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THE STRUCTURE OF GROUPS WHOSE SUBGROUPS ARE PERMUTABLE-BY-FINITE

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Abstract

A subgroup H of a group G is said to be *permutable* if $HX = XH$ for each subgroup X of G, and the group G is called *quasihamiltonian* if all its subgroups are permutable. We shall say that G is a QF*-group* if every subgroup H of G contains a subgroup K of finite index which is permutable in G. It is proved that every locally finite QF-group contains a quasihamiltonian subgroup of finite index. In the proof of this result we use a theorem by Buckley, Lennox, Neumann, Smith and Wiegold concerning the corresponding problem when permutable subgroups are replaced by normal subgroups: if G is a locally finite group such that H/H_G is finite for every subgroup H, then G contains an abelian subgroup of finite index.

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1. Introduction

A subgroup X of a group G is said to be *permutable* if $XH = HX$ for every subgroup H of G. This concept was introduced by Ore, and the behaviour of permutable subgroups was later investigated by several authors. A group is called *quasihamiltonian* if all its subgroups are permutable. It was proved by Stonehewer [12] that permutable subgroups of arbitrary groups are ascendant, so that quasihamiltonian groups are locally nilpotent. The structure of quasihamiltonian groups was described by Iwasawa ([6, 7]). In particular, quasihamiltonian primary groups are either abelian or contain an abelian subgroup of finite index and finite exponent. For a detailed account of results concerning permutable subgroups we refer to [11, Chapters 5 and 6].

Some classical theorems of Neumann [9] deal with groups in which every subgroup is close to being normal in a suitable sense. In particular, Neumann described groups in which every subgroup is normal in a subgroup of finite index and those in which all subgroups have finite index in their normal closures; corresponding results to these theorems for generalized permutable subgroups have been obtained in [2] and [5], respectively. Recently, the structure of groups in which all subgroups are normal-by-finite has been investigated; here a subgroup H of a group G is said to be *normal-by-finite* if the core H_G of H in G has finite index in H, and G is called a *CF-group* if all its subgroups are normal-by-finite. It has been proved in [1] that every locally finite CF-group is abelian-by-finite. The aim of this paper is to extend this latter theorem, replacing normality by permutability.

We shall say that a subgroup H of a group G is *permutable-by-finite* if it contains a permutable subgroup K of G such that the index $|H : K|$ is finite; the group G is called a *QF-group* if all its subgroups are permutable-by-finite. Our result here is the following.

THEOREM. *Let G be a locally finite group in which every subgroup is permutable-by-finite. Then G contains a quasihamiltonian subgroup of finite index.*

Consideration of any Tarski group shows that the local finiteness hypothesis cannot be omitted in this theorem.

Most of our notation is standard and can be found in [10]. For any subgroup H of a group G, we denote by H_* the subgroup of H generated by all permutable subgroups of G contained in H. Since the conjugates and the join of any collection of permutable subgroups are likewise permutable, the subgroup H_* is normal in H and permutable in G; it will be called the *permutable core* of H in G.

2. Some preliminary results

In this short section we collect some general results that will be useful in the proof of our main theorem.

We say that a group G is an *iterated semidirect product* of a sequence $(K_n)_{n\in\mathbb{N}}$ of its subgroups if the following conditions are satisfied:

(i) $G = \langle K_n \mid n \in \mathbb{N}\rangle$;
(ii) the subgroup K_{n+1} is normalized by $K_1, \dots, K_n$ for each positive integer n;
(iii) $\langle K_1, \dots, K_n\rangle \cap K_{n+1} = \{1\}$ for all n.

Our first lemma shows that iterated semidirect products of infinitely many non-abelian subgroups cannot occur in the structure of abelian-by-finite groups.

LEMMA 2.1. *Let the group G be an iterated semidirect product of infinitely many non-abelian subgroups $K_1, \dots, K_n, K_{n+1}, \dots$. Then G is not abelian-by-finite.*

PROOF. Assume for a contradiction that the lemma is false, and consider a counterexample G containing an abelian normal subgroup A with smallest possible index. Clearly, G has infinite derived subgroup, so that A cannot be contained in the centre of G by Schur's theorem (see for instance [10, Part 1, Theorem 4.12]). For each positive integer n, put $L_n = \langle K_1, \dots, K_n \rangle$, so that

$$G = \bigcup_{n \in \mathbb{N}} L_n \quad \text{and} \quad A = \bigcup_{n \in \mathbb{N}} (A \cap L_n).$$

Therefore there exists some $m \geq 1$ such that $G = AL_m$ and $A \cap L_m$ is a non-central subgroup of G. In particular, the centralizer $C_G(A \cap L_m)$ is a proper subgroup of G containing A. On the other hand, $A \cap L_m$ is a normal subgroup of G which normalizes and so centralizes every subgroup K_n with $n \geq m+1$. Thus $C_G(A \cap L_m)$ contains the subgroup

$$H = \langle K_n \mid n \geq m+1 \rangle,$$

and hence

$$|H : A \cap H| \leq |C_G(A \cap L_m) : A| < |G : A|.$$

As H is the iterated semidirect product of its non-abelian subgroups $K_{m+1}, K_{m+2}, \dots$, this contradicts the minimal choice of G. □

If $\mathfrak{X}$ is a class of groups, the *$\mathfrak{X}$-residual* of a group G is the intersection of all normal subgroups N of G such that G/N belongs to $\mathfrak{X}$, and G is called a *residually* $\mathfrak{X}$-group if it has trivial $\mathfrak{X}$-residual. Clearly, the $\mathfrak{X}$-residual of any group G is the smallest normal subgroup J of G such that the factor group G/J is residually $\mathfrak{X}$. The proof of the following result is straightforward.

LEMMA 2.2. *Let G be a residually (central-by-finite) group. Then the factor group $G/Z(G)$ is residually finite.*

Recall that if H is a subgroup of a group G, the *profinite closure* of H in G is the intersection of all subgroups of finite index of G containing H.

LEMMA 2.3. *Let G be a periodic residually finite group, and let H be a subgroup of G with finite exponent e. Then the profinite closure $\hat{H}$ of H in G has exponent e.*

PROOF. Let $(G_i)_{i \in I}$ be the collection of all normal subgroups of finite index of G. Then $\hat{H} = \bigcap_{i \in I} HG_i$, and hence for every $h \in \hat{H}$ and for each $i \in I$, there exist

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elements $h_i \in H$ and $g_i \in G_i$ such that $h = h_i g_i$. Since

$$h^e = h_i^e g_i^{h_i^{e-1}} \cdots g_i^{h_i} g_i = g_i^{h_i^{e-1}} \cdots g_i^{h_i} g_i$$

belongs to G_i for all $i \in I$, it follows that $h^e = 1$, and $\hat{H}$ has exponent e. □

3. Primary groups

The aim of this section is to prove that primary locally finite QF-groups are abelian-by-finite. The case of metabelian groups plays a central role in our considerations.

LEMMA 3.1. *Let G be a metabelian p-group with the property QF. If the derived subgroup G' of G has finite exponent, then all subgroups of G are subnormal.*

PROOF. If x is any element of G, the normal subgroup $\langle x, G' \rangle$ is nilpotent (see [10, Part 2, Lemma 6.34]), and so G is a Fitting group. Let X be any subgroup of G, and let X_* be the permutable core of X in G. Then $X = X_* E$, where E is a suitable finite subgroup of G. Since X_* is subnormal in G (see [4, Lemma 2.2]), it follows that X is subnormal in G. □

LEMMA 3.2. *Let G be a metabelian p-group with the property QF. If the derived subgroup G' of G has finite exponent, then G is abelian-by-finite.*

PROOF. Suppose first that G' has exponent p, and let E be a finite non-abelian subgroup of G. Assume that the group $EG'/Z(EG')$ is infinite, so that

$$G' = C_{G'}(E) \times K,$$

where K is an infinite subgroup. Then the permutable core K_* of K in G is also infinite. Clearly, K_* has finite index in EK_*, and so it contains a subgroup of finite index L such that $L^E = L$. As the group EL is nilpotent (see [10, Part 2, Lemma 6.34]), we obtain that $C_L(E) \neq \{1\}$. This contradiction shows that the normal subgroup EG' of G is central-by-finite, so that by Schur's theorem its derived subgroup is a finite non-trivial normal subgroup of G (see [10, Part 1, Theorem 4.12]). It follows that G is hypercentral. Moreover, all subgroups of G are subnormal by Lemma 3.1, and hence G is nilpotent by a result of Möhres [8]. Therefore G is abelian-by-finite (see [4, Lemma 4.2]).

Suppose now that G' has exponent p^n with $n > 1$. The above case yields that $G/(G')^p$ contains an abelian subgroup of finite index $H/(G')^p$. Thus $H' \leq (G')^p$ has exponent at most p^{n-1}, and by induction on the exponent of G' we have that the QF-group H is abelian-by-finite. Therefore G itself is abelian-by-finite. □

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If G is any p-group, we put $\Omega_0(G) = \{1\}$ and for each non-negative integer n, we define inductively the normal subgroup $\Omega_n(G)$ of G by choosing $\Omega_{n+1}(G)/\Omega_n(G)$ as the subgroup generated by all elements of order p of $G/\Omega_n(G)$.

LEMMA 3.3. *Let G be a metabelian p-group whose derived subgroup G' is residually finite and has infinite exponent. If G is a QF-group and the factor groups $\Omega_{n+1}(G)/\Omega_n(G)$ are abelian for all $n \geq 0$, then the normal closure E^G of every finite subgroup E of G is central-by-finite.*

PROOF. Clearly, the subgroup EG' is abelian-by-finite, and hence it is a CF-group (see [4, Lemma 4.1]). Thus there exists a subgroup A of finite index in G' such that $|E : C_E(A)| \leq 2$ and every subgroup of A is normal in EG' (see [1, Lemma 2.1 and Lemma 3.8]). Assume that $|E : C_E(A)| = 2$, so that there exists $g \in E$ such that $g^{-1}ag = a^{-1}$ and so $(ga)^2 = g^2$ for all $a \in A$. If the element g has order 2^n, it follows that A is contained in $\Omega_n(G)$, contrary to the hypothesis. Thus $C_E(A) = E$, and hence the subgroup EG' is central-by-finite. As E^G is contained in EG', the lemma is proved. □

Our next result allows us to apply the above lemma in the case of residually finite QF-groups.

LEMMA 3.4. *Let G be a locally finite p-group with the property QF. If G is residually finite, then there exists a normal subgroup H of finite index in G such that the factor groups $\Omega_{n+1}(H)/\Omega_n(H)$ are abelian for all $n \geq 0$.*

PROOF. Assume that the statement is false. We shall prove that there exist in G a sequence $(K_n)_{n\in\mathbb{N}}$ of finite non-abelian subgroups and a descending series

$$G = G_0 > G_1 > \cdots > G_n > \cdots$$

of normal subgroups of finite index such that for each $n \geq 1$ the following conditions hold:

(a) if s_n is the smallest non-negative integer such that the factor group

$$\Omega_{s_n+1}(G_{n-1})/\Omega_{s_n}(G_{n-1})$$

is non-abelian, then $K_n \leq \Omega_{s_n+1}(G_{n-1})$ and $K_n/(K_n \cap \Omega_{s_n}(G_{n-1}))$ is generated by non-permutable subgroups of order p;

(b) K_n is normal in the subgroup $L_n = \langle K_1, \ldots, K_n\rangle$;

(c) $G_n \cap L_n \leq \Omega_{s_n}(G_{n-1})$.

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In fact, let s_1 be the smallest non-negative integer such that the group $\Omega_{s_1+1}(G)/\Omega_{s_1}(G)$ is non-abelian, and let x and y be elements of G such that each of the cosets $x\Omega_{s_1}(G)$ and $y\Omega_{s_1}(G)$ generates a non-permutable subgroup of order p of $G/\Omega_{s_1}(G)$. Put $K_1 = \langle x, y\rangle$, and suppose that for some $n \geq 1$ the subgroups $K_1, \dots, K_n$ and $G_1, \dots, G_{n-1}$ have been chosen. Clearly, the subgroup $\Omega_{s_n}(G_{n-1})$ has exponent p^{s_n}, and so by Lemma 2.3 it coincides with its profinite closure in G_{n-1}. Thus the factor group $G_{n-1}/\Omega_{s_n}(G_{n-1})$ is residually finite, and hence there exists a normal subgroup G_n of finite index in G such that

$$\Omega_{s_n}(G_{n-1}) \leq G_n < G_{n-1}$$

and

$$G_n \cap L_n \leq \Omega_{s_n}(G_{n-1}).$$

Let s_{n+1} be the least non-negative integer such that $\Omega_{s_{n+1}+1}(G_n)/\Omega_{s_{n+1}}(G_n)$ is non-abelian. Then there exists a finite subgroup K of $\Omega_{s_{n+1}+1}(G_n)$ such that the group $K/K \cap \Omega_{s_{n+1}}(G_n)$ is generated by non-permutable subgroups of order p. Put $K_{n+1} = K^{L_n}$. Then $K_{n+1} \leq \Omega_{s_{n+1}+1}(G_n)$ and $K_{n+1}/(K_{n+1} \cap \Omega_{s_{n+1}}(G_n))$ is likewise generated by non-permutable subgroups of order p. This completes the construction of the subgroups K_{n+1} and G_n.

Clearly, $s_n \leq s_{n+1}$ for each n, so that $\Omega_{s_n}(G_{n-1}) = \Omega_{s_n}(G_n) \leq \Omega_{s_{n+1}}(G_n)$, and hence

$$N = \bigcup_{n\in\mathbb{N}} \Omega_{s_n}(G_{n-1})$$

is a normal subgroup of G. Moreover, since $\Omega_{s_{n+1}}(G_n) \leq \Omega_{s_{m+1}}(G_m) \leq G_m$ for all $m \geq n$, it follows that N is contained in any G_n. As

$$K_n \cap \Omega_{s_n}(G_{n-1}) \leq K_n \cap N \leq L_n \cap G_n \leq \Omega_{s_n}(G_{n-1}),$$

we have that $K_n \cap N = K_n \cap \Omega_{s_n}(G_{n-1})$ and hence $K_n/K_n \cap N$ is generated by non-permutable subgroups of order p. Finally, the relation

$$K_{n+1} \leq \Omega_{s_{n+1}+1}(G_n) \leq G_n$$

yields that

$$K_{n+1}N \cap L_n \leq G_n \cap L_n \leq \Omega_{s_n}(G_{n-1}) \leq N.$$

Consider in G the subgroup $L = \bigcup_{n\in\mathbb{N}} L_n$. Then the factor group LN/N is the iterated semidirect product of its non-abelian subgroups $K_1N/N, \dots, K_nN/N, \dots$, and in particular LN/N cannot be abelian-by-finite by Lemma 2.1. On the other hand, LN/N is generated by elements of order p, so that all its permutable subgroups are normal and LN/N is a CF-group. Thus LN/N is abelian-by-finite (see [1]) and this contradiction completes the proof. □

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LEMMA 3.5. *Let G be a metabelian p-group with the property QF. If the derived subgroup G' of G is residually finite, then the factor group $G/Z(G)$ is also residually finite.*

PROOF. Let H be any subgroup of finite index of G'. Then G'/H_G has finite exponent, so that G/H_G is abelian-by-finite by Lemma 3.2, and hence it also has the property CF (see [4, Lemma 4.1]). Thus H/H_G is finite and G/H_G is finite-by-abelian, so that G/H_G is even central-by-finite. As G' is residually finite, it follows that G is residually (central-by-finite), and hence $G/Z(G)$ is residually finite by Lemma 2.2. □

The condition QF is essential in the above lemma, as the consideration of the standard wreath product of a group of order p with a group of type p^∞ shows.

Recall that the *FC-centre* of a group G is the subgroup consisting of all elements of G with finitely many conjugates, and G is called an *FC - group* if it coincides with its FC-centre.

LEMMA 3.6. *Let G be a residually finite p-group with the property QF. If the normal closure of every finite subgroup of G has finite exponent, then there exists a subgroup of finite index in G whose derived subgroup has finite exponent.*

PROOF. Assume that the statement is false. We shall prove that there exist in G a sequence $(K_n)_{n\in\mathbb{N}}$ of finite non-abelian subgroups and a descending series

$$G = G_0 > G_1 > \cdots > G_n > \cdots$$

of normal subgroups of finite index such that for each n the following conditions hold:

(a) $K_n < G_{n-1}$;

(b) if $H_0 = \{1\}$ and H_n is the profinite closure of $K_n^G H_{n-1} \cap G_n$ in G, then the factor group $K_n/(H_{n-1} \cap K_n)$ is non-abelian;

(c) $L_n \cap G_n \leq H_{n-1}$, where $L_n = \langle K_1, \ldots, K_n \rangle$.

In fact, let K_1 be a finite non-abelian subgroup of G, and let G_1 be a normal subgroup of finite index of G such that $K_1 \cap G_1 = \{1\}$. Suppose that for some $n \geq 1$ the subgroups $K_1, \ldots, K_n$ and $G_1, \ldots, G_n$ have been chosen. As the normal closure K_n^G has finite exponent, it follows from Lemma 2.3 that H_n is a normal subgroup of G with finite exponent and that G_n/H_n is a residually finite non-abelian group. Thus there exist a finite non-abelian subgroup K_{n+1} of G_n and a normal subgroup of finite index G_{n+1} of G such that $\langle K_1, \ldots, K_n, K_{n+1} \rangle \cap G_{n+1} \leq H_n$, $K_{n+1}/H_n \cap K_{n+1}$ is non-abelian and $H_n \leq G_{n+1} < G_n$. The construction of the subgroups $K_1, \ldots, K_n, \ldots$ and $G_0, G_1, \ldots, G_n, \ldots$ is complete.

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Consider the subgroups

$$L = \bigcup_{n\in\mathbb{N}} L_n \quad \text{and} \quad H = \bigcup_{n\in\mathbb{N}} H_n.$$

Since $H_n \leq G_{n+1}$ for each n, we have that

$$H \leq \bigcap_{m\in\mathbb{N}} G_m.$$

In particular, $H \cap K_n \leq G_n \cap K_n \leq H_{n-1}$ and so the factor group $K_n/(H \cap K_n)$ is non-abelian for each n. It follows also that

$$K_{n+1}H \cap L_n \leq G_n \cap L_n \leq H_{n-1} \leq H.$$

Finally, if m and n are positive integers with $m < n$, then

$$[K_n, K_m] \leq [G_{n-1}, K_m^G] \leq [G_m, K_m^G] \leq G_m \cap K_m^G \leq H.$$

Therefore the factor group LH/H is the direct product of its finite non-abelian subgroups $K_1H/H, K_2H/H, \ldots, K_nH/H, \ldots$ On the other hand, LH/H is an FC-group with the property QF, and hence it cannot be decomposed in such a product (see [4, Lemma 4.5]). This contradiction completes the proof. □

LEMMA 3.7. *Let G be a residually finite metabelian p-group with the property QF. Then G is abelian-by-finite.*

PROOF. By Lemma 3.4, G contains a normal subgroup H of finite index such that the factor groups $\Omega_{n+1}(H)/\Omega_n(H)$ are abelian for all $n \geq 0$. Moreover, Lemma 3.2 allows us to suppose that the derived subgroup H' of H has infinite exponent. Thus the normal closure E^H of any finite subgroup E of H is central-by-finite by Lemma 3.3, and in particular E^H has finite exponent. Lemma 3.6 yields that there exists in H a subgroup K of finite index whose derived subgroup K' has finite exponent. As K is abelian-by-finite by Lemma 3.2, it follows that G is itself abelian-by-finite. □

Our next lemma proves that the theorem holds for soluble primary groups.

LEMMA 3.8. *Let G be a soluble p-group with the property QF. Then G is abelian-by-finite.*

PROOF. Let K be the smallest non-trivial term of the derived series of G. By induction on the derived length of G we may suppose that the factor group G/K is abelian-by-finite, so that G is metabelian-by-finite and hence without loss of generality it can be assumed that G is metabelian. Let J be the finite residual of the derived subgroup G' of G, and put $\bar{G} = G/J$. As $\bar{G}'$ is residually finite, by Lemma 3.5 the factor group $\bar{G}/Z(\bar{G})$ is also residually finite, so that $\bar{G}/Z(\bar{G})$ is abelian-by-finite by Lemma 3.7 and $\bar{G}$ is nilpotent-by-finite. If x is any element of G, the subgroup $\langle x\rangle G'$ has the property CF (see [4, Lemma 4.1]) and hence x normalizes all subgroups of J (see [1, Lemma 4.1]). Thus G acts on J as a group of power automorphisms, and in particular $G/C_G(J)$ is finite. As the subgroup $C_G(J)$ is nilpotent-by-finite, it follows that G itself is nilpotent-by-finite and so it is abelian-by-finite (see [4, Lemma 4.2]). □

LEMMA 3.9. *Let G be a p-group with the property QF and let F be the Fitting subgroup of G. Then the subgroup F'' is contained in the FC-centre of G. In particular, the Fitting subgroup of G is soluble.*

PROOF. Let E be any finite subgroup of F. The normal closure $N = E^G$ of E in G is nilpotent, and so it has finite exponent. It follows that N is abelian-by-finite and hence it is a CF-group (see [4, Lemma 4.3 and Lemma 4.1]). Thus N contains an abelian normal subgroup of finite index A such that all subgroups of A are normal in N (see [1, Lemma 2.1]). Let L be a finite subgroup such that $N = AL$. Since L induces on A a group of power automorphisms, we have that $L/C_L(A)$ is abelian and so $N' \leq AC_L(A)$. Therefore N'' is contained in $C_L(A)$, and in particular N'' is a finite normal subgroup of G containing E''. It follows that F'' is contained in the FC-centre of G. As primary FC-groups with the property QF are soluble (see [4, Lemma 4.5]), we have also that F is soluble. □

We can now prove the main result of this section.

THEOREM 3.10. *Let G be a locally finite p-group with the property QF. Then G is abelian-by-finite.*

PROOF. Let n be a non-negative integer such that $\Omega_n(G)$ is soluble. Since $\Omega_{n+1}(G)/\Omega_n(G)$ is generated by elements of order p, all its permutable subgroups are normal. In particular, $\Omega_{n+1}(G)/\Omega_n(G)$ is a CF-group, so that it is abelian-by-finite (see [1]) and hence $\Omega_{n+1}(G)$ is also soluble. It follows now from Lemma 3.8 that each $\Omega_n(G)$ is abelian-by-finite. If A is an abelian normal subgroup of finite index of $\Omega_n(G)$, then the normal closure A^G is generated by finitely many conjugates of A and so its centre $Z(A^G)$ has finite index in $\Omega_n(G)$. As $Z(A^G)$ is contained in the Fitting subgroup F of G, the factor group $\Omega_n(G)F/F$ is finite for every n, so that G/F is an FC-group, and hence it is soluble (see [4, Lemma 4.5]). Therefore G is soluble by Lemma 3.9, and hence it is abelian-by-finite by Lemma 3.8. □

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4. Proof of the theorem

The first result of this section deals with the case of periodic locally nilpotent groups.

LEMMA 4.1. *Let G be a periodic locally nilpotent QF-group. Then G contains a quasihamiltonian subgroup of finite index.*

PROOF. By Theorem 3.10, the Sylow p-subgroup G_p of G is abelian-by-finite for each prime p. Let π be the set of all prime numbers p such that G_p is not quasihamiltonian, and assume for a contradiction that π is infinite. If $p \in \pi$, let X_p be a non-permutable subgroup of G_p, and put

$$X = \mathrm{Dr}_{p\in\pi} X_p.$$

As G is a QF-group, the permutable core X_* of X contains X_q for some $q \in \pi$, and so $X_q = X_* \cap G_q$ is permutable in G_q. This contradiction shows that all but finitely many G_q are quasihamiltonian, and hence G is quasihamiltonian-by-finite. □

LEMMA 4.2. *Let G be a periodic soluble QF-group. If G is countable, then it contains a quasihamiltonian subgroup of finite index.*

PROOF. In the set $\pi = \pi(G)$ consider the relation $\sim$ defined by $p \sim q$ if and only if $p \neq q$ and each p-element of G commutes with all q-elements of G. Let ρ be any subset of π and write $\rho' = \pi \setminus \rho$. Since G is countable, there exist a ρ-subgroup H and a ρ'-subgroup K of G such that $G = HK$ (see for instance [3, Lemma 5.1.6]). The normal closure H_*^G of the permutable core H_* of H in G is likewise a ρ-subgroup, and from

$$H_*^G H = H(H_*^G H \cap K) = H$$

it follows that $H_*^G = H_*$. Hence H_* and similarly K_* are normal subgroups of G, so that $H_*K_* = H_* \times K_*$. As G is a QF-group, the subgroup H_*K_* has finite index in G. If p_1 and p_2 are prime numbers in ρ and ρ', respectively, which do not divide the order of G/H_*K_*, it follows that $p_1 \sim p_2$. If π is infinite, then any infinite subset σ of π contains distinct elements p and q such that $p \sim q$, because we can always choose an infinite subset ρ of σ such that the intersection $\rho' \cap \sigma$ is infinite. Application of Ramsey's theorem yields that in this case σ contains an infinite subset τ such that $p \sim q$ for all distinct primes $p, q \in \tau$.

Let θ be the set of all prime numbers q for which the Sylow q-subgroups of G are not normal, and assume that θ is infinite. Then θ contains an infinite subset

$\{q_n \mid n \in \mathbb{N}\}$ such that $q_i \sim q_j$ if $i \neq j$. For each positive integer n, let Q_n be a Sylow q_n-subgroup of G, so that

$$Q = \langle Q_n \mid n \in \mathbb{N} \rangle = \mathrm{Dr}_{n\in\mathbb{N}}\, Q_n.$$

As the permutable core Q_* of Q has finite index in Q, there is a positive integer m such that Q_m is contained in Q_*. Then Q_m is ascendant, and so even normal in G. This contradiction shows that the set θ is finite; in particular, if R is the Hirsch-Plotkin radical of G, the set $\pi(G/R)$ is finite. Moreover, by Theorem 3.10 each non-trivial Sylow subgroup of G contains a non-trivial permutable abelian subgroup of G, and hence all Sylow subgroups of G/R are finite. It follows that G/R is finite, so that by Lemma 4.1, G is quasihamiltonian-by-finite. □

A group G is called a *BQF-group* if there exists a positive integer k such that $|H/H_*| \leq k$ for all subgroups H of G. It has been proved in [4] that the statement of the theorem is true in the case of BQF-groups. As we will use this fact in the proof of our main theorem, we state it here as a lemma.

LEMMA 4.3. *Let G be a locally finite BQF-group. Then G contains a quasihamiltonian subgroup of finite index.*

PROOF OF THE THEOREM. Suppose first that G is soluble, and assume that it is not quasihamiltonian-by-finite, so that by Lemma 4.3 the property BQF does not hold for G. If X is any subgroup of G, and E is a finite subgroup such that $X = X_*E$, then

$$|X : X_*| = |E : E \cap X_*| \leq |E : E_*|.$$

It follows that there exists a sequence $(E_n)_{n\in\mathbb{N}}$ of finite subgroups of G such that

$$|E_1 : (E_1)_*| < |E_2 : (E_2)_*| < \cdots < |E_n : (E_n)_*| < \cdots.$$

For each positive integer n put $K_n = (E_n)_*$ and let $\{U_{n,1}/K_n, \ldots, U_{n,r_n}/K_n\}$ be the set of all non-trivial subgroups of E_n/K_n. Clearly, there exist elements $g_{n,1}, \ldots, g_{n,r_n}$ of G such that $\langle g_{n,i}\rangle U_{n,i} \neq U_{n,i}\langle g_{n,i}\rangle$ for all $i = 1, \ldots, r_n$. Consider the countable subgroup $H = \langle E_n, g_{n,1}, \ldots, g_{n,r_n} \mid n \in \mathbb{N}\rangle$. Clearly, each E_n has the same permutable core in H and in G, so that in particular H is not a BQF-group. Moreover, H contains a quasihamiltonian normal subgroup of finite index L by Lemma 4.2. Let π be the set of all prime numbers p such that the unique Sylow p-subgroup L_p of L contains a subgroup V_p which is not permutable in G, and put

$$V = \mathrm{Dr}_{p\in\pi}\, V_p.$$

If π is infinite, there exists $p \in \pi$ such that V_p is contained in the permutable core V_* of V; then V_p is a direct factor of V_* and hence is permutable in G (see [11,

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Lemma 6.2.16]). This contradiction shows that the set π is finite. For each prime $p \in \pi(L)$, $H/L_{p'}$ is a finite extension of a primary subgroup, and so it is abelian-by-finite by Theorem 3.10. It follows that $H/L_{p'}$ is a CF-group (see [4, Lemma 4.1]), and hence a BCF-group (see [1]), so that there exists a positive integer k_p such that $|WL_{p'} : (WL_{p'})_H| \leq k_p$ for every subgroup W of L_p. Moreover, the unique Sylow p-subgroup P of $(WL_{p'})_H$ is normal in H and hence

$$|W : W_H| \leq |W : P| = |WL_{p'} : (WL_{p'})_H| \leq k_p.$$

Put $\pi = \{p_1, \dots, p_t\}$, $k = k_{p_1} \dots k_{p_t}$, $|H : L| = m$, and let Y be any subgroup of H. Then

$$Y \cap L = (Y \cap L_{p_1}) \times \cdots \times (Y \cap L_{p_t}) \times \mathrm{Dr}_{p \notin \pi}(Y \cap L_p)$$

and the subgroup

$$Z = (Y \cap L_{p_1})_H \times \cdots \times (Y \cap L_{p_t})_H \times \mathrm{Dr}_{p \notin \pi}(Y \cap L_p)$$

is permutable in H. Since $|Y : Z| = |Y : Y \cap L| \cdot |Y \cap L : Z| \leq mk$, it follows that H is a BQF-group, and this contradiction proves the statement when G is soluble.

Suppose now that G is an arbitrary locally finite QF-group. It follows from the Hall-Kulatilaka theorem that every infinite homomorphic image of G contains an infinite abelian permutable subgroup, so that G is radical-by-finite, and without loss of generality it can be assumed that G is a radical group. Let R_1 be the Hirsch-Plotkin radical of G and R_2/R_1 the Hirsch-Plotkin radical of G/R_1. Then R_2 is soluble-by-finite by Lemma 4.1, and it follows from the soluble case that R_2 contains a quasihamiltonian normal subgroup of finite index. Therefore R_2/R_1 is finite and so G/R_1 is also finite and hence G is quasihamiltonian-by-finite. □

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