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PROPERTIES OF THE FIXED POINT SET OF CONTRACTIVE MULTI-FUNCTIONS

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1. **Introduction.** A well known theorem by S. Banach states that a contractive function $f: X \to X$ on a complete metric space X has a fixed point, and that this fixed point is unique. This result has a partial extension to multi-functions: every contractive compact-valued multi-function on a complete metric space has a fixed point (see Definition 1 and Theorem 1 below). But simple examples show that this fixed point is no longer unique. We investigate some questions concerned with the properties of the fixed point set Φ of a contractive multi-function φ. Is, e.g., Φ connected if φ is connected-valued? Is Φ convex if φ is convex-valued? The answer is yes if X is the real line (§2), but examples in §3 and §4 show that in general the answer is no.

Let D be the Hausdorff metric generated by the metric of the space X.

DEFINITION 1. A multi-function $\varphi: X \to X$ is contractive if there exists a $k \in [0, 1)$ such that for every distinct pair $p, q \in X$

$$D(\varphi(p), \varphi(q)) \le kD(p, q).$$

p is a fixed point of φ if $p \in \varphi(p)$. The following theorem asserts the existence of a fixed point for contractive multi-functions.

THEOREM 1. *Every compact-valued contractive multi-function $\varphi: X \to X$ on a complete metric space X has a fixed point.*

The proof is a fairly easy modification of the corresponding proof in the single-valued case. We omit it as Theorem 1 will not be needed in the following, and as a very similar theorem was announced in Markin [1].

2. **Contractive Multi-functions on the Real Line.** As the only subsets on the real line R^1 which are connected or convex are the intervals, properties of the fixed point set of a contractive multi-function on R^1 with connected or convex images are easy to investigate.

THEOREM 2. *Let $\varphi: R^1 \to R^1$ be a contractive multi-function such that $\varphi(x)$ is compact and connected for all $x \in R^1$. Then the fixed point set of φ is compact and connected.*

Proof. As $\varphi(x)$ is compact and connected, it is either a compact interval or a single point, i.e.

$$\varphi(x) = [m, M], \quad -\infty < m \le M < \infty, \text{for all } x \in R^1.$$

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Then we have for any distinct pair $x_1, x_2 \in R^1$ with $\varphi(x_i)=[m_i, M_i]$ for $i=1, 2$,

$$D(\varphi(x_1), \varphi(x_2)) = \max(|M_2-M_1|, |m_2-m_1|),$$

so that for some $k \in [0, 1)$ both

$$|M_2-M_1| \leq k\,|x_2-x_1| \quad \text{and} \quad |m_2-m_1| \leq k\,|x_2-x_1|.$$

Therefore

$$a(x)=\min \varphi(x) \quad \text{and} \quad b(x) = \max \varphi(x)$$

are two contractive single-valued functions on R^1, and hence have each a unique fixed point a_0 resp. b_0. A look at the graph of φ in $R^1 \times R^1$ shows that clearly $a_0 \leq b_0$ and that $[a_0, b_0]$ is the fixed point set of φ.

NOTE. The word "connected" in Theorem 2 can obviously be replaced by "convex".

QUESTION. If the image of the contractive function $\varphi: R^1 \to R^1$ consists of exactly n points for all $x \in R^1$, does the fixed point set of φ consist of n points?

3. **The fixed point set of a connected-valued function need not be connected.** Define first a multi-function $\psi: R^2 \to R^2$ on the Euclidean plane as follows. For every $p=(x, y) \in R^2$ let $\psi(p)$ be the boundary of the square with sides parallel to the axes, side length one, and centre at $p'=(\frac{1}{4}x, \frac{1}{4}y)$. Then ψ is a contractive function as

$$\text{(1)} \qquad D(\psi(p_1), \psi(p_2)) = D(p_1', p_2') = \tfrac{1}{4}D(p_1, p_2),$$

and its fixed point set Ψ is the boundary of the square with centre at the origin, side length $\frac{4}{3}$ and sides parallel to the axes.

We now modify ψ to a contractive multi-function φ which is still connected-valued, but has a nonconnected fixed point set. Let X be the closed strip of R^2 defined by

$$X = \{(x, y) \in R^2 \mid x-1 \leq y \leq x+1\},$$

and define $\varphi: X \to X$ by $\varphi(p)=\psi(p) \cap X$ for all $p \in X$. Then the fixed point set Φ of φ is given by $\Phi=\Psi \cap X$ and hence has two components. Clearly $\varphi(p)$ is compact for all $p \in X$, and it is still connected as a square with side length one and sides parallel to the axes cannot intersect both $y=x-1$ and $y=x+1$, so that at most one corner is cut off $\psi(p)$ to obtain $\varphi(p)$. It remains to show that $\varphi: X \to X$ is contractive. We assert that, with p' defined as above,

$$\text{(2)} \qquad D(\varphi(p_1), \varphi(p_2)) \leq 2D(p_1', p_2')$$

and hence

$$D(\varphi(p_1), \varphi(p_2)) \leq kD(p_1, p_2) \quad \text{with} \quad k = \tfrac{1}{2}.$$

(In fact $k=\frac{\sqrt{2}}{4}$, but the proof is longer and the result not needed.)

If $\varphi(p_i)=\psi(p_i)$ for both $i=1$ and $i=2$ then (2) follows from (1). Therefore it is

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only necessary to consider the case where for at least one p_i, say for p_1, we have $\varphi(p_1) \neq \psi(p_1)$. Then (see Figure 1) $\varphi(p_1)$ consists of the boundary of a square of which one corner with vertex v_1 is cut off by a line l parallel to a diagonal of the square. Let v_2 be the vertex of $\psi(p_2)$ corresponding to v_1, and let v_3 be the point such that $v_3v_1 \parallel l$ and $v_3v_2 \perp l$. Then clearly

$$D(v_1, v_3) \leq D(v_1, v_2) \quad \text{and} \quad D(v_3, v_2) \leq D(v_1, v_2).$$

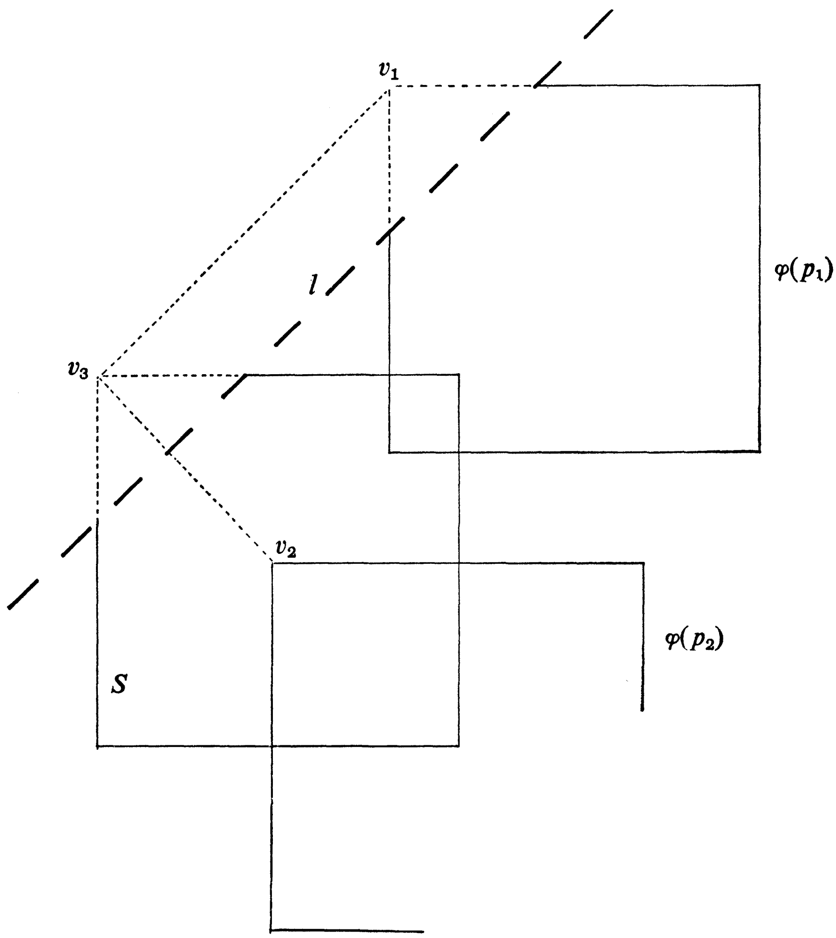

Figure 1.

Denote by S the part of the boundary of the square with vertex v_3 cut off by l. Then

$$D(\varphi(p_1), S) = D(v_1, v_3).$$

One can also check that, whether v_2 is cut off by l or not,

$$D(S, \varphi(p_2)) \leq D(v_3, v_2).$$

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Hence

$$\begin{aligned} D(\varphi(p_1), \varphi(p_2)) &\le D(\varphi(p_1), S) + D(S, \varphi(p_2)) \\ &\le D(v_1, v_3) + D(v_3, v_2) \\ &\le 2D(v_1, v_2) = 2D(p_1', p_2'), \end{aligned}$$

so that (2) holds and φ is contractive.

Therefore φ is an example of a contractive function with compact and connected images which has a nonconnected fixed point set.

4. **The fixed point set of a convex-valued function need not be convex.** We define a multi-function $\varphi: R^2 \to R^2$ on the Euclidean plane R^2 in the following way. For any $p=(x, y) \in R^2$, let $\varphi(p)$ be the closed disc with radius $\frac{1}{2}$ and centre $p'=(x', y')$, where

$$\text{(3)} \qquad \begin{aligned} x' &= \tfrac{2}{3}x, \\ y' &= \tfrac{3}{4}-\tfrac{1}{2}\,|x|. \end{aligned}$$

Then $\varphi(p)$ is clearly convex and compact for all $p \in R^2$. To see that φ is contractive note that for any $p_1, p_2 \in R^2$

$$D(\varphi(p_1), \varphi(p_2)) = D(p_1', p_2').$$

Hence, if $p_i=(x_i, y_i)$ are such that $x_i \ge 0$ for $i=1, 2$, it follows from (3) that

$$D^2(\varphi(p_1), \varphi(p_2)) = \tfrac{4}{9}(x_1-x_2)^2+\tfrac{1}{4}(x_1-x_2)^2,$$

therefore

$$\text{(4)} \qquad D(\varphi(p_1), \varphi(p_2)) = \tfrac{5}{6}\,|x_1-x_2| \le kD(p_1, p_2) \quad \text{with} \quad k = \tfrac{5}{6}.$$

If $x_i \le 0$ for $i=1, 2$, we obtain (4) similarly. If $x_1 > 0$ and $x_2 < 0$, let $p_3=(0, y_3)$ be the point such that $p_1p_3p_2$ are collinear. Then, using the cases previously discussed,

$$\begin{aligned} D(\varphi(p_1), \varphi(p_2)) &\le D(\varphi(p_1), \varphi(p_3))+D(\varphi(p_3), \varphi(p_2)) \\ &\le kD(p_1, p_3)+kD(p_3, p_2) \\ &= kD(p_1, p_2). \end{aligned}$$

So (4) holds generally, and φ is contractive.

But the fixed point set of φ is not convex. To see this, test the points $(\pm 1, 0)$ and $(0, 0)$. For $p=(1, 0)$ the distance of p from the centre $p'=(\frac{2}{3}, \frac{1}{4})$ of its image is $<\frac{1}{2}$. The same is true for $(-1, 0)$, so that $(\pm 1, 0)$ are two fixed points. But the distance of $p=(0, 0)$ from the centre $p'=(0, \frac{3}{4})$ of its image is $>\frac{1}{2}$, so that $(0, 0)$ is not a fixed point.

Hence φ is an example of a contractive function with compact and convex images which has a fixed point set which is not convex.

QUESTION: Is the fixed point set of φ connected if φ is a contractive multi-function with compact and convex images?

Added in Proof. A proof of Theorem 1 has since been published, see [2, Theorem 5].

References

1. J. T. Markin, *A fixed point theorem for set valued mappings*, Bull. Amer. Math. Soc. **74** (1968), 639–640.

2. S. B. Nadler, Jr., *Multi-valued contractive mappings*, Pacific J. Math. **30** (1969), 475–488.

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