \chi_v(p), 1), \chi_v(p))$ *which corresponds to* $(1, |p|, \chi_v(p))$*.*

We denote by σ the unique refinement in the inert case, and the refinement denoted by $(3,2)$ in the split case.

Proof. The action of u_1 on $\pi^n(\chi)_p$ as been calculated in a previous section. □

10.6 Modular and classical Hecke operators

In order to understand how the refinements vary on the eigenvariety, we need to make explicit the link between Hecke operators (at p) constructed in § 8 and classical Hecke acting on automorphic forms, as above. Here we work at Iwahori level at p and identify matrices with the corresponding Iwahori double classes. If p is inert in E, the Atkin–Lehner algebra we consider at p is generated by the two (so-called classical) operators U_p^c and S_p^c described above. If p is split in E, we consider the Atkin–Lehner algebra $\mathcal{A}(p)$ of $\mathrm{GL}_3(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$ (see [BC09, § 6.4.1]); it is generated by the Hecke operators, up to identification of $E \otimes \mathbb{Q}_p \overset{i_v \times i_{\overline{v}}}{\simeq} \mathbb{Q}_p \times \mathbb{Q}_p$,

$$(pI_3, I_3), \quad \left(\begin{pmatrix} p & & \\ & p & \\ & & 1 \end{pmatrix}\right), \left(\begin{pmatrix} p & & \\ & 1 & \\ & & 1 \end{pmatrix}\right), \quad \left(\begin{pmatrix} p & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} p & & \\ & p & \\ & & 1 \end{pmatrix}\right), \quad (I_3, pI_3),$$

that we denote respectively U_0^c, U_1^c, U_2^c and U_3^c (c stands for classical in the sense 'not normalized'). If we use i_v to identify $GU(2,1)(\mathbb{Q}_p)$ with $\mathrm{GL}_3 \times \mathrm{GL}_1(\mathbb{Q}_p)$, then these operators are identified respectively with

$$(pI_3, p), \quad \left(\begin{pmatrix} p & & \\ & p & \\ & & 1 \end{pmatrix}, p\right), \quad \left(\begin{pmatrix} p & & \\ & 1 & \\ & & 1 \end{pmatrix}, p\right), \quad (I_3, p).$$

In § 8 we defined Hecke operators modularly, U_p and S_p in the inert case, and both Brasca [Bra16] and Bijakowski [Bij16] defined U_0, U_1, U_2, U_3 in the split case (see remark in § 8.3). These Hecke operators have been normalized and correspond to the above Iwahori double classes, so that we have the following result.

Let $\Pi = \pi_\infty \otimes \bigotimes_\ell \pi_\ell$ be an algebraic, regular, cuspidal automorphic representation of $GU(2,1)$ of level $K^{Np}I$ whose Hecke eigenvalues appear in the global sections of a coherent automorphic sheaf (of weight κ) and $f \in \Pi \cap H^0(X_I, \omega^\kappa)$ an eigenform for $\mathcal{H} = \mathcal{H}^{Np} \otimes \mathcal{A}(p)$, such that, if p is inert,

$$U_p f = p^{-k_2} U_p^c f \quad \text{and} \quad S_p f = p^{-k_1-k_2-k_3} S_p^c f,$$

and if p splits,

$$U_0 f = p^{-k_3} U_0^c f, \quad U_1 f = U_1^c f, \quad U_2 f = p^{-k_2} U_2^c f, \quad U_3 f = p^{-k_1-k_2} U_3^c f,$$

where the action of the double classes U_p^c, S_p^c and U_i^c is given by convolution on π_p.

Proposition 10.18. *Suppose p is inert, f is a classical automorphic form of classical weight $\kappa = (k_1, k_2, k_3)$ of Iwahori level at p (i.e. $f \in H^0(X_{Iw}, \omega^\kappa)$), which is an eigenform for the action of $\mathcal{H} \otimes \mathcal{A}(p)$, and denote by λ, μ the eigenvalues of f for U_p, S_p respectively.*

Let Π be a irreducible factor of the associated automorphic representation (generated by Φ_f). Then $\Pi_p^I \neq \{0\}$ and thus the algebra $\mathcal{A}(p)$ acts on Π_p^I with U_p^c of eigenvalue $p^{k_2}\lambda$ and S_p^c of eigenvalue $p^{|k|}\mu$.

Proof. To prove the statement, we remark that the association $f \mapsto \Phi_f$ is Hecke equivariant for the classical Hecke operators U_p^c, S_p^c acting on f. But we defined the Hecke operators U_p, S_p geometrically by $U_p = p^{-k_2} U_p^c$ and $S_p = p^{-|k|} S_p^c$ to make them vary p-adically. Thus we get the result. □

Using the previous refinements for representations of $GU(2,1)$, we can prove the following result on density of crystalline points on the eigenvariety $\mathcal{E}$ of Theorem 9.15,

Proposition 10.19. *Suppose p is inert. Let $x \in \mathcal{E}(F)$. There exists a neighborhood V of x and a constant $C > 0$ such, that for all classical points $y \in V$, if $|w_2(y) + w_3(y)| > C$, then ρ_y is crystalline and of Hodge–Tate weights $(1 + w_3(y), -1 - w_1(y), -w_2(y))$.*

In particular crystalline points are dense in $\mathcal{E}$ by classicity Proposition 8.16 (as we can also assume $w_1(y) - w_2(y) > C$) and Theorem 8.17 (as we can moreover assume $-w_1(y) - w_3(y) > C$).

Proof. Denote by F_1, F_2 the two invertible functions of $\mathcal{E}$ given by the eigenvalues under

$$U_p = \begin{pmatrix} p^2 & & \\ & p & \\ & & 1 \end{pmatrix} \quad \text{and} \quad S_p = \begin{pmatrix} p & & \\ & p & \\ & & p \end{pmatrix}.$$

The valuations of F_1, F_2 are locally constant on $\mathcal{E}$, and thus there exists V a neighborhood of x where these valuations are constant. As y corresponds to f a classical form of (p-adic) weight $w(y)$ and level K proper under $\mathcal{H} \otimes \mathcal{A}(p)$, we can look at Π an irreducible component of the representation generated by Φ_f, which is thus algebraic, regular, and its associated representation ρ_y does not depend on Π as it only depends on the eigenvalues of $\mathcal{H}$ on f. As Π_p, the pth component of Π, is generated by its I-invariants, Π_p is a subquotient of the induction ind $-n_B^G(\mathbb{Q}_p)(\psi)$ for some unramified character ψ ([BC09, Proposition 6.4.3] and the adjunction property of induction). We need to show that Π_p is unramified, but as Π_p appears as a subquotient of $\mathrm{ind}_B(\psi)$, which has a unique unramified subquotient, it suffices to prove that $\mathrm{ind}_B(\psi)$ is irreducible, which happens in particular when $|\psi_1(p)| \neq p^{\pm 1}$ when p is inert (cf. the key result of Keys, see [Rog90, 12.2]).

In the inert case, we have that if $w = (-k_2, -k_1, -k_3)$ i.e. f is of automorphic classical weight (k_1, k_2, k_3), then by Proposition 10.18

$$\psi_1^\sigma(p) = p^{-k_1-k_3} F_1(y)/F_2(y),$$

for a certain choice $\sigma \in W_{GU}$ (see § 10.5 for example), but as the valuations of F_1, F_2 are constant on V, there is a constant C such that if $|k_1 + k_3| > C$, Π_p is unramified. Thus, by local–global compatibility at p for Π (cf. [Ski12, Theorem B]), ρ_y is crystalline. □

Remark 10.20. In the split case, the same proposition is true under the assumption $\delta(w(y)) := \min_i(|w_i(y) - w_{i+1}(y)|) > C$ as the same proof of proposition 8.2 of [BC04], together with classicity results of [BPS16], and [Bra16, Proposition 6.6, Theorem 6.7] applies.

10.7 Types at ramified primes for χ

In order to control the ramification at $\ell |\, \mathrm{Cond}(\chi)$, Bellaïche and Chenevier introduced a particular type (K_0, J_0), which we can slightly modify to suit our situation.

Proposition 10.21. *Let $\ell |\, \mathrm{Cond}(\chi)$ a prime. There exists a compact subgroup K_ℓ of $GU(2,1)(\mathbb{Q}_\ell)$ and a representation J_ℓ of K_ℓ such that the following hold:*

(i) $\mathrm{Hom}_{K_\ell}(J_\ell, \widetilde{\pi_\ell^n(\chi)} \otimes (\chi_{0,\ell} \circ \det)) \neq 0$;

(ii) *For all smooth admissible representation π of $GU(2,1)(\mathbb{Q}_\ell)$ such that $\mathrm{Hom}_{K_\ell}(J_\ell, \pi) \neq 0$ and for all place $v|\ell$, there exist four unramified characters $\phi_1, \phi_2, \phi_3, \phi_4 : E_v^\times \longrightarrow \mathbb{C}^\times$ such that the Langlands semi-simple class in $\mathrm{GL}_3 \times \mathrm{GL}_1$ corresponds to*

$$L(\pi_{E_v}) = (\phi_1 \oplus \phi_2 \oplus \phi_3\chi_0^{-1}, \phi_4\chi_0^{-1})$$

or to the (unpolarized) Langlands class in GL_3,

$$L(\pi_{E_v}) = \phi_1\overline{\phi_4}\chi_0 \oplus \phi_1\overline{\phi_4}\chi_0 \oplus \phi_3\overline{\phi_4}.$$

Proof. Let (K_ℓ^0, J_ℓ^0) be the type defined by Bellaïche and Chenevier in [BC04]. If $\ell = v_1v_2$ is split, let K_ℓ be the subgroup of matrices congruent to

$$\begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & 0 & y \end{pmatrix}, e$$

modulo ℓ^m, the ℓ-adic valuation of $\mathrm{Cond}(\chi)$. Let J_ℓ be the representation that sends the matrices in K_ℓ to $\chi_{0,v_1}^{-1}(y)\chi_{0,v_1}(\overline{e})$. As every matrix in $GU(2,1)(\mathbb{Q}_\ell) = \mathrm{GL}_3 \times \mathrm{GL}_1(\mathbb{Q}_\ell)$ can be written as

$M = \lambda U$ where $U \in U(2,1)(\mathbb{Q}_\ell) = \mathrm{GL}_3(\mathbb{Q}_\ell)$ and $\lambda = (1, \lambda)$ is in the center, we can check that $\mathrm{Hom}_{K_\ell}(J_\ell, \widetilde{\pi}_\ell^n \otimes \chi_{0,v_1} \circ \det^{-1}) \neq 0$.

Now, if $\mathrm{Hom}_{K_\ell}(J_\ell, \pi) \neq 0$, then $\mathrm{Hom}_{K_\ell^0}(J_\ell^0, \pi_{|U(2,1)}) \neq 0$ when restricted to $U(2,1) = \mathrm{GL}_3$, and thus by [BC04] we have the conclusion up to a character. But, as $K' = (\mathrm{Id} \times \mathrm{GL}_1) \cap K_\ell^0 \simeq \mathbb{Z}_\ell^\times$, $\pi_{|K'} = \chi_{0,v_1}^{-1} \otimes \psi$ where ψ is an unramified character, and thus

$$L(\pi_{E,v}) = \phi_1 \overline{\psi} \chi_0 \oplus \phi_2 \overline{\psi} \chi_0 \oplus \phi_3 \overline{\psi}.$$

If ℓ is prime, write $K_\ell = \mathcal{O}_{E_\ell}^\times K_\ell^0$ and define J_ℓ by

$$J_\ell(\lambda M^0) = \chi_\ell(\lambda)^{-2} J_\ell^0(M^0).$$

As $\mathcal{O}_{E_\ell}^\times \cap K^0 \subset \mathcal{O}_{E_\ell}^1$, this is well-defined because the central character of J_ℓ^0 is up to an unramified character equal to χ_ℓ^{-2}. Moreover, $\mathrm{Hom}_{K_\ell}(J_\ell, \widetilde{\pi}_\ell^n)(\chi) \otimes (\chi_{0,\ell} \circ \det) \neq 0$ as it is the case for (K_ℓ^0, J_ℓ^0) by [BC04] which refers back to Blasco [Bla02], and the central character of $\widetilde{\pi^n}_\ell(\chi)$ is equal to χ_ℓ (up to a unramified character).

Conversely, if π is a representation of $GU(3)(\mathbb{Q}_\ell)$ such that $\mathrm{Hom}_{K_\ell}(J_\ell, \pi) \neq 0$, then $\mathrm{Hom}_{K_\ell^0}(J_\ell^0, \pi_{|U(3)(\mathbb{Q}_\ell)}) \neq 0$ and thus $L(\pi_{|U(3)}) = \phi_1 \oplus \phi_2 \oplus \phi_3 \chi_0^{-1}$ by [BC04], and its central character corresponds to χ_ℓ up to an unramified character, and we thus get the result on the Langlands Base change of π. $\square$

11. Deformation of $\widetilde{\pi^n}$

By Proposition 10.21, we can find for every $\ell |\, \mathrm{Cond}(\chi)$ K_ℓ a subgroup of $GU(2,1)(\mathbb{Q}_\ell)$ and an irreducible representation J_ℓ such that

$$\mathrm{Hom}_{K_\ell}(J_\ell, \widetilde{\pi_\ell^n}(\chi) \otimes (\chi_{0,\ell} \circ \det)) \neq 0,$$

and for all $\widetilde{\pi}_\ell$ of type (K_ℓ, J_ℓ) its base change to $\mathrm{GL}_3(E_v)$, for all $v|\ell$, gives the representation (normalized as in Proposition 9.17, Theorem 9.19)

$$L(\pi_{\ell,E_v}) = \phi_1 \chi_0^{-1} \oplus \phi_2 \oplus \phi_3,$$

where $\phi_j : E_v^\times \longrightarrow \mathbb{C}^\times$ are unramified characters.

11.1 Choosing the level

Up to choosing compatibly places at ∞ and embeddings of $\mathbb{Q}_{p^2}$, we can make $\chi : G_E \longrightarrow \overline{\mathbb{Q}_p}$, the p-adic realization of χ at p, have τ-Hodge–Tate weight $-a = -(k+1)/2$ and thus $\overline{\chi}$ τ-Hodge–Tate weight $a - 1 = (k-1)/2$.

Let $N = \mathrm{Cond}(\chi)$; suppose $p \neq 2$ if p is inert, $p \mid N$, and is unramified in E. Define $K_f = \prod_\ell K_\ell$ by the following.

(i) If ℓ is prime to pN, K_ℓ is the maximal compact subgroup defined previously such that $\widetilde{\pi}^n$ as invariants by K_ℓ (hyperspecial at unramified ℓ, very special otherwise).

(ii) If $\ell = p$, K_p is the Iwahori subgroup of $GU(2,1)(\mathbb{Q}_p)$.

(iii) If $\ell | N$, K_ℓ is the type as defined before.

We then set

$$J = \bigotimes_{\ell|N} J_\ell \otimes (\chi_{0,\ell} \circ \det),$$

as representation of K_f.

By construction of K_f, there is $\phi \in \widetilde{\pi^n}(\chi)^{K_f}$, an automorphic eigenform for $\mathcal{H}^{Np}$ and of character under $\mathcal{A}(p)$ corresponding to the refinement σ of Proposition 10.17 and which is associated a classical Picard modular form $f \in H^0(X_I, \omega^\kappa)$ (by Proposition D.2 or Proposition 10.11 if $a > 1$) which is an eigenform for $\mathcal{A}(p)$, whose eigenvalues for $\mathcal{A}(p)$ corresponds to the refinement σ too (with the normalization explained in Proposition 10.18), and $\kappa = (0, 1-a, a)$.

Thus, setting $w_0 = (a-1, 0, -a)$ (corresponding to automorphic weight $(0, 1-a, a)$), to f is associated a point $x_0 \in \mathcal{E}$ such that $w(x_0) = w_0$ and $\rho^{ss}_{x,\mathrm{Ski}} = \overline{\chi}\varepsilon^{-2}(1 \oplus \chi \oplus \chi^\perp)$, and, with normalization of Proposition 9.17,

$$\rho^{ss}_x = 1 \oplus \chi \oplus \chi^\perp,$$

which is of (ordered) Hodge–Tate weights $(1-a, -a, 0)$.

11.2 A family passing through f

As we have normalized the pseudocharacter T of Proposition 9.19 in order to have the 'right' representation at x_0 (corresponding to $1 \oplus \chi \oplus \chi^\perp$), the map w from the eigenvariety gives the p-adic (automorphic) weight, and neither the classical automorphic weight nor the Hodge–Tate weights of T; thus we will normalize this map accordingly.

The τ-Hodge–Tate weights of $1 \oplus \chi \oplus \chi^\perp$ are given by $(1-a, -a, 0) =: h_0$. Let $F/\mathbb{Q}_p$ be a finite extension such that f is defined over F.

Proposition 11.1. *If p is inert, there exist:*

(i) *a dimension-1 regular integral affinoid Y over F, and $y_0 \in Y(F)$;*

(ii) *a semi-simple continuous representation,*

$$\rho_{K(Y)} : \mathrm{Gal}(\overline{E}/E)_{Np} \longrightarrow \mathrm{GL}_3(K(Y)),$$

satisfying $\rho^\perp_{K(Y)} \simeq \rho_{K(Y)}$, the property (ABS) of [BC04], and $\mathrm{tr}(\rho_{K(Y)})(\mathrm{Gal}(\overline{E}/E)) \subset \mathcal{O}_Y$;

(iii) *an F-morphism, $h = (h_1, h_2, h_3) : Y \longrightarrow \mathbb{A}^3$ such that $h_3 = 0$, $h(y_0) = h_0$;*

(iv) *a subset $Z \subset Y(F)$ such that $h(Z) \subset h_0 + (p-1)(p+1)^2\mathbb{Z}^3_{\mathrm{dom}}$ (i.e. the weights are regular);*

(v) *a function F_1 in $\mathcal{O}(Y)^\times$ of constant valuation*

such that we have the following.

(1) *For every affinoid Ω containing y_0, $\Omega \cap Z$ is Zariski dense in Ω.*

(2) *For all $z \in Z \cup \{y_0\}$ ρ^{ss}_z is the Galois representation associated to a cuspidal (algebraic) automorphic representation Π of $GU(2,1)$ such that*

$$\mathrm{Hom}_{K_f}(J, \Pi) \neq 0.$$

(3) *$\rho^{ss}_{y_0} \simeq 1 \oplus \chi \oplus \chi^\perp$.*

(4) *For $z \in Z$, $(\rho^{ss}_z)_{G_K}$ is crystalline of τ-Hodge–Tate weights $h_1(z) < h_2(z) < h_3(z)$, and its τ-refinement given by F_1 is*

$$(p^{h_1 - h_3(z)} F_1(z), 1, p^{h_3 - h_1(z)} F_1^{-1}(z)).$$

In particular,

$$D_{\mathrm{crys}}(\rho^{ss}_z)_\tau^{\varphi^2 = p^{h_1 - h_3} F_1(z)} \neq 0.$$

(5) *At y_0, the refinement is $(\chi^\perp_p(p), 1, \chi_p(p))$.*

Proof. Recall that p is inert here. The modular form f corresponds to a point $x_0 \in \mathcal{E}$, the eigenvariety defined in Theorem 9.15, associated to the type (K_f, J). Let $B \subset B(w_0, r) \subset \mathcal{W}$ be the closed subset defined in the same fashion as in [BC04] by,

$$\begin{cases} w_2 = 0, \\ 2w_1 + w_3 = a - 2. \end{cases} \tag{1}$$

Thus $w_0 \in B$. Define X to be an irreducible component of $\mathcal{E} \otimes_{\mathcal{W}} B$ containing x_f. We get $w_B : X \longrightarrow B$ which is finite (if r small enough) surjective. We can thus look at the universal pseudo-character T on $\mathcal{E}$ and compose it with $\mathcal{O}_{\mathcal{E}} \longrightarrow \mathcal{O}(X)$. Applying [BC04, Lemma 7.2], we get an affinoid Y, regular of dimension 1, $y_0 \in Y$ and a finite surjective morphism $m : Y \longrightarrow X$ such that $m(y_0) = x_f$ and there exist a representation $\rho : G_E \longrightarrow \mathrm{GL}_3(K(Y))$ of trace $G_E \xrightarrow{T} \mathcal{O}_X \longrightarrow \mathcal{O}_Y$ satisfying (ABS). At y_0, the representation $\rho_{y_0}^{ss}$ is given by $1 \oplus \chi \oplus \chi^{\perp}$. The map h is given as follow. First, write $\nu = (\nu_1, \nu_2, \nu_3) = (1 + w_3, -1 - w_1, -w_2)$. Then h is given by composition of m with the map ν (the shift of w) of $\mathcal{E}$; it is still finite and surjective on B,[5] and for every $y \in Y$ such that $m(y) = x_f$, then $h(y) = h_0$. In terms of automorphic weight (k_1, k_2, k_3) the previous map is given by $(1-k_3, k_2-1, k_1)$, and thus gives the Hodge–Tate weights for regular discrete series. In terms of Hodge–Tate weights, the equations (1) giving rise to B become

$$\begin{cases} h_3 = 0, \\ h_1 - 2h_2 = 1 + a. \end{cases} \tag{2}$$

Write

$$\mathcal{Z} = \{\underline{h} \in B \cap h_0 + (p-1)(p+1)^2\mathbb{Z}^{3,\mathrm{dom}} : -h_2 > C, h_2 - h_1 > C', |h_1 - h_3| > C''\},$$

where $C'' > 0$ is bigger than the bound given (up to reducing r and thus B) in Proposition 10.19 for crystallinity, C' is the bound given by classicity Theorem 8.17, and C is the bound given in classicity at the level of sheaves, Proposition 8.16 (we remark that h_3 is constant). Then $\mathcal{Z}$ is strongly Zariski dense in B. Then $Z := \kappa^{-1}(\mathcal{Z}) \subset Y(F)$ contains only classical (and regular) points by Proposition 8.16 and the classicity result of Bijakowski (Theorem 8.17). Moreover they are all crystalline by Proposition 10.19. It is strongly Zariski dense by flatness (thus openness) of κ. Let us define F_1. The action on a point x_f associated to modular form f – of (classical automorphic) weight (k_1, k_2, k_3) associated to π that is a quotient of $\mathrm{ind}_b^{GU(3)}(\psi)$ – of the operator

$$U_p S_p^{-1},$$

which corresponds up to a normalization by $1/p^{h_1 - h_3 - 1} = p^{k_1 + k_3}$ to the classical Iwahori double coset

$$p^{-1}\begin{pmatrix} p^2 & & \\ & p & \\ & & 1 \end{pmatrix} = \begin{pmatrix} p & & \\ & 1 & \\ & & p^{-1} \end{pmatrix},$$

and corresponds to

$$p^{k_1+k_3}\psi_1^{\sigma}(p),$$

where $\psi = (\psi_1, \psi_2)$ is a character of $(\mathcal{O}^{\times})^2$ and σ the refinement of f associated to the action of $\mathcal{A}(p)$. Indeed, the eigenvalue of U_p coincide with $p^{-k_2}\psi_1^{\sigma}(p)^2\psi_2^{\sigma}$ and the one of S_p with $p^{-k_1-k_2-k_3}\psi_1^{\sigma}(p)\psi_2^{\sigma}(p)$. Thus, $U_p S_p^{-1}$ has eigenvalue $p^{k_1+k_3}\psi_1^{\sigma}(p) = p^{h_3 - h_1} p \psi_1^{\sigma}(p)$. Thus, we set

[5] We did a slight abuse of notation, it should be $B' \simeq B$, centered in h_0, and whose equations are given below.

F_1 the function on $\mathcal{E}$ given by $p^{-1}U_pS_p^{-1}$. We have that $p^{h_1-h_3}F_1 = \psi_1^\sigma(p)$. Property (2) comes from the construction of the eigenvariety $\mathcal{E}$. Part (3) is the calculation of the Galois representation associated to $\pi^n(\chi)$. Part (4) is local–global compatibility at $\ell = p$ ([Ski12] as recalled in § 9.6) and Proposition 10.19 as the eigenvalues of the crystalline Frobenius class φ^2 coincide with $\psi_i^\sigma(p)$.

The last assertion is the calculation made in Proposition 10.17. □

Proposition 11.2. *If $p = v\overline{v}$ is split, there exist:*

(i) *a dimension-1 regular integral affinoid Y over F, and $y_0 \in Y(F)$;*

(ii) *a semi-simple continuous representation,*

$$\rho_{K(Y)} : \mathrm{Gal}(\overline{E}/E)_{Np} \longrightarrow \mathrm{GL}_3(K(Y)),$$

satisfying $\rho_{K(Y)}^\perp \simeq \rho_{K(Y)}$, the property (ABS) of [BC04], and $\mathrm{tr}(\rho_{K(Y)})(\mathrm{Gal}(\overline{E}/E)) \subset \mathcal{O}_Y$;

(iii) *an F-morphism, $h = (h_1, h_2, h_3) : Y \longrightarrow \mathbb{A}^3$ such that $h_3 = 0$, $h(y_0) = h_0$;*

(iv) *a subset $Z \subset Y(F)$ such that $h(Z) \subset h_0 + (p-1)\mathbb{Z}^3_{\mathrm{dom}}$;*

(v) *three functions F_1, F_2, F_3 in $\mathcal{O}(Y)$ of constant valuation*

such that we have the following.

(1) *For every affinoid Ω containing y_0, $\Omega \cap Z$ is Zariski dense in Ω.*

(2) *For all $z \in Z \cup \{y_0\}$ ρ_z^{ss} is the Galois representation associated to a cuspidal (algebraic) automorphic representation Π of $GU(2,1)$ such that*

$$Hom_{K_f}(J, \Pi) \neq 0.$$

(3) $\rho_{y_0}^{ss} \simeq 1 \oplus \chi \oplus \chi^\perp$.

(4) *For $z \in Z$, $(\rho_z^{ss})_{G_v}$ is crystalline of Hodge–Tate weights $h_1(z) < h_2(z) < h_3(z)$, and*

$$(p^{h_1(z)}F_1(z), p^{h_2(z)}F_2(z), p^{h_3(z)}F_3(z))$$

is an accessible refinement of ρ_z^{ss}.

(5) *At y_0, this refinement is $(\chi_v^\perp(p), 1, \chi_v(p))$.*

Proof. As the proof is almost the same as [BC04] and we chose to detail the inert case, we will just sketch it. Choose x_f the point in $\mathcal{E}$ associated to $\pi^n(\chi)$ and the accessible refinement $(\chi_v^\perp(p), 1, \chi_v(p))$. Denote by $B \subset \mathcal{W}$ the closed subset defined as in the inert case by

$$\begin{cases} k_1 = 0, \\ 2k_2 + k_3 = 2 - a, \end{cases} \tag{3}$$

and choose X an irreducible component of $\mathcal{E} \times_{\mathcal{W}} B$ containing x_f. Apply [BC04, Lemma 7.2], and get Y regular and y_0 and a representation,

$$\rho : G_{E,Np} \longrightarrow \mathrm{GL}_3(\mathcal{O}_Y),$$

such that $\rho^\perp = \rho$. Denote h as in the inert case ($\nu = (1 - k_3, k_2 - 1, k_1)$), and idem for Z (classicity at the level of sheaves is given by [Bra16, 6.2], and classicity by Pilloni and Stroh

[PS12] or (in greater generality) [BPS16].). The four Hecke operators living on $\mathcal{E}$, $U_i, i = 0, \dots, 3$ are normalized as in 8.3, then set, for $i = 1, 2, 3$,

$$F_i = U_{i-1}U_3^{-1}.$$

By § 10.6, and local–global compatibility at v (with the fact that v coincide with τ_∞), h_i are the Hodge–Tate weights of $(\rho_z)_{|G_v}$, and the normalization of the Hecke Operators recalled in 10.6 assure that $(p^{h_i}F_i)_i$ is a refinement at v for all classical forms. $\square$

11.3 Constructing the extension

Proposition 11.3. *We are in one of the following two cases:*

(i) $\rho_{K(Y)}$ *is absolutely irreducible;*

(ii) *there exists a two-dimensional representation $r \subset \rho$ such that $r_{K(Y)}$ is absolutely irreducible and*

$$r_{y_0}^{ss} = \begin{pmatrix} \chi & \\ & \chi^\perp \end{pmatrix}.$$

Remark 11.4. As showed by the proof, the second case never happens if $a \geqslant 2$ and p is split.

Proof. The proof uses similar ideas as in [BC04, Proposition 9.1], but unfortunately in our case the refinement authorizes a two-dimensional subrepresentation and a one-dimensional quotient. Let us first suppose p splits, and suppose $\rho_{K(Y)}$ is reducible. As $\rho^\perp \simeq \rho$, we can suppose that there exists a character $\psi \subset \rho$ and a two-dimensional representation r such that

$$0 \longrightarrow \psi \longrightarrow \rho \longrightarrow r \longrightarrow 0.$$

Thus there exists i such that the (generalized) Hodge–Tate weight of ψ at v is h_i. Moreover, for all $z \in Z$, by weak admissibility of ψ, we must have that there exists a j such that $v(p^{h_j}F_j(z)) = h_i(z)$. As the valuation of F_j is constant on Y, we can calculate it at y_0 and

$$\alpha = (v(F_1), v(F_2), v(F_3)) = (0, a, -a).$$

In particular, at $z \in Z$ such that $|h_i(z) - h_j(z)| > a$ for all $i \neq j$, we find $\alpha_j = 0$ and $i = j = 1$. Thus, by density ψ is of Hodge–Tate weight h_1. In particular, at y_0, ψ is of Hodge–Tate weight $h_1(y_0) = 1 - a$. Now, if $a \neq 1$, $\psi_{y_0} = \chi$, and if r were reducible, then by weak admissibility we would find $i, j \neq 1$ such that $h_i(z) = h_j(z) + \alpha_j$, for a Zariski dense subset of $z \in Z$, which is absurd. Thus there is a unique subquotient of ρ which is of rank 1, it is ψ. As $\rho^\perp = \rho$, this means that $\psi^\perp = \psi$, which is impossible as $\psi_{y_0} = \chi$. If $a = 1$ then ψ_{y_0} has v-Hodge–Tate weight 0, and if $\psi_{y_0} = \chi$ the same as previously happens; thus suppose that $\psi_{y_0} = 1$. In this case, r is still irreducible but $r_{y_0}^{ss} = \chi \oplus \chi^\perp$.

Now, we focus on p inert, which is similar. Suppose we are not in the case where $\rho_{K(Y)}$ is irreducible. We can thus find a two-dimensional subrepresentation $r \subset \rho$ (if r is one-dimensional, take the quotient and apply $(.)^\perp$, as $\rho^\perp = \rho$). Suppose that r is reducible. Take $z \in \mathcal{Z}$; as the valuation α_1 of F_1 is constant, we can calculate it at y_0, and we get, from $p^{h_1 - h_3}F_1(y_0) = \chi_p^\perp(p)$,

$$\alpha_1 = a.$$

But if r_z^{ss} is not irreducible, this means, following Rogawski's classification recalled in [BC04, § 3.2.3], and the fact that the representations associated by Blasius and Rogawski [BR92] are irreducible (but not necessarily three-dimensional), that z is either endoscopic-tempered of type

$(1,1,1)$, endoscopic non-tempered or stable non-tempered. In the case endoscopic non-tempered, looking at the Arthur parameter at infinity, the Hodge–Tate weights verifies $h_1 = h_2$ or $h_2 = h_3$, which is not possible by choice of Y and Z. In the stable non-tempered case, the Hodge–Tate weights are (k,k,k), which is not allowed in $\mathcal{Z}$. So we need to check that z is not endoscopic of type $(1,1,1)$. But in this case, this would mean by weak admissibility for ρ_z^{ss} (which would thus be a totally split sum of three characters) that

$$\{h_1 - h_3 + \alpha_1, 0, h_3 - h_1 - \alpha_1\} = \{h_1 - h_3, 0, h_3 - h_1\},$$

but the previous equality is impossible for $-h_1 < -a$ (which is the generic situation). Thus $z \in \mathcal{Z}$ is endoscopic, tempered, of type $(2,1)$, and r is irreducible. By weak admissibility, and the previous calculations (or because $\rho^\perp = \rho$), $r_{y_0}^{ss}$ has to be $\chi^\perp \oplus \chi$. $\square$

11.4 Good reduction outside p

Proposition 11.5. *In the previous case (i) of Proposition 11.3, write $\rho' = \rho_{K(Y)} \otimes (\chi_p^\perp)^{-1}$. Let $v|\ell \neq p$ be a place of E. Then we have the following.*

(i) *If $v \nmid \mathrm{Cond}(\chi)$, then $\rho_{K(Y)}$ and ρ' are unramified at v.*
(ii) *If $v|\mathrm{Cond}(\chi)$, then $\dim_{K(Y)}(\rho'_{K(Y)})^{I_v} = 2$.*

In case (ii) of Proposition 11.3, write $r' = r_{K(Y)} \otimes (\chi_p^\perp)^{-1}$. Let $v|\ell \neq p$ be a place of E. Then we have the following.

(i) *If $v \nmid \mathrm{Cond}(\chi)$, then $r_{K(Y)}$ and r' are unramified at v.*
(ii) *If $v|\mathrm{Cond}(\chi)$, then $\dim_{K(Y)}(r')^{I_v} = 1$.*

Proof. After all the constructions, this can be deduced as in [BC04]. First there exists $g \in \mathcal{O}_Y$ such that $g(y_0) \neq 0$ and $\rho_{K(Y)}$ has a $\mathcal{O}_{Y,(g)}$ stable lattice. Denote by ρ the representation valued in $\mathcal{O}_{Y,(g)}$, and for all $y \in \mathrm{Spm}(\mathcal{O}_{Y,(g)}) = Y(g^{-1})$, ρ_y the reduction at y. In case (i) of Proposition 11.3, as $\rho_{K(Y)}$ is semi-simple, ρ_z is semi-simple for $z \in Z'$, a cofinite subset of $Z \cap Y(g^{-1})$. But now, for $z \in Z'$, $\rho_z = \rho_z^{ss}$ is the Galois representation associated to a regular automorphic representation Π_z of $GU(2,1)$. In case (ii) of Proposition 11.3, $r_{K(Y)}$ is semi-simple, thus for all $z \in Z'$, still cofinite in Z, $r_z^{ss} = r_z \subset \rho_z^{ss}$, and

$$\dim_{K(Y)} {r'}^{I_v} \geqslant \dim_{K(Y)} ({\rho'}^{ss})^{I_v} - 1,$$

and $\dim_{K(Y)}(\rho')^{I_v}$ is related to the ramification of a (tempered endoscopic of type $(2,1)$) automorphic representation of $GU(2,1)$. Thus, to show the result, we only need to control ramification at v of (the base change of) Π_z.

If $v \nmid \mathrm{Cond}(\chi)$, by construction of the eigenvariety and choice of the maximal compact, $(\Pi_z)_v$ has a vector fixed by K_ℓ. We can thus finish because if ℓ is unramified, then K_ℓ is hyperspecial, and if ℓ ramifies, then K_ℓ is chosen very special, and [BC04, Proposition 3.1] gives the result for the base change. Now by local–global compatibility (for example [Ski12]), ρ_z^{ss} (and thus r_z in case (ii) of Proposition 11.3) is unramified at v.

If $v|\mathrm{Cond}(\chi)$, by construction Π_z has type $(K_\ell, J_\ell \otimes \chi_{0,\ell}^{-1} \circ \det)$, and thus by Proposition 10.21 the local Langlands representation associated to $(\Pi_z)_v$ is $\phi_1 \oplus \phi_2 \oplus \phi_3 \chi_{0,\ell}$ for three unramified characters ϕ_i. Thus, by local–global compatibility again, there exists I_v' a finite index subgroup of I_v such that $\rho_z'(I_v') = 1$. Thus, $(\rho')(I_v') = 1$. But up to extending scalars, $\rho'_{|I_v}$ is a finite

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representation θ of I_v/I'_v, defined on F' a finite extension of F. Thus, $\rho'_{I_v} \otimes_F F'$ is well-defined and semi-simple, and, evaluating the trace, we get

$$1 \oplus 1 \oplus (((\chi_p)^{\perp})^{-1})_{I_v} = (\rho'_{|I_v} \otimes F')^{ss}_{y_0} \simeq \theta.$$

We thus get the result. □

11.5 Elimination of case (ii) of Proposition 11.3

We want to prove that $\rho_{K(Y)}$ is always irreducible, and thus prove that case (ii) of Proposition 11.3 can never happen. Thus suppose we are in case (ii).

Proposition 11.6. *There exists a continuous representation $\overline{r} : G_E \longrightarrow \mathrm{GL}_3(F)$ such that $\overline{r}$ is a non-split extension of $\chi^{\perp}$ by χ,*

$$\overline{r} = \begin{pmatrix} \chi & \star \\ & \chi^{\perp} \end{pmatrix}$$

such that $\overline{r} = \overline{r}^{\perp}$ and verifying:

(i) $\dim_F(\overline{r} \otimes \overline{\chi})^{I_v} = 2$ *if* $v \not| \mathrm{Cond}(\chi)$;

(ii) $\dim_F(\overline{r} \otimes \overline{\chi})^{I_v} \geqslant 1$ *if* $v | \mathrm{Cond}(\chi)$;

(iii) *if p splits,* $D_{\mathrm{cris},\overline{v}}(\overline{r})^{\phi = \chi^{\perp}_{\overline{v}}(p)} \neq 0$ *and* $D_{\mathrm{cris},v}(\overline{r})^{\phi = \chi^{\perp}_{v}(p)} \neq 0$;

(iv) *if p is inert,* $D_{\mathrm{cris},\tau}(\overline{r})^{\phi^2 = \chi^{\perp}(p)} \neq 0$.

Proof. We first sketch the proof in case p inert we will detail a bit the argument in Proposition 11.8. First, by Ribet's Lemma (see [Bel03, Corollaire 1] or [Che03, Appendice, Lemma 3.1] and [BC04, Lemme 7.3]) there exists a $g \neq 0 \in \mathcal{O}_Y$ and a $\mathcal{O}_{Y,(g)}$-lattice Λ stable by $r_{K(Y)}$ such that $\overline{r} := \overline{r_{\Lambda}} = \begin{pmatrix} \chi & \star \\ & \chi^{\perp} \end{pmatrix}$ is a non-split extension. We can moreover easily assure that this lattice is a direct factor of a lattice stable by $\rho_{K(Y)}$. Then, conditions (i) and (ii) follow from the Proposition 11.5. For condition (iii), we can use the analog of Kisin's argument as extended by Liu, see [Liu15], as in the proof of the next proposition, as for all $z \in Z$,

$$D_{\mathrm{cris},\tau}(r_z)^{\phi^2 = p^{h_1} F_1} = D_{\mathrm{cris},\tau}(\rho_z)^{\phi^2 = p^{h_1} F_1} \neq 0$$

as shown by Proposition 11.3. The split case is similar to [BC04, Proposition 9.3 and Lemme 9.1] (see also Remarque 6.3.8 or apply the results to $\overline{r}_{\overline{v}}$). □

Write $r' = r \otimes (\chi^{\perp}_p)^{-1} = r \otimes \overline{\chi_p}$, which is an extension of 1 by $\chi_p \overline{\chi_p} = \omega_p$ (the cyclotomic character).

Lemma 11.7. *The representation r' is crystalline at p.*

Proof. As $\chi^{\perp}$ is crystalline (at v and $\overline{v}$ if p splits, at p if p is inert), it is enough to prove that r is crystalline. Suppose first p is inert. As $D_{\mathrm{crys},\tau}$ is left-exact, because $\overline{r}$ is extension of $\chi^{\perp}_p$ by χ_p we have

$$D_{\mathrm{crys},\tau}(\chi_p) \subset D_{\mathrm{crys},\tau}(\overline{r}),$$

but on $D_{\mathrm{crys},\tau}(\chi_p)$ φ^2 acts as $\chi_p(p) = \chi^{\perp}_p(p)p^{-2}$, thus this line is distinct from $D_{\mathrm{crys},\tau}(r)^{\varphi^2 = \chi^{\perp}(p)}$, and thus $D_{\mathrm{crys},\tau}(r)$ is of dimension 2. But because of the action of φ, $D_{\mathrm{crys}}(r)$ is a $K \otimes_{\mathbb{Z}_p} F$-module of dimension 2, i.e. r is crystalline. If p splits, as $\overline{r}^{\perp} \simeq r$, it suffices to prove that $\overline{r}$ is crystalline at $\overline{v}$. But we use $D_{\mathrm{cris},\overline{v}}(\overline{r})^{\phi = \chi^{\perp}_{\overline{v}}(p)}$ as in the inert case to get the result. □

Thus r' gives a non-zero element in $H^1_f(E, \omega_p)$, but by [BC04, Lemme 9.3], which is a well-known result, $H^1_f(E, \omega_p) = \{0\}$, and thus r' must be trivial, which gives a contradiction. We are thus in the case where $\rho_{K(Y)}$ is irreducible.

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11.6 Good reduction at p

Suppose p is inert. The result for p split is analogous to [BC04, Proposition 9.3]. Write $u = \chi_p^{\perp}(p)$.

Proposition 11.8. *There exists a continuous representation $\overline{\rho} : G_E \longrightarrow \mathrm{GL}_3(F)$ such that we have the following.*

(i) *For every place v of E not dividing p, we have:*
 (a) $\dim_F(\overline{\rho} \otimes (\chi_p^{\perp})^{-1})^{I_v} \geqslant 2$ *if* $v|\mathrm{Cond}(\chi)$;
 (b) $\dim_F(\overline{\rho} \otimes (\chi_p^{\perp})^{-1})^{I_v} = 3$ *if* $v \not| \mathrm{Cond}(\chi)$.

(ii) $D_{\mathrm{cris},\tau}(\overline{\rho})^{\varphi^2=u}$ *is non-zero.*

(iii) $\overline{\rho}^{ss} \simeq \chi_p \oplus \chi_p^{\perp} \oplus 1$ *and one of the two following assertions is true.*
 (a) *Either $\overline{\rho}$ has a subquotient r of dimension 2, such that $r^{\perp} \simeq r$ and r is a non-trivial extension of $\chi_p^{\perp}$ by χ_p.*
 (b) *Either $\overline{\rho} \simeq \overline{\rho}^{\perp}$; $\overline{\rho}$ has a unique subrepresentation r_1 of dimension 2 and a unique subquotient r_2 of dimension 2, with r_1 a non-trivial extension of 1 by χ_p and r_2 a non-trivial extension of $\chi_p^{\perp}$ by 1, and $r_1^{\perp} \simeq r_2$.*

Proof. Denote by $\mathcal{O}$ the rigid local ring of Y at y_0, a discrete valuation ring of residual field F, denote by L its fraction field, and ρ_L the representation which is the scalar extension of ρ to L. As $\overline{\rho_L}^{ss} = 1 \oplus \chi_p \oplus \chi_p^{\perp}$ which are pairwise distinct characters, we can also use [BC04, Proposition 7.1], the analog to Ribet's theorem, to find $\Lambda \subset L^3$ a lattice stable by ρ_L such that the reduced representation $\overline{\rho} = \overline{\rho_\Lambda}$ satisfies condition (iii)(a) or (iii)(b). Condition (i) is true by Proposition 11.5. We can argue as in [BC04] to get condition (ii), but we will need a generalization to G_K if p is inert. Fortunately what we need is in [Liu15]. As in [BC04, Lemma 7.3] there is an affinoid $Y \supset \Omega \ni y_0$ such that ρ_L as a $\mathcal{O}_\Omega$-stable lattice Λ_Ω such that $\overline{\rho}_{\Lambda_\Omega,y_0} = \overline{\rho}$. Write $\rho = \rho_{\Lambda_\Omega}$. Let thus $Z' \subset \Omega$ the points that are in $Z \subset Y$, in Ω, and such that $\overline{\rho_z}$ is semi-simple (it is a cofinite subset of $Z \cap \Omega$ as $\rho_{K(\Omega)}$ is semi-simple (irreducible)). By choice of Z, we have that for all $z \in Z'$

$$D_{\mathrm{crys},\tau}(\rho_z)^{\phi^2=p^{h_1(z)-h_3(z)}F_1(z)} \neq 0.$$

As ρ is polarized, its $\sigma\tau$-Hodge–Tate weights are $h^\sigma(z) = (-h_3, -h_2, -h_1)$. Set $h_i^K = (h_i, h_{4-i}) \in F_\tau \times F_{\sigma\tau} = K \otimes_{\mathbb{Q}_p} F$. Thus $(\Omega, \rho, (h_i^K)_i, F_1, Z)$ is a weakly refined (polarized) p-adic representation of G_K of dimension 3 in the sense of [Liu15, Definition 0.3.1]. To verify property (f) of [Liu15, Definition 0.3.1], recall that over the weight space $\mathcal{W}$ we had a universal character $\chi = \chi_1 \times \chi_2 : \mathcal{O}^\times \times \mathcal{O}^1 \longrightarrow \mathcal{O}(\mathcal{W})^\times$, and, as $\mathcal{W}$ is regarded over K, we can split $\mathcal{O}(\mathcal{W}) \otimes_{\mathbb{Q}_p} K = \mathcal{O}(W)_\tau \times \mathcal{O}(W)_{\sigma\tau}$. Under this isomorphism, the derivative at 1 of χ_1, denoted $(wt_\tau(\chi_1), wt_{\sigma\tau}(\chi_1))$, is at every $\kappa \in \mathbb{Z}^3 \subset \mathcal{W}$ given by

$$(k_1, k_3) = (h_3, 1 - h_1).$$

Thus set $\psi = \tau^{-1}(\chi_1 \circ c)$. Its derivative at 1 is given by $(k_3 - 1, k_1) = (-h_1, h_3)$ at classical points $\kappa \in \mathbb{Z}^3$. Thus, the character,

$$\mathcal{O}^\times \xrightarrow{\psi} \mathcal{O}(\mathcal{W})^\times \longrightarrow \mathcal{O}(B)^\times \longrightarrow \mathcal{O}(\Omega)^\times,$$

has the desired property (f).

Write $\rho' = \rho \otimes \psi$ (where ψ is precomposed by product of the two Lubin–Tate characters of K, $G_K \longrightarrow O^\times$). Thus ρ' has $\kappa_1^{K'} = (0, 0)$ as smallest Hodge–Tate weight. In particular, by [Liu15, Theorem 0.3.2], $\Omega_{fs} = \Omega$. But then by Theorem 0.1.2 applied to $S = \Omega$, k, n big enough, and ρ', we have that

$$D^+_{\mathrm{crys}}(\rho'_{y_0})^{\varphi^2=F_1} \simeq D^+_{\mathrm{Sen}}(\rho'_{y_0})^{\Gamma},$$

(see [Liu15, Remark 3.3.5 and Corollary 1.5.4]; as 0 is the only non-positive Hodge–Tate weight[6] of ρ', [Liu15, Corollary 1.5.4] applies), and $D^+_{\mathrm{Sen}}(\rho'_{y_0})^{\Gamma} \neq 0$.

Thus $D_{\mathrm{crys},\tau}(\rho'_{y_0})^{\varphi^2=F_1} \neq 0$, which means

$$D_{\mathrm{crys},\tau}(\rho_{y_0})^{\varphi^2=p^{h_1(y_0)-h_3(y_0)}F_1} = D_{\mathrm{crys},\tau}(\rho_{y_0})^{\varphi^2=u} \neq 0. \qquad \square$$

11.7 Elimination of case (iii)(a) of Proposition 11.8

We can do as in [BC04], and as we eliminated case (ii) of Proposition 11.3, to eliminate case (iii)(a) of Proposition 11.8. Suppose we are in case (iii)(a); there is thus a subquotient r of $\overline{\rho}$ such that $r^{\perp} \simeq r$ and r is an extension of $\chi_p^{\perp}$ by χ_p. Write $r' = r \otimes (\chi_p^{\perp})^{-1} = r \otimes \overline{\chi_p}$, which is an extension of 1 by $\chi_p\overline{\chi_p} = \omega_p$ (the cyclotomic character).

Lemma 11.9. *The representation r' is crystalline at p (at $v_1, v_2|p$ is p is split).*

Proof. The split case is identical to [BC04, Lemma 9.1] and Lemma 11.7. Suppose p is inert. As $\chi_p^{\perp}$ is crystalline, it is enough to prove that r is crystalline. But $V \mapsto D_{\mathrm{crys},\tau}(V)^{\varphi^2=u}$ is left-exact, thus, if we write $u = \chi^{\perp}(p)$,

$$\dim_F D_{\mathrm{crys},\tau}(\overline{\rho})^{\varphi^2=u} \leqslant \dim_F D_{\mathrm{crys},\tau}(r)^{\varphi^2=u} + D_{\mathrm{crys},\tau}(1)^{\varphi^2=u}.$$

But $D_{\mathrm{crys},\tau}(1)^{\varphi^2=u} = 0$, and thus $D_{\mathrm{crys},\tau}(r)^{\varphi^2=u} \neq 0$. The end of the proof is identical to Lemma 11.7. $\square$

Lemma 11.10. *The representation r' is unramified at every place $w \nmid p$.*

Proof. This is exactly identical to [BC04, Lemme 9.2]. $\square$

Thus by [BC04, Lemme 9.3], r must be trivial, which contradicts Proposition 11.8(iii)(a).

11.8 Conclusion

We are thus in case (iii)(b) of Proposition 11.8, with r_1 a non-trivial extension of 1 by χ_p.

Lemma 11.11. *The representation r_1 is crystalline at p if p is inert, and at $v_1, v_2|p$ if p splits.*

Proof. Suppose p inert. As $r_1 \simeq r_2^{\perp}$, we only need to prove that r_2 is crystalline. Because $D_{\mathrm{crys},\tau}(\cdot)^{\varphi^2=u}$ is left-exact, we again have

$$\dim_F D_{\mathrm{crys},\tau}(\overline{\rho})^{\varphi^2=u} \leqslant \dim_F D_{\mathrm{crys},\tau}(r_2)^{\varphi^2=u} + \dim_F D_{\mathrm{crys},\tau}(\chi_p)^{\varphi^2=u}.$$

As $D_{\mathrm{crys},\tau}(\chi_p)^{\varphi^2=u} = \{0\}$ and $\dim_F D_{\mathrm{crys},\tau}(\overline{\rho})^{\varphi^2=u} \neq 0$, we have $\dim_F D_{\mathrm{crys},\tau}(r_2)^{\varphi^2=u} \neq \{0\}$. Moreover,

$$D_{\mathrm{crys},\tau}(1) \subset D_{\mathrm{crys},\tau}(r_2),$$

[6] In [Liu15] the convention of the Hodge–Tate weights is opposite to ours: there the Hodge–Tate weight of the cyclotomic character is 1.

by left-exactness of $D_{\mathrm{crys},\tau}$, which gives a line where φ^2 acts as $1 \neq u$. Thus there are at least two different lines in $D_{\mathrm{crys},\tau}(r_2)$ which means this is two-dimensional and, by existence of φ, r_2 (thus r_1) is crystalline.

Now suppose p splits. Then the proof is identical to [BC04, Lemme 9.4], as $1 \neq \chi_v(p) \neq \chi_v^{\perp}(p) \neq 1$ (recall $|\chi_v(p)|_{\mathbb{C}} = |\chi_{\overline{v}}(p)|_{\mathbb{C}} = p^{-1/2}$). □

Theorem 11.12. *The representation r_1 gives a non-zero element of $H^1_f(E, \chi_p)$.*

Proof. We only need to prove that r_1 has good reduction outside p. But then as ρ is unramified outside $p\,\mathrm{Cond}(\chi)$, by Proposition 11.8, we only need to check for $v|\mathrm{Cond}(\chi)$. We have shown in the proof of Proposition 11.5 that there exists an open subgroup $I'_w \subset I_w$ such that $\rho'_{|I_w}$ factors through I_w/I'_w and $\rho'_{|I_w} = 1 \oplus 1 \oplus (\chi_p^{\perp})^{-1}_{|I_w}$. Thus $r_1^{I_w}$ is then of dimension 1. □

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Appendix A. Calculations on the weight space

In this appendix we explain a bit more the structure of the weight space $\mathcal{W}$ defined in § 3. $\mathcal{W}$ is represented by a disjoint union of $((p+1)(p^2-1))$ three-dimensional open balls over $\mathcal{O}$. Indeed (if $p \neq 2$)

$$\mathcal{O}^\times \simeq (\mathbb{F}_{p^2})^\times \times (1 + p\mathcal{O}),$$

which induced, up to the choice of a basis of $\mathcal{O}$ over $\mathbb{Z}_p$, an isomorphism,

$$\mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}^\times, \mathbb{G}_m) \simeq (\mathbb{Z}/\widehat{(p^2-1)}\mathbb{Z}) \times B_2(1, 1^-),$$

where $B_2(1,1^-)$ is the open two-dimensional ball centered in 1, of radius 1. Also, as a $\mathbb{Z}_p$-module,

$$\mathcal{O}^1 \simeq S \times \mathbb{Z}_p,$$

where S is a finite group of cardinal $p+1$.

Proof. We have the exact sequence,

$$0 \longrightarrow \mathcal{O}^1 \longrightarrow \mathcal{O}^\times \xrightarrow{Nm} \mathbb{Z}_p^\times \longrightarrow 0,$$

(surjectivity is given by local class field theory for example). Reducing modulo p, we have the surjectivity of $\mathbb{F}_{p^2} \xrightarrow{\overline{Nm}} \mathbb{F}_p$. We thus have the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \{x \in \mathcal{O}^1 : x \equiv 1 \pmod{p}\} \arrow[r] \arrow[d] & 1+p\mathcal{O} \arrow[r, "Nm^1"] \arrow[d] & 1+p\mathbb{Z}_p \arrow[d] & \\
0 \arrow[r] & \mathcal{O}^1 \arrow[r] \arrow[d] & \mathcal{O}^\times \arrow[r, "Nm"] \arrow[d] & \mathbb{Z}_p^\times \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \{x \in \mathbb{F}_{p^2} : x^{p+1} = 1\} \arrow[r] & \mathbb{F}_{p^2}^\times \arrow[r, "\overline{Nm}"] \arrow[d] & \mathbb{F}_p^\times \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

The application $Nm^1 = 1+p\mathcal{O} \longrightarrow 1+p\mathbb{Z}_p$ is surjective; indeed, for all z inside $1+p\mathbb{Z}_p$, because Nm is surjective, there exists $u \in \mathcal{O}^\times$ such that $uu^\sigma = 1+pz$ (denote by σ the conjugation, and $\overline{\bullet}$ reduction modulo p). We deduce that $\overline{u} \in \{x \in \mathbb{F}_{p^2} : x^{p+1} = 1\}$. We then set $u' = u/[\overline{u}]$, where $[.]$ denotes the Teichmuller lift. Then $u' \in 1+p\mathcal{O}$ and $(u')(u')^\sigma = uu^\sigma/([\overline{u}][\overline{u}]^\sigma) = uu^\sigma/([\overline{u}^{p+1}]) = 1+pz$. The second equality is because $[.]$ commutes with Frobenius morphisms. (We could also prove the surjectivity by a method of successive approximations.) The map $\mathcal{O}^1 \longrightarrow \{x \in \mathbb{F}_{p^2} : x^{p+1} = 1\}$ is also surjective: for all $x \in \{x \in \mathbb{F}_{p^2} : x^{p+1} = 1\}$, $[x][x]^\sigma = [x^{p+1}] = [1] = 1$. Thus, up to choosing a base of $\mathcal{O}$ over $\mathbb{Z}_p$, we can with the logarithm identify $1+p\mathcal{O}$ to $\mathbb{Z}_p^2$; this assures that $\{x \in \mathcal{O}^1 : x \equiv 1 (\mathrm{mod}\ p)\} \simeq \mathbb{Z}_p$ (because logarithm exchanges trace and Nm). □

In particular,

$$\mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}^1, \mathbb{G}_m) \simeq \coprod_{\hat{S}} B_1(1, 1^-).$$

Thus, $\mathcal{W}$ is isomorphic to a union of $(p+1)(p^2-1)$ open balls of dimension 3. There is also a universal character,

$$\kappa^{\mathrm{un}} : T^1(\mathbb{Z}_p) \longrightarrow \mathbb{Z}_p[[T^1(\mathbb{Z}_p)]].$$

The following lemma is essential.

LEMMA A.1. *Every weight $\kappa \in \mathcal{W}(K)$ is automatically locally ($\mathbb{Q}_p$-) analytic.*

Actually we can be more precise.

LEMMA A.2. *Let $\mathcal{U} \subset \mathcal{W}$ a quasi-compact open; then there exists $w_{\mathcal{U}}$ such that $\kappa^{\mathrm{un}}_{|\mathcal{U}}$ is $w_{\mathcal{U}}$-analytic.*

Proof. It is [Urb11, Lemma 3.4.6]. □

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We will construct $\mathcal{W}(w)$, an open subset of $\mathcal{W}$ containing the w-analytic κ (it is an affinoid). Set $w \in]n-1,n] \cap v(\overline{\mathbb{Q}}_p)$. We define it this way, following [AIP15]. First set $\mathfrak{W}(w)^0$ to be $\operatorname{Spf} \mathcal{O}_K\langle\langle S_1, S_2, S_3\rangle\rangle$ where K is a finite extension of $\mathbb{Q}_p$ containing an element p^w of valuation w. Define $\mathfrak{T}_w$ the subtorus of $\mathfrak{T}$ the formal torus associated to T^0, given by

$$\mathfrak{T}_w(R) = \operatorname{Ker}(\mathfrak{T}(R)) \longrightarrow \mathfrak{T}(R/p^w R),$$

for any flat, p-adically complete $\mathcal{O}_K$-algebra R. Denote by X_i' the coordinates on $\mathfrak{T}_w$, so that $1 + p^w X_i' = 1 + X_i$ on $\mathfrak{T}$, and define the universal character,

$$\kappa^{0,\mathrm{un}} : \begin{array}{ccc} \mathfrak{T}_w \times \mathfrak{W}(w)^0 & \longrightarrow & \widehat{\mathbb{G}_m} \\ (1+p^w X_1', 1+p^w X_2', 1+p^w X_3', S_1, S_2, S_3) & \longmapsto & \displaystyle\prod_{i=1}^{3} (1+p^w X_i)^{S_i p^{-w+2/p-1}}. \end{array}$$

Then define $\mathcal{W}(w)^0$ to be the rigid fiber of $\mathfrak{W}(w)^0$ and finally, $W(w)$ to be the fiber product,

$$\mathcal{W} \times_{\operatorname{Hom}_{\mathrm{cont}}((1+p\mathcal{O})\times(1+p\mathcal{O})^1, \mathbb{C}_p^\times)} \mathcal{W}(w)^0,$$

where the map $\mathcal{W}(w)^0 \longrightarrow \operatorname{Hom}_{\mathrm{cont}}(1+p\mathcal{O}) \times (1+p\mathcal{O})^1, (\mathbb{C}_p^\times)$ is given by

$$(s_1, s_2, s_3) \longmapsto (1+p^n x_1, 1+p^n x_2, 1+p^n x_3) \mapsto \prod_{i=1}^{3} (1+p^n x_i)^{s_i p^{-w+2/p-1}}.$$

Then we can write $\mathcal{W} = \bigcup_{w \geqslant 0} \mathcal{W}(w)$ as an increasing union of affinoids.

Appendix B. Kernel of Frobenius

Proposition B.1. *On the stack $\mathcal{BT}^{\mathcal{O}}_{(2,1),\mathrm{pol}}$ and $\overline{X}$, the Cartier divisor ha_τ is reduced.*

Proof. This is [dSG16, Theorem 2.8], which can be proved by considering the deformation space at a point. Unfortunately we cannot use the result of [Her18] because of the polarization (but a similar proof works). □

Proposition B.2. *Let $G/\operatorname{Spec}(\mathcal{O}_C)$ be a p-divisible $\mathcal{O}$-module. Suppose $\mathrm{ha}_\tau(G) < 1/2p^2$, and let K_1 the first Frobenius-subgroup of G (see Theorem 5.8). Then*

$$K_1 \times_{\operatorname{Spec}(\mathcal{O}_C)} \operatorname{Spec}(\mathcal{O}_C/p^{1/2p^2}) = \operatorname{Ker} F^2 \times_{\operatorname{Spec}(\mathcal{O}_C/p)} \operatorname{Spec}(\mathcal{O}_C/p^{1/2p^2}).$$

Denote, for $K/\mathbb{Q}_{p^2}$, by $\mathfrak{X}/\operatorname{Spf}(\mathcal{O}_K)$ a (smooth) presentation of $\mathcal{BT}^{\mathcal{O}}_{r,(2,1),\mathrm{pol}}/\operatorname{Spf}(\mathcal{O}_K)$ (which is smooth, see for example [Wed01]), and, for $v \in v(K)$, $\mathfrak{X}(v)$ is the open subset of the blow-up along $I_v = (p^v, \mathrm{ha}_\tau)$ where I_v is generated by ha_τ. As $\mathfrak{X}$ is smooth and ha_τ is reduced, $\mathfrak{X}(v)$ is normal and its special fiber (modulo π_K) is reduced.

Take $v = 1/2p^2$ and K a totally ramified extension of $\mathbb{Q}_p$ of degree $1/2p^2$ (so that $v(\pi_K) = 1/2p^2$).

Then over $X(v)$, the rigid fiber over K of $\mathfrak{X}(v)$, we have a subgroup $K_1 \subset G[p^2]$, and by the Proposition 5.11 this subgroup extend to a subgroup over $\mathfrak{X}(v)$. Now, over $\mathfrak{X}(v) \otimes \mathcal{O}_K/\pi_K = \mathfrak{X}(v) \otimes \kappa_K$ the rigid fiber of $\mathfrak{X}(v)$, we have two subgroups, K_1 and $\operatorname{Ker} F^2$, which coincide on every point (by [Her19, § 9] or the very proof of the Proposition 5.11) but as $\mathfrak{X}(v) \otimes \mathcal{O}_K/\pi_K$ is reduced, $K_1 = \operatorname{Ker} F^2$ over $X(v) \otimes \mathcal{O}_K/\pi_K$. As every $\mathcal{O}_C$-point of $\mathcal{BT}^{\mathcal{O}}_{(2,1),\mathrm{pol}}$ gives a point of $\mathfrak{X}$, we have the result using $G[p^r]$ for r big enough (bigger than 3 is enough).

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Corollary B.3. *Let G as in the previous proposition, but suppose* $\mathrm{ha}_\tau(G) < 1/2p^4$. *Then* $\mathrm{ha}_\tau(G/K_1) = p^2\,\mathrm{ha}_\tau(G)$.

Proof. Recall that $\mathrm{ha}_\tau = \mathrm{ha}_{\sigma\tau}$ and ha_τ is given by $\det(V^2)$ without any division. By the previous proposition, the map $G[p^2] \longrightarrow G[p^2]/K_1$ coincide modulo π_K with the map $G[p^2] \xrightarrow{F^2} G[p^2]^{(p^2)}$. Thus, there is an isomorphism modulo π_K : $\det(\omega_{(G/K_1)^D,\sigma\tau}) \simeq \det(\omega_{G^D,\sigma\tau}^{\otimes p^2})$ which identifies (modulo π_K) $\widetilde{\mathrm{ha}_{\sigma\tau}}(G/K_1)$ with $\widetilde{\mathrm{ha}_{\sigma\tau}}(G)^{\otimes p^2}$. Thus we get

$$\inf\left\{p^2\,\mathrm{ha}_\tau(G), \frac{1}{2p^2}\right\} = \inf\left\{\mathrm{ha}_\tau(G/K_1), \frac{1}{2p^2}\right\}.$$

As $p^2\,\mathrm{ha}_\tau(G) < 1/2p^2$, we get the result. □

Appendix C. Dévissage of the formal coherent locally analytic sheaves

Let $\kappa \in \mathcal{W}(w)$ a character and κ^0 its restriction to $\mathcal{W}(w)^0$, and $w < m - (p^2m-1)/(p^2-1)$. Denote on $\mathfrak{X}_1(p^{2m})(v)$ the sheaf $\mathfrak{w}_w^{\kappa^0\dagger}$ defined as

$$\zeta_*\mathcal{O}_{\mathfrak{I}W_w^+}[\kappa^0], \quad \text{where } \zeta : \mathfrak{IW}_w^+ \longrightarrow \mathfrak{X}_1(p^{2m})(v).$$

If we set $\pi : \mathfrak{X}_1(p^{2m})(v) \longrightarrow \mathfrak{X}(v)$, then the sheaf $\mathfrak{w}_w^{\kappa\dagger}$ of overconvergent forms is given by

$$(\pi_*\mathfrak{w}_w^{\kappa^0\dagger})(-\kappa)^{B(\mathbb{Z}_p)\mathfrak{B}_w},$$

where $(-)(-\kappa')$ denotes a twist of the action of $B(\mathbb{Z}_p)\mathfrak{B}_w$ and $(-)^{B(\mathbb{Z}_p)\mathfrak{B}_w}$ means taking invariants. Remark that after the twist, the action of $B(\mathbb{Z}_p)\mathfrak{B}_w$ factors through B_n.

Consider the projection 'in family'

$$\zeta \times 1 : \mathfrak{IW}_w^+ \times \mathfrak{W}(w)^0 \longrightarrow \mathfrak{X}_1(p^{2m})(v) \times \mathfrak{W}(w)^0,$$

and denote by

$$\mathfrak{w}_w^{\kappa^{0,\mathrm{un}}\dagger} = (\zeta \times 1)_*\mathcal{O}_{\mathfrak{IW}_w^+ \times \mathfrak{W}(w)^0}[\kappa^{0,\mathrm{un}}]$$

the family of sheaves over $\mathfrak{X}_1(p^{2m})(v) \times \mathfrak{W}(w)^0$.

Let $\mathrm{Spf}(R)$ a small enough open in $\mathfrak{X}_1(p^{2m})(v)$. Recall that we denote by ψ the universal polarized trivialization of K_m^D, denote by e_1, e_2 a basis of $\mathcal{O}/p^m\mathcal{O} \oplus \mathcal{O}/p^{2m}\mathcal{O}$, $e_1^{\sigma\tau} = \mathrm{HT}_{\sigma\tau,w}(e_1)$, $e_2^{\sigma\tau} = \mathrm{HT}_{\sigma\tau,w}(e_2), e^\tau = \mathrm{HT}_{\tau,w}(e_2)$ the images of this basis in $\mathcal{F}_{\sigma\tau}/p^w, \mathcal{F}_\tau/p^w$. Denote by $f_1^{\sigma\tau}, f_2^{\sigma\tau}$, f^τ a lift of this basis in $\mathcal{F}_{\sigma\tau}, \mathcal{F}_\tau$.

With these choices we can identify $\mathfrak{IW}_{w|\,\mathrm{Spf}(R)}^+$ with matrices,

$$\begin{pmatrix} 1 & & \\ p^w\mathfrak{B}(0,1) & 1 & \\ & & 1 \end{pmatrix} \times \begin{pmatrix} 1 + p^w\mathfrak{B}(0,1) \\ 1 + p^w\mathfrak{B}(0,1) \\ 1 + p^w\mathfrak{B}(0,1) \end{pmatrix} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(R).$$

Denote by X_0 the coordinate in the 3×3 matrix and by X_1, X_2, X_3 the coordinates of the balls inside the column. Thus, we can identify a function f on $\mathfrak{IW}_{w|\,\mathrm{Spf}(R)}^+$ to a formal series in $R\langle\langle X_0, X_1, X_2, X_3\rangle\rangle$.

Now, let $\kappa^0 \in \mathfrak{W}(w)^0$; then $f \in \mathfrak{w}_w^{\kappa^0\dagger}$ if it verifies

$$f(X_0, \lambda X_1, \lambda X_2, \lambda X_3) = (\kappa^0)'(\lambda) f(X_0, X_1, X_2, X_3), \quad \forall \lambda \in \mathfrak{T}_w(R).$$

In particular, we deduce that there exists a unique $g \in R\langle\langle X_0\rangle\rangle$ such that

$$f(X_0, X_1, X_2, X_3) = g(X_0)\kappa^0(X_1, X_2, X_3),$$

and thus there is a bijection $\mathfrak{w}_w^{\kappa^0\dagger} \simeq R\langle\langle X_0\rangle\rangle$. The same holds in family.

Lemma C.1. *For all $f \in \mathfrak{w}_w^{\kappa^{0,\mathrm{un}}\dagger}(R \hat{\otimes} \mathcal{O}_K\langle\langle S_1, S_2, S_3\rangle\rangle)$, there exists a unique $g \in R\langle\langle S_1, S_2, S_3, X_0\rangle\rangle$ such that*

$$f(X_0, X_1, X_2, X_3) = g(X_0)(\kappa^{0,\mathrm{un}})'(1 + p^w X_1, 1 + p^w X_2, 1 + p^w X_3).$$

This decomposition induces a bijection

$$\mathfrak{w}_w^{\kappa^{0,\mathrm{un}}\dagger}(R \hat{\otimes} \mathcal{O}_K\langle\langle S_1, S_2, S_3\rangle\rangle) \simeq R\langle\langle S_1, S_2, S_3, X_0\rangle\rangle.$$

Lemma C.2. *Let π be a uniformizer of $\mathcal{O}_K$. Then*

$$\kappa^{0,\mathrm{un}}((1 + p^w X_i)) \in 1 + \pi\mathcal{O}_K\langle\langle S_1, S_2, S_3, X_1, X_2, X_3\rangle\rangle.$$

Proof. The calculation is made in [AIP15, Lemma 8.1.5.3]. □

Corollary C.3. *Denote by $\mathfrak{w}_{w,1}^{\kappa^{0,\mathrm{un}}\dagger}$ the reduction modulo π of $\mathfrak{w}_w^{\kappa^{0,\mathrm{un}}\dagger}$. Then the sheaf $\mathfrak{w}_{w,1}^{\kappa^{0,\mathrm{un}}\dagger}$ is constant on $(\mathfrak{X}_1(p^{2n}) \times \mathfrak{W}(w)^0) \times \mathrm{Spec}(\mathcal{O}_K/\pi)$: it is the inverse image of a sheaf on $\mathfrak{X}_1(p^{2n}) \times \mathrm{Spf}(\mathcal{O}_K/\pi)$.*

Let $f_1^{\sigma\tau'}, f_2^{\sigma\tau'}, f^{\tau'}$ be another lift of the basis image of $\mathrm{HT}_{\star,w}$. Let

$$P = \begin{pmatrix} 1 + p^w a_1 & p^w a_2 & \\ p^w a_3 & 1 + p^w a_4 & \\ & & 1 + p^w a_5 \end{pmatrix}$$

be the base change matrix from $\underline{f}$ to $\underline{f}'$ and $\underline{X}'$ the coordinates on $\mathfrak{IW}^+_{w|\,\mathrm{Spf}(R)}$ relatively to $\underline{f}'$.

Lemma C.4. *We have the following congruences:*

$$\begin{aligned} X_0 &\equiv X_0' + a_3 \pmod{p^w}; \\ X_1 &\equiv X_1' + a_1 \pmod{p^w}; \\ X_2 &\equiv X_2' + a_4 \pmod{p^w}; \\ X_3 &\equiv X_3' + a_5 \pmod{p^w}. \end{aligned}$$

Proof. Indeed, as seen inside $\mathcal{T}_a^\times n/U_a n$, we have that the two systems of coordinates verifies

$$P(I_3 + p^w \underline{X})U = I_3 + p^w \underline{X}',$$

where $U \in \mathrm{GL}_2 \times \mathrm{GL}_1$ is a unipotent matrix of the form $I_3 + p^w N$, N upper triangular nilpotent, and

$$\underline{X} = \begin{pmatrix} X_1 & & \\ X_0 & X_2 & \\ & & X_3 \end{pmatrix}.$$

Thus, write $P = I_3 + p^w P_0$; then $I_3 + p^w(P_0 + \underline{X} + N) \equiv I_3 + p^w \underline{X}' \pmod{p^{2w}}$. □

We can thus deduce the following corollary for the family of sheaves,

Corollary C.5. *Let $\kappa^0 \in \mathcal{W}(w)(K)$. The quasi-coherent sheaf $\mathfrak{w}_w^{\kappa^{0,\mathrm{un}}\dagger}$ on $\mathfrak{X}_1(p^{2m}) \times \mathfrak{W}(w)^0$ is a small Banach sheaf.*

Proof. We just have to check that on $\mathfrak{X}_1(p^{2m}) \times \mathfrak{W}(w)^0 \times \mathrm{Spec}(O_K/\pi)$ the sheaf $\mathfrak{w}_{w,1}^{\kappa^{0,\mathrm{un}}\dagger}$ is an inductive limit of coherent sheaves which are extensions of the trivial sheaf. Write $\mathfrak{w}_{w,1}^{\kappa^{0,\mathrm{un}}\dagger,\geqslant r}$ the subsheaf of sections that are locally polynomials in X_0 of total degree smaller than r. This makes sense globally by Lemma C.4, and moreover, $\mathfrak{w}_{w,1}^{\kappa^{0,\mathrm{un}}\dagger}$ is the inductive limit over r of these sheaves. But then $\mathfrak{w}_{w,1}^{\kappa^{0,\mathrm{un}}\dagger,\geqslant r}(\mathrm{mod})\mathfrak{w}_{w,1}^{\kappa^{0,\mathrm{un}}\dagger,\geqslant r-1}$ is isomorphic to the trivial sheaf. □

Appendix D. Non-tempered representations and $(\mathfrak{q}, K)$-cohomology

We are interested in calculating the $(\mathfrak{q}, K)$-cohomology of the representation $\pi^n(\chi)$ defined in Proposition 10.12 to show it appears in the global sections of a coherent automorphic sheaf on the Picard modular surface.

We have the following theorem of Harris ([Har90b, Lemma 5.2.3 and Proposition 5.4.2], [Gol14, Theorem 2.6.1])

Theorem D.1. *Let $\pi = \pi_\infty \otimes \pi_f$ be an automorphic representation of $U(2,1)$ of Harish-Chandra parameter λ, and such that $H^0(\mathfrak{q}, K, \pi_\infty \otimes V_\sigma^\vee) \neq 0$; then there is a $U(2,1)(\mathbb{A}_f)$ equivariant embedding,*

$$\pi_f \hookrightarrow H^0(X, \mathcal{V}_\sigma^\vee),$$

where $\mathcal{V}_\sigma^\vee$ is the automorphic vector bundle associated to the representation $V_\sigma^\vee$ of $K = K_\infty$.

Thus we only need to calculate the $(\mathfrak{q}, K)$-cohomology of $\pi^n(\chi)_\infty$, and even the one of the restriction of the representation to $SU(2,1)$. Fortunately we can explicitly do so, rewriting the induction $\mathrm{ind}-n_B^{SU(2,1)}(\mathbb{R})(\chi_\infty)$, as a space of functions, and determining the quotient corresponding to $\pi^n(\chi)$. In [Wal76], Wallach calculated all the representations of $SU(2,1)(\mathbb{R})$ using this description of the induction. As explained in [Wal76, p. 181], the induction space $\mathrm{ind}-n_B^{SU(2,1)(\mathbb{R})}(\chi)$ corresponds to X^Λ with $\Lambda = (a-1)\Lambda_1 + (-a)\Lambda_2$ (which is thus reducible). The shift by $-\Lambda_1 - \Lambda_2$ is due to the normalization by the modulus character in the induction. Its discrete series subobject corresponds to one of the discrete series $D_{\widetilde{\Lambda}}^-$ described in [Wal76, p. 183], and its quotient corresponds to the non-tempered representation (T_{a-2}^-, Z_{a-2}^-) (defined in [Wal76, p. 184], and the fact that it appears in the said induction is [Wal76, Lemma 7.12]). As the name does not suggest, T_{a-2}^-, which coincides with the restriction of $\pi^n(\chi)_\infty$ to $SU(2,1)(\mathbb{R})$, will be holomorphic (but we can exchange holomorphic and anti-holomorphic by changing the complex structure of the Picard surface).

Proposition D.2. *Let $(\sigma, V_\sigma) = \mathrm{Sym}^{a-1} \otimes \det^{-a} : M \mapsto \mathrm{Sym}^{a-1}(\overline{M}) \otimes \det(\overline{M})^{-a}$ the representation of $U(2) = SK_\infty \subset SU(2,1)(\mathbb{R})$. Then*

$$H^0(\mathfrak{q}, K_\infty, T_{a-2}^- \otimes V_\sigma^\vee) \neq 0.$$

To show the previous proposition, denote by

$$J_0 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

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the hermitian form of signature $(2,1)$ used in [Wal76]. Denote by

$$P=\begin{pmatrix}\frac{1}{\sqrt{2}} & & \frac{1}{\sqrt{2}}\\ & 1 & \\ \frac{1}{\sqrt{2}} & & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

the base change matrix (so that $PJ_0P=J$, $\overline{P}=P^{-1}=P$). In this new presentation, the complex structure is given by $h'=PhP$, i.e.,

$$h':z\in\mathbb{C}\mapsto\begin{pmatrix}z & & \\ & z & \\ & & \overline{z}\end{pmatrix}\in U_{J_0}(\mathbb{R}).$$

In this form, the Lie algebra of $U_{J_0}(\mathbb{R})$ is given by

$$\mathfrak{g}=\left\{\begin{pmatrix}ia_0 & b & c\\ -\overline{b} & ie_0 & f\\ \overline{c} & \overline{f} & il_0\end{pmatrix},a_0,e_0,l_0\in\mathbb{R}\right\}.$$

Using the action of $h'(i)$ we can decompose $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ with

$$\mathfrak{p}=\left\{\begin{pmatrix}0 & 0 & c\\ & 0 & f\\ \overline{c} & \overline{f} & 0\end{pmatrix},c_0,a_0\in\mathbb{R}\right\}.$$

Extending scalars to $\mathbb{C}$, we can further decompose, $\mathfrak{p}_{\mathbb{C}}=\mathfrak{p}^+\oplus\mathfrak{p}^-$, where conjugacy by $h'(z)$ on $\mathfrak{p}^+$ and $\mathfrak{p}^-$ is given by $z/\overline{z}$ and $\overline{z}/z$ respectively. Explicitly, $\mathfrak{p}^-$ is generated by

$$X^-=N^+\otimes i-N^-\otimes 1\quad\text{and}\quad Y^-=M^+\otimes i-M^-\otimes 1,$$
$$N^-=\begin{pmatrix}0 & & i\\ & 0 & \\ -i & & 0\end{pmatrix},\quad N^+=\begin{pmatrix}0 & & 1\\ & 0 & \\ 1 & & 0\end{pmatrix},$$
$$M^-=\begin{pmatrix}0 & & 0\\ & 0 & i\\ 0 & -i & 0\end{pmatrix},\quad M^+=\begin{pmatrix}0 & & 0\\ & 0 & 1\\ 0 & 1 & 0\end{pmatrix},$$

and $\mathfrak{p}^+$ is generated

$$X^+=N^+\otimes i+N^-\otimes 1\quad\text{and}\quad Y^+=M^+\otimes i+M^-\otimes 1.$$

To calculate the action of $\mathfrak{p}^-$ on our representation, we use the following formula for a matrix X and $f\in\mathfrak{g}$:

$$X\cdot f=\left(\frac{d}{dt}\exp(tX)\bullet f\right)_{t=0}.$$

As Z^-_{a-2} is a space of holomorphic functions, we get the following exponentials for the matrices $M^\pm,N^\pm$:

$$\exp(tM^-)=\begin{pmatrix}1 & & 0\\ & \operatorname{ch}t & i\operatorname{sh}t\\ & -i\operatorname{sh}t & \operatorname{ch}t\end{pmatrix},\quad \exp(tM^+)=\begin{pmatrix}1 & & 0\\ & \operatorname{ch}t & \operatorname{sh}t\\ & \operatorname{sh}t & \operatorname{ch}t\end{pmatrix},$$

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$$\exp(tN^-) = \begin{pmatrix} \operatorname{ch} t & & i \operatorname{sh} t \\ & 1 & 0 \\ -i \operatorname{sh} t & & \operatorname{ch} t \end{pmatrix}, \quad \exp(tN^+) = \begin{pmatrix} \operatorname{ch} t & & \operatorname{sh} t \\ & 1 & 0 \\ \operatorname{sh} t & & \operatorname{ch} t \end{pmatrix},$$

and the actions of the matrices $M^{\pm}, N^{\pm}$ is given by

$$\begin{aligned} N^+ f &= -(a-2)\overline{z_1} f + (\overline{z_1}^2 - 1)\frac{df}{d\overline{z_1}} + \overline{z_1 z_2}\frac{df}{d\overline{z_2}}, \\ N^- f &= -i(a-2)\overline{z_1} f + i(\overline{z_1}^2 + 1)\frac{df}{d\overline{z_1}} + i\overline{z_1 z_2}\frac{df}{d\overline{z_2}}, \\ M^+ f &= -(a-2)\overline{z_2} f + \overline{z_1 z_2}\frac{df}{d\overline{z_1}} + (\overline{z_1}^2 - 1)\frac{df}{d\overline{z_2}}, \\ M^- f &= -i(a-2)\overline{z_2} f + i\overline{z_1 z_2}\frac{df}{d\overline{z_1}} + i(\overline{z_1}^2 + 1)\frac{df}{d\overline{z_2}}. \end{aligned}$$

We deduce that the action of $\mathfrak{p}^-$ is given by

$$Y^- f \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -2i\frac{df}{d\overline{z_2}},$$

and

$$X^- f \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -2i\frac{df}{d\overline{z_1}}$$

and the action of $\mathfrak{p}^+$ by

$$Y^+ f \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -2i(a-1)\overline{z_2} f + 2i\overline{z_1 z_2}\frac{df}{d\overline{z_1}} + 2i\overline{z_2}^2\frac{df}{d\overline{z_2}},$$

and

$$X^+ f \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -2i(a-1)\overline{z_1} f + 2i\overline{z_1 z_2}\frac{df}{d\overline{z_2}} + 2i\overline{z_1}^2\frac{df}{d\overline{z_1}}.$$

As Z^-_{a-2} is defined as a completion of the quotient of holomorphic polynomials in variables $\overline{z_1}, \overline{z_2}$ by the subspace of polynomials of degrees less or equal to $(a-2)$, $H^0(\mathfrak{p}^-, Z^-_{a-2}) = (Z^+_{a-2})^{\mathfrak{p}^-=0}$ is identified with homogeneous polynomials in $\overline{z_1}, \overline{z_2}$ of degree $a-1$.

As for a representation τ of K_∞ we have

$$H^q(\mathfrak{q}, K, V \otimes V_\tau) = (H^q(\mathfrak{p}^-, V) \otimes V_\tau)^K$$

(cf. [Har90b, 4.14]), we have that $H^0(\mathfrak{q}, K, Z^+_{a-1} \otimes V_\sigma^\vee) \neq 0$.

Remark D.3. Using a slightly more precise calculation for $U(2,1)$ instead of $SU(2,1)$, we could show for $U(2,1)$ that

$$H^0(\mathfrak{q}, K_\infty, \pi^n(\chi) \otimes V_{(0,1-a,a)}) \neq 0;$$

in particular, the Hecke eigenvalues of $\pi^n(\chi)$ appear in the global sections over X, the Picard modular variety, of the automorphic sheaf $\omega^{(0,1-a,a)}$.

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