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Biflatness and Pseudo-Amenability of Segal Algebras

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Abstract. We investigate generalized amenability and biflatness properties of various (operator) Segal algebras in both the group algebra, $L^1(G)$, and the Fourier algebra, $A(G)$, of a locally compact group G.

Barry Johnson introduced the important concept of amenability for Banach algebras in [17], where he proved, among many other things, that a group algebra $L^1(G)$ is amenable precisely when the locally compact group G is amenable. For other Banach algebras, it is often useful to relax some of the conditions in the original definition of amenability, and a popular theme in abstract harmonic analysis has been to find, for various classes of Banach algebras associated to locally compact groups, a "correct notion" of amenability in the sense that it singles out the amenable groups. For example, the measure algebra $M(G)$ is amenable if and only if G is both amenable and discrete [5]; however, $M(G)$ is Connes-amenable (a definition of amenability for dual Banach algebras) exactly when G is amenable [28]. As another example, the Fourier algebra, $A(G)$, can fail to be amenable even for compact groups [19] but is operator amenable (a version of amenability that makes sense for Banach algebras with an operator space structure) if and only if G is amenable [27].

The purpose of this paper is to examine the amenability properties of Segal algebras in both $L^1(G)$ and $A(G)$. All of the aforementioned versions of amenability imply the existence of a bounded approximate identity (or identity in the case of Connes-amenability); however, a proper Segal algebra never has a bounded approximate identity [2]. Ghahramani, Loy, and Zhang have introduced several notions of "amenablility without boundedness", including approximate and pseudo-amenability, which do not a priori imply the existence of bounded approximate identities [13, 15]. It is thus natural to try to determine when a Segal algebra is approximately/pseudo-amenable. Indeed, this has already been considered in [13] and [15]. In particular, Ghahramani and Zhang showed that if $S^1(G)$ is a Segal algebra in $L^1(G)$ with an approximate identity which "approximately commutes with orbits" (this includes all [SIN]-groups) and G is amenable, then $S^1(G)$ is pseudo-amenable and that when G is compact, $S^1(G)$ is pseudo-contractible [15, Propos-

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tion 4.4 and Theorem 4.5] (also see [13, Corollary 7.1]). At present, there is no known example of an approximately amenable Banach algebra without a bounded approximate identity, so in our study of Segal algebras we will only consider pseudo-amenability and pseudo-contractibility. We note that the approximate and pseudo-amenability of $L^1(G)$, $M(G)$, and $A(G)$ are studied in [13–15].

An important property that is related to amenability is the homological notion of biflatness (see, for example, [4, Theorem 2.9.65]). In Section 2, we provide a natural generalization of biflatness, in the spirit of the definitions of approximate and pseudo-amenability: approximate biflatness. Our definition is inspired by A.Yu. Pirkovskii's recent characterization of biflatness [23]. We prove that a sufficient condition for A to be pseudo-amenable is that it is approximately biflat and has an approximate identity (Theorem 2.4). The section concludes with an examination of some hereditary properties of (approximately) biflat Banach algebras that are needed for our study of the approximate cohomology of Segal algebras.

In Section 3, we study Segal algebras, $S^1(G)$, in $L^1(G)$. We prove that G is amenable when $S^1(G)$ is pseudo-amenable (Theorem 3.1) and prove that for [SIN]-groups, $S^1(G)$ is either pseudo-amenable or approximately biflat if and only if G is amenable. For symmetric Segal algebras, we show that G is amenable exactly when $S^1(G)$ is a flat $L^1(G)$-bimodule, which happens exactly when $S^1(G)$ has a type of approximate diagonal in $L^1(G)\widehat{\otimes}S^1(G)$ (Theorem 3.3). This idea is then used in Theorem 3.4 to give an alternative approach to that of [12] for describing continuous derivations from $S^1(G)$ into $L^1(G)$-modules when G is amenable. We show in Theorem 3.5 that $S^1(G)$ is compact when $S^1(G)$ is pseudo-contractible (the converse to [15, Theorem 4.5]). Finally, in Theorem 3.6 we prove, for any group G and every continuous derivation $D\colon S^1(G) \to S^1(G)^*$, that $\pi^* \circ D$ is w*-approximately inner, where π is the product map from $S^1(G)\widehat{\otimes}S^1(G)$ into $S^1(G)$.

In Sections 4 and 5, we turn our attention to (operator) Segal algebras in $A(G)$. We first show in Theorem 4.2 that an arbitrary Segal algebra $SA(G)$ in $A(G)$ is pseudo-contractible if and only if G is discrete and $SA(G)$ has an approximate identity. We then focus on the Lebesgue–Fourier algebra $S^1A(G) = A(G) \cap L^1(G)$ that was introduced by Ghahramani and Lau in [11] and was recently studied by Forrest, Wood, and Spronk in [10]. As well, we will examine Feichtinger's Segal algebra $S_0(G)$, which was shown by the second author to have many remarkable properties [29]. When $S^1A(G)$ has an approximate identity and G contains an abelian open subgroup, Theorem 4.6 shows that $S^1A(G)$ is approximately biflat (and therefore pseudo-amenable). Supposing that G contains an open subgroup H that is weakly amenable and such that Δ_H, the diagonal subgroup of $H \times H$, has a bounded approximate indicator (this is true for example whenever G_e, the connected component of the identity, is amenable), then $S^1A(G)$ is operator approximately biflat (and therefore operator pseudo-amenable) whenever it has an approximate identity (Theorem 4.7). We conclude with Theorem 5.3 that shows that under these same hypotheses, the Feichtinger Segal algebra $S_0(G)$ is actually operator biflat. This, in particular, implies that it is operator pseudo-amenable.

1 Preliminaries

1.1 Banach Algebras of Harmonic Analysis

Let G be a locally compact group and let $M(G)$ be the Banach space of complex-valued, regular, Borel measures on G. The space $M(G)$ is a unital Banach algebra with the convolution multiplication, and $L^1(G)$, the group algebra on G, is a closed ideal in $M(G)$. We write δ_s for the point mass at $s \in G$; the element δ_e is the identity of $M(G)$, and $l^1(G)$ is the closed subalgebra of $M(G)$ generated by the point masses.

Let G be a locally compact group, let $P(G)$ be the set of all continuous positive definite functions on G, and let $B(G)$ be its linear span. The space $B(G)$ can be identified with the dual of the group C^*-algebra $C^*(G)$, this latter being the completion of $L^1(G)$ under its largest C^*-norm. With pointwise multiplication and the dual norm, $B(G)$ is a commutative, regular, semisimple, Banach algebra. The Fourier algebra $A(G)$ is the closure of $B(G) \cap C_c(G)$ in $B(G)$. It is shown in [7] that $A(G)$ is a commutative, regular, semisimple Banach algebra whose carrier space is G. Also, up to isomorphism, $A(G)$ is the unique predual of $VN(G)$, the von Neumann algebra generated by the left regular representation of G on $L^2(G)$.

Let H be a closed subgroup of G, and let $I(H) = \{v \in A(G) : v\big|_H = 0\}$. A net (u_γ) in $B(G)$ is called an *approximate indicator* for H if

(i) $\lim v(u_\gamma\big|_H) = v$ for all $v \in A(H)$, and
(ii) $\lim wu_\gamma = 0$ for all $w \in I(H)$.

Approximate indicators were introduced in [1].

1.2 Operator Spaces

Our standard reference for operator spaces is [6]. We summarize some basic definitions below.

Let V be a Banach space. An *operator space structure* on V is a family of norms $\{\|\cdot\|_n : M_n(V) \to \mathbb{R}^{\geq 0}\}_{n\in\mathbb{N}}$ that satisfy Ruan's axioms, where each $M_n(V)$ is the space of $n \times n$ matrices with entries in V. The natural morphisms between operator spaces are the *completely bounded maps, i.e.,* those linear maps $T: V \to W$ which satisfy $\|T\|_{cb} = \sup_{n\in\mathbb{N}} \|T_n\| < \infty$, where $T_n: M_n(V) \to M_n(W)$ is given by $T_n[v_{ij}] = [Tv_{ij}]$. We say that T is *completely contractive* if $\|T\|_{cb} \leq 1$. Operator spaces admit an analogue of the projective tensor product $\widehat{\otimes}$, which we call the operator projective tensor product $\widehat{\otimes}_{op}$.

If A is a Banach algebra which is also an operator space, and V is a left A-module and an operator space, we say that V is a *completely contractive left A-module* if the product map $\pi_0: A \otimes V \to V$ extends to a complete contraction $\pi: A\widehat{\otimes}_{op}V \to V$. Completely contractive right and bi-modules are defined similarly. We say that A is a completely contractive Banach algebra if it is a completely contractive bimodule over itself. Natural examples include $L^1(G)$, which inherits the maximal operator space structure as the predual of a commutative von Neumann algebra; and $A(G)$, which inherits its operator space structure as the predual of $VN(G)$.

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1.3 Amenability Properties

Let A be a (completely contractive) Banach algebra.

Following Johnson [18], we say that A is (*operator*) *amenable* if A admits a *bounded approximate diagonal*, *i.e.,* a bounded net (m_α) in $A\widehat{\otimes}A$ (resp. in $A\widehat{\otimes}_{op}A$) such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0, \qquad \pi(m_\alpha)a \to a \tag{1.1}$$

for all $a \in A$, where $a \cdot (b \otimes c) = (ab) \otimes c$, $(b \otimes c) \cdot a = b \otimes (ca)$, and π is the product map. (Operator) amenability of A is equivalent to having every (completely) bounded derivation from A into a(n operator) dual bimodule be inner; see [18]. A natural relaxation of amenability is to allow A to admit a diagonal net, as in (1.1) above, but not insist that it is bounded. In doing so we obtain (*operator*) *pseudo-amenability*, as defined in [13]. If A admits a net in $A\widehat{\otimes}A$ (resp. in $A\widehat{\otimes}_{op}A$) which satisfies (1.1) and the additional property that $a \cdot m_\alpha = m_\alpha \cdot a$, then A is said to be (*operator*) *pseudo-contractible*, as defined in [15].

We say that A is (*operator*) *biflat* if there is a (completely bounded) bounded A-bimodule map $\theta\colon (A\widehat{\otimes}A)^* \to A^*$ (resp. $(A\widehat{\otimes}_{op}A)^* \to A^*$) such that $\theta \circ \pi^* = \mathrm{id}_{A^*}$. A. Ya. Helemskii proved that A is amenable if and only if A is biflat and admits a bounded approximate identity; see [16] or [3]. The analogous characterization of operator amenability follows similarly. A (completely contractive) left A-module is said to be (*operator*) *projective* if there is a (completely bounded) bounded left A-module map $\xi\colon V \to A\widehat{\otimes}V$ (resp. $V \to A\widehat{\otimes}_{op}V$) such that $\pi \circ \xi = \mathrm{id}_V$. A similar definition holds for right modules. A is (*operator*) *biprojective* if there is a (completely bounded) bounded A-bimodule map $\xi\colon A \to A\widehat{\otimes}A$ (resp. $A \to A\widehat{\otimes}_{op}A$) such that $\pi \circ \xi = \mathrm{id}_A$.

1.4 Segal Algebras

Segal algebras were first defined by H. Reiter for group algebras; see [26], for example. The definition of operator Segal algebras appeared in [10]. However, our abstract definition deviates from the one given in [10] in the sense that we demand that Segal algebras be essential modules.

Let A be a (completely contractive) Banach algebra. An (*operator*) *Segal algebra* is a subspace B of A such that

(i) B is dense in A,
(ii) B is a left ideal in A,
(iii) B admits a norm (operator space structure) $\|\cdot\|_B$ under which it is complete and a (completely) contractive A-module, and
(iv) B is an essential A-module: $A \cdot B$ is $\|\cdot\|_B$-dense in B.

We further say that B is *symmetric* if it is also a (completely) contractive essential right A-module.

In the case that $A = L^1(G)$ we will write $S^1(G)$ instead of B and further require that

(v) $S^1(G)$ is closed under left translations: $L_x f \in S^1(G)$ for insist that all x in G and f in $S^1(G)$, where $L_x f(y) = f(x^{-1}y)$ for y in G. By well-known techniques, condition (iii) on $B = S^1(G)$ is equivalent to

(iii′) the map $(x, f) \mapsto L_x f : G \times S^1(G) \to S^1(G)$ is continuous with $\|L_x f\|_{S^1} = \|f\|_{S^1}$ for all x in G and f in $S^1(G)$.

Moreover, symmetry for $S^1(G)$ is equivalent to having $S^1(G)$ be closed under right actions, $R_x f \in S^1(G)$ for x in G and f in $S^1(G)$, where $R_x f(y) = f(yx^{-1})\Delta(x^{-1})$, with the actions being continuous and isometric.

We will discuss two specific types of operator Segal algebras in the Fourier algebra $A(G)$. One is the Lebesgue-Fourier algebra, $S^1A(G)$, whose study was initiated in [11] and which was shown to be an operator Segal algebra in [10]. The second is Feichtinger's algebra $S_0(G)$, whose study in the non-commutaive case was taken up in [29]. This study included an exposition of the operator space structure. Though slightly different terminology was used in that article, it was proved there that $S_0(G)$ is an operator Segal algebra in $A(G)$, in the sense defined above.

2 Approximate biflatness and pseudo-amenability

Throughout this section, A is a Banach algebra. Recall that if E, F are Banach spaces, then the weak* operator topology (W*OT) on $\mathcal{B}(E, F^*)$ is the locally convex topology determined by the seminorms $\{p_{e,f} : e \in E,\ f \in F\}$, where $p_{e,f}(T) = |\langle f, Te\rangle|$. On bounded sets, the W*OT is exactly the w^*-topology of $\mathcal{B}(E, F^*)$ when identified with $(E \widehat{\otimes} F)^*$, so closed balls of $\mathcal{B}(E, F^*)$ are W*OT compact. When E and F are operator spaces, $\mathcal{C}B(E, F^*)$ is identified with $(E \widehat{\otimes}_{op} F)^*$ [6, Corollary 7.1.5]. On $\|\cdot\|_{cb}$-bounded subsets of $\mathcal{C}B(E, F^*)$, the W*OT agrees with the weak* topology.

Suppose that X and Y are Banach A-bimodules. Following A.Yu. Pirkovskii [23], a net $(\theta_\delta)_\delta$ of bounded linear maps from X into Y, satisfying

$$\|\theta_\delta(a \cdot x) - a \cdot \theta_\delta(x)\| \to 0 \quad \text{and} \quad \|\theta_\delta(x \cdot a) - \theta_\delta(x) \cdot a\| \to 0 \tag{2.1}$$

for all a in A, will be called an *approximate A-bimodule morphism* from X to Y. If Y is a dual Banach space, and instead of norm convergence we have w^*-convergence in (2.1), we call $(\theta_\delta)_\delta$ a *w^*-approximate A-bimodule morphism*.

The following proposition may be compared with [23, Corollary 3.2].

Proposition 2.1 *The following statements are equivalent:*

(i) *A is biflat;*

(ii) *there is a net $\theta_\delta : (A \widehat{\otimes} A)^* \to A^*$ ($\delta \in \Delta$) of A-bimodule morphisms such that $(\theta_\delta)_\delta$ is uniformly bounded in $\mathcal{B}((A \widehat{\otimes} A)^*, A^*)$ and $W^*OT\text{-}\lim_\delta \theta_\delta \circ \pi^* = \mathrm{id}_{A^*}$;*

(iii) *there is a w^*-approximate A-bimodule morphism $\theta_\delta : (A \widehat{\otimes} A)^* \to A^*$ ($\delta \in \Delta$) such that $(\theta_\delta)_\delta$ is uniformly bounded in $\mathcal{B}((A \widehat{\otimes} A)^*, A^*)$ and $W^*OT\text{-}\lim_\delta \theta_\delta \circ \pi^* = \mathrm{id}_{A^*}$.*

Proof The implications (i) ⇒ (ii) and (ii) ⇒ (iii) are trivial. Let $(\theta_\delta)_\delta$ be a w^*-approximate morphism satisfying the properties of statement (iii). As bounded subsets of $\mathcal{B}((A \widehat{\otimes} A)^*, A^*)$ are relatively W^*OT compact, $(\theta_\delta)_\delta$ has a W^*OT limit point, θ; we may assume that $W^*OT\text{-}\lim_\delta \theta_\delta = \theta$. Routine calculations show that θ is an A-bimodule map such that $\theta \circ \pi^* = \mathrm{id}_{A^*}$. ■

Remark 2.2 When A is a quantized Banach algebra, one can similarly prove an operator space version of Proposition 2.1:

A is operator biflat if and only if there is a net $\theta_\delta : (A\widehat{\otimes}_{op}A)^* \to A^*$ $(\delta \in \Delta)$ of completely bounded A-bimodule morphisms such that $\sup_\delta \|\theta_\delta\|_{cb} < \infty$ and $W^*OT\text{-}\lim_\delta \theta_\delta \circ \pi^* = \mathrm{id}_{A^*}$.

By dropping the condition of uniform boundedness from statement (ii) of Proposition 2.1, we obtain our definition of (operator) approximate biflatness. Remark 4.9 gives examples of approximately biflat Banach algebras which are not biflat.

Definition 2.3 We call a (quantized) Banach algebra, A, (*operator*) *approximately biflat* if there is a net $\theta_\delta\colon (A\widehat{\otimes}A)^* \to A^*$ (respectively, $\theta_\delta\colon (A\widehat{\otimes}_{op}A)^* \to A^*$) $(\delta \in \Delta)$ of (completely) bounded A-bimodule morphisms such that $W^*OT\text{-}\lim_\delta \theta_\delta \circ \pi^* = \mathrm{id}_{A^*}$.

Note that statement (iii) in the following theorem agrees with statement (iii) of Proposition 2.1, except that we have dropped the condition of uniform boundedness. Statement (ii) may be seen as an approximate biprojectivity condition.

Theorem 2.4 *Consider the following conditions for a Banach algebra A:*

(i) *A is pseudo-amenable;*
(ii) *there is an approximate A-bimodule morphism (β_δ) from A into $A\widehat{\otimes}A$ such that*

$$\|\pi \circ \beta_\delta(a) - a\| \to 0 \qquad (a \in A);$$

(iii) *there is a w^*-approximate A-bimodule morphism $\theta_\delta\colon (A\widehat{\otimes}A)^* \to A^*$ $(\delta \in \Delta)$ such that $W^*OT\text{-}\lim_\delta \theta_\delta \circ \pi^* = \mathrm{id}_{A^*}$;*
(iv) *A is approximately biflat.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *and if A has a central approximate identity, then* (iii) $\Rightarrow$ (i). *If A has an approximate identity, then* (iv) $\Rightarrow$ (i).

Proof Assuming that condition (i) holds, let (m_δ) be an approximate diagonal for A. Then it is easy to check that

$$\beta_\delta\colon A \to A\widehat{\otimes}A\colon a \mapsto a \cdot m_\delta$$

satisfies the properties of condition (ii). The dual maps $\theta_\delta = \beta_\delta^*$ satisfy the conditions of statement (iii).

Suppose that $\theta_\delta\colon (A\widehat{\otimes}A)^* \to A^*$ $(\delta \in \Delta)$ satisfies the conditions of statement (iii) and let $(e_\lambda)_{\lambda\in\Lambda}$ be a central approximate identity for A. Then for any $a \in A$ and $\psi \in (A\widehat{\otimes}A)^*$

$$\begin{aligned}
\lim_\lambda \lim_\delta \langle \psi, a \cdot \theta_\delta^*(e_\lambda) - \theta_\delta^*(e_\lambda) \cdot a\rangle &= \lim_\lambda \lim_\delta \langle e_\lambda, \theta_\delta(\psi \cdot a) - \theta_\delta(a \cdot \psi)\rangle \\
&= \lim_\lambda \lim_\delta \langle e_\lambda, \theta_\delta(\psi \cdot a) - \theta_\delta(\psi) \cdot a + \theta_\delta(\psi) \cdot a - \theta_\delta(a \cdot \psi)\rangle \\
(*) \qquad &= \lim_\lambda \lim_\delta \langle e_\lambda, \theta_\delta(\psi \cdot a) - \theta_\delta(\psi) \cdot a\rangle + \langle e_\lambda, a \cdot \theta_\delta(\psi) - \theta_\delta(a \cdot \psi)\rangle \\
&= \lim_\lambda (0 + 0) = 0,
\end{aligned}$$

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where we have used the centrality of (e_λ) at line $(*)$. Also, for $a \in A$ and $\phi \in A^*$,

$$\begin{aligned}\lim_\lambda \lim_\delta \langle \phi, \pi^{**}(\theta_\delta^*(e_\lambda)) \cdot a\rangle &= \lim_\lambda \lim_\delta \langle e_\lambda, \theta_\delta(\pi^*(a\cdot\phi))\rangle \\ &= \lim_\lambda \langle e_\lambda, a\cdot\phi\rangle = \lim_\lambda \langle e_\lambda a, \phi\rangle \\ &= \langle a, \phi\rangle.\end{aligned}$$

Let $E = \Lambda \times \Delta^\Lambda$ be directed by the product ordering, and for each $\beta = (\lambda, (\delta_{\lambda'})) \in E$, let $m_\beta = \theta_{\delta_\lambda}(e_\lambda) \in (A\widehat{\otimes}A)^{**}$. Using the iterated limit theorem [20, p. 69], the above calculations give for each a in A

$$a\cdot m_\beta - m_\beta \cdot a \to 0,\ w^* \text{ in } (A\widehat{\otimes}A)^{**} \quad \text{and} \quad \pi^{**}(m_\beta)a \to a,\ w^* \text{ in } A^{**}. \tag{2.2}$$

As in the proof of [15, Proposition 2.3], we can use Goldstine's theorem to obtain (m_β) in $A\widehat{\otimes}A$, and we can replace weak* convergence in equation (2.2) by weak convergence. This implies, via Mazur's theorem, that A is pseudo-amenable (again see [15, Proposition 2.3]).

The proof that A is pseudo-amenable when A is approximately biflat and has an approximate identity (e_λ) is the same as that given above except that we reverse the order in which we calculate the iterated limits and use the fact that each θ_δ is now an A-bimodule map:

$$\lim_\delta \lim_\lambda a\cdot\theta_\delta^*(e_\lambda) - \theta_\delta^*(e_\lambda)\cdot a = \lim_\delta \lim_\lambda \theta_\delta^*(a\cdot e_\lambda - e_\lambda \cdot a) = \lim_\delta 0 = 0$$

and

$$\begin{aligned}\lim_\delta \lim_\lambda \langle \phi, \pi^{**}(\theta_\delta^*(e_\lambda)) \cdot a\rangle &= \lim_\delta \lim_\lambda \langle e_\lambda a, \theta_\delta(\pi^*(\phi))\rangle \\ &= \lim_\delta \langle a, \theta_\delta(\pi^*(\phi))\rangle = \langle a, \phi\rangle.\end{aligned}$$

This completes the proof. ■

One can similarly prove the analogous relationship between operator pseudo-amenability and operator approximate biflatness. Our motivation in writing this paper has been to obtain information about the approximate (co)homology of Segal algebras, so we will not attempt to exhaustively determine the relationship between approximate biflatness and other forms of amenability. Instead, we have chosen to only examine approximate biflatness versus pseudo-amenability (Theorem 2.4) and refer the reader to [15] for a detailed study of the relationship between pseudo-amenability and several other amenability properties. We will, however, conclude this section with an examination of some hereditary properties of (approximately) biflat Banach algebras that are needed in the sequel.

Proposition 2.5 *Let B be an (operator) Segal algebra in A, and suppose that B contains a net $(e_\lambda)_{\lambda\in\Lambda}$ in its centre such that $(e_\lambda^2)_{\lambda\in\Lambda}$ is an approximate identity for B. If A is (operator) approximately biflat, then so is B.*

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Proof We will prove the operator space version of the proposition; the other case is similar. Let T_λ be the completely bounded map specified by

$$T_\lambda \colon A\widehat{\otimes}_{op}A \to B\widehat{\otimes}_{op}B \colon a \otimes b \mapsto ae_\lambda \otimes be_\lambda .$$

As e_λ is central in B, T_λ is a B-bimodule map. Let $\theta_\delta \colon (A\widehat{\otimes}A)^* \to A^*$ $(\delta \in \Delta)$ be a net of completely bounded A-bimodule maps such that $W^*OT\text{-}\lim_\delta \theta_\delta \circ \pi_A^* = \mathrm{id}_{A^*}$, and consider the completely bounded B-bimodule map, $p \colon A^* \to B^* \colon \phi \mapsto \phi\big|_B$. Let $E = \Lambda \times \Delta^\Lambda$ be directed by the product ordering, and for each $\beta = (\lambda, (\delta_{\lambda'})_{\lambda'}) \in E$, define $\theta_\beta \colon (B\widehat{\otimes}_{op}B)^* \to B^*$ so that the following diagram commutes.

$$\begin{tikzcd}
(A\widehat{\otimes}_{op}A)^* \arrow[r, "\theta_{\delta_\lambda}"] & A^* \arrow[d, "p"] \\
(B\widehat{\otimes}_{op}B)^* \arrow[u, "T_\lambda^*"] \arrow[r, "\theta_\beta"] & B^*
\end{tikzcd}$$

That is, $\theta_\beta = p \circ \theta_{\delta_\lambda} \circ T_\lambda^*$, a completely bounded B-bimodule map. Note that because e_λ lies in the centre of B,

$$T_\lambda^* \circ \pi_B^*(\phi) = \pi_A^* \circ R_\lambda^*(\phi) \qquad (\lambda \in \Lambda,\ \phi \in B^*),$$

where $R_\lambda \colon A \to B \colon a \mapsto ae_\lambda^2$. Let $\phi \in B^*$, $b \in B$. By the iterated limit theorem we have

$$\begin{aligned}
\lim_\beta \langle b, \theta_\beta \circ \pi_B^*(\phi)\rangle &= \lim_\lambda \lim_\delta \langle b, (p \circ \theta_\delta \circ T_\lambda^* \circ \pi_B^*)(\phi)\rangle \\
&= \lim_\lambda \lim_\delta \langle b, (\theta_\delta \circ \pi_A^* \circ R_\lambda^*)(\phi)\rangle \\
&= \lim_\lambda \langle b, R_\lambda^*(\phi)\rangle \\
&= \lim_\lambda \langle be_\lambda^2, \phi\rangle \\
&= \langle b, \phi\rangle .
\end{aligned}$$

Hence, $W^*OT\text{-}\lim_\beta \theta_\beta \circ \pi_{B^*} = \mathrm{id}_{B^*}$. ∎

Note that if $(e_\lambda)_\lambda$ is an approximate identity that is bounded in the multiplier norm on B, then $(e_\lambda^2)_\lambda$ is also an approximate identity for B.

Definition 2.6 The (*operator*) *biflatness constant* of an (operator) biflat (quantized) Banach algebra A is the number $BF_A = \inf_\gamma \|\theta\|$ (respectively, $BF_A^{op} = \inf_\gamma \|\theta\|_{cb}$), where the infimum is taken over all (completely) bounded A-bimodule maps

$$\theta \colon (A\widehat{\otimes}A)^* \to A^* \quad (\text{resp. } \theta \colon (A\widehat{\otimes}_{op}A)^* \to A^*,)$$

such that $\theta \circ \pi^* = \mathrm{id}_{A^*}$.

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Proposition 2.7 *Let A be a (quantized) Banach algebra containing a directed family of closed ideals $\{A_\gamma : \gamma \in \Gamma\}$ such that for each $\gamma \in \Gamma$ there is a (completely) bounded homomorphic projection P_γ of A onto A_γ. Suppose that either:*

(i) *A has a central approximate identity $(e_\lambda)_\lambda$ in $\cup_\gamma A_\gamma$; or*

(ii) *for each $a \in A$, $\|P_\gamma a - a\| \to 0$.*

(a) *If each A_γ is (operator) approximately biflat, then so is A.*

(b) *If each A_γ is (operator) biflat with $\sup_\gamma BF_{A_\gamma} < \infty$ (resp., $\sup_\gamma BF^{op}_{A_\gamma} < \infty$), and* (i) *holds with $(e_\lambda)_\lambda$ bounded in the (completely bounded) multiplier norm of A, or* (ii) *holds with $\sup_\gamma \|P_\gamma\| < \infty$ (resp., $\sup_\gamma \|P_\gamma\|_{cb} < \infty$), then A is (operator) biflat.*

Proof We first prove (a). Given $\alpha = (F, \Phi, \epsilon)$, where $F \subset A$, $\Phi \subset A^*$ are finite, and $\epsilon > 0$, we will find an A-bimodule map $\theta_\alpha : (A\widehat{\otimes}A)^* \to A^*$ such that

$$|\langle a, (\theta_\alpha \circ \pi_A^*)(\phi) - \phi\rangle| < \epsilon \qquad (a \in F,\ \phi \in \Phi). \tag{2.3}$$

Assuming first that condition (i) holds, take $e_{\lambda_0} = e_0$ such that

$$\|ae_0 - a\| < \epsilon/2M \qquad (a \in F) \tag{2.4}$$

where $M = \sup\{\|\phi\| : \phi \in \Phi\}$. Choose $\gamma_0 \in \Gamma$ such that $e_0 \in A_{\gamma_0}$. Consider the maps

$$\iota_0 : A_{\gamma_0}\widehat{\otimes}A_{\gamma_0} \to A\widehat{\otimes}A : a \otimes b \mapsto a \otimes b \quad \text{and} \quad T_0 : A \to A_{\gamma_0} : a \mapsto ae_0$$

and let $\pi_0 : A_{\gamma_0}\widehat{\otimes}A_{\gamma_0} \to A_{\gamma_0}$ be the multiplication map. As A_{γ_0} is approximately biflat, there is an A_{γ_0}-bimodule map $\theta_0 : (A_{\gamma_0}\widehat{\otimes}A_{\gamma_0})^* \to A_{\gamma_0}^*$ such that

$$|\langle T_0 a, (\theta_0 \circ \pi_0^*)(\phi|_{A_{\gamma_0}}) - \phi|_{A_{\gamma_0}}\rangle| < \epsilon/2 \qquad (a \in F,\ \phi \in \Phi). \tag{2.5}$$

Define θ_α so that the following diagram commutes:

$$\begin{tikzcd}
(A_{\gamma_0}\widehat{\otimes}A_{\gamma_0})^* \arrow[r, "\theta_0"] & A_{\gamma_0}^* \arrow[d, "T_0^*"] \\
(A\widehat{\otimes}A)^* \arrow[u, "\iota_0^*"] \arrow[r, "\theta_\alpha"] & A^*
\end{tikzcd}$$

That is, let $\theta_\alpha = T_0^* \circ \theta_0 \circ \iota_0^*$. For $a \in F$ and $\phi \in \Phi$, equations (2.4) and (2.5) give

$$\begin{aligned}
|\langle a, (\theta_\alpha \circ \pi_A^*)(\phi) - \phi\rangle| &\le |\langle T_0 a, (\theta_0(\iota_0^*(\pi_A^*(\phi))) - \phi\rangle| + |\langle T_0 a - a, \phi\rangle| \\
&\le |\langle T_0 a, (\theta_0 \circ \pi_0^*)(\phi|_{A_{\gamma_0}}) - \phi|_{A_{\gamma_0}}\rangle| + \|ae_0 - a\|\|\phi\| \\
&< \epsilon.
\end{aligned}$$

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If condition (ii) holds, we instead choose γ_0 such that $\|P_{\gamma_0}a-a\| < \epsilon/2M$ ($a \in F$). By replacing T_0 in the above paragraph by P_{γ_0}, we again obtain equation (2.3).

Because we only know that θ_0 is an A_{γ_0}-bimodule map, the argument showing that θ_α is an A-bimodule map requires some care. Note that

$$\iota_0^*(a \cdot \psi) = P_{\gamma_0}(a) \cdot \iota_0^*(\psi) \qquad (a \in A,\ \psi \in (A\widehat{\otimes}A)^*),$$

where on the left and right we respectively have A-module and A_{γ_0}-module actions. Let $a, b \in A$, $\psi \in (A\widehat{\otimes}A)^*$ and assume first that $\theta_\alpha = T_0^* \circ \theta_0 \circ \iota_0^*$. Then

$$\begin{aligned}
\langle b, \theta_\alpha(a\cdot\psi)\rangle &= \langle T_0 b, \theta_0(\iota_0^*(a\cdot\psi))\rangle \\
&= \langle T_0 b, \theta_0(P_{\gamma_0}(a)\cdot\iota_0^*(\psi))\rangle \\
&= \langle T_0 b, P_{\gamma_0}(a)\cdot\theta_0(\iota_0^*(\psi))\rangle \\
&= \langle T_0(b)P_{\gamma_0}(a), \theta_0(\iota_0^*(\psi))\rangle \\
&= \langle T_0(ba), \theta_0(\iota_0^*(\psi))\rangle \\
&= \langle ba, T_0^*(\theta_0(\iota_0^*(\psi)))\rangle \\
&= \langle b, a\cdot\theta_\alpha(\psi)\rangle,
\end{aligned}$$

where we have used the fact that $T_0 b P_{\gamma_0} a = P_{\gamma_0}((T_0 b)a) = be_0 a = bae_0 = T_0(ba)$. As well, $P_{\gamma_0} b P_{\gamma_0} a = P_{\gamma_0}(ba)$, so the same argument works when $\theta_\alpha = P_{\gamma_0}^* \circ \theta_0 \circ \iota_0^*$. A symmetric argument shows that θ_α is also a right A-module map. The operator biflatness version of part (a) is proved in exactly the same way.

Under the hypotheses of the non-bracketed part of statement (b), the maps θ_α can be chosen to be uniformly bounded in $\mathcal{B}((A\widehat{\otimes}A)^*, A^*)$, so biflatness follows from Proposition 2.1. If A is a quantized Banach algebra, then the bracketed hypotheses of statement (b) yield completely bounded maps θ_α in $\mathcal{CB}((A\widehat{\otimes}_{op}A)^*, A^*)$ such that $\sup_\alpha \|\theta_\alpha\|_{cb} < \infty$. Operator biflatness of A follows from Remark 2.2. ∎

If $\{V_i : i \in I\}$ is a family of operator spaces, we let $\bigoplus_{i\in I}^p V_i$ ($1 \le p < \infty$) have the operator space structure it attains as the predual of the direct product of dual spaces in the case $p = 1$, and through interpolation in the case $p > 1$. See [24].

Proposition 2.8 *Let $\{A_i : i \in I\}$ be a family of (quantized) Banach algebras.*

(i) *If each A_i is (operator) approximately biflat, then for $1 \le p < \infty$, $\bigoplus_{i\in I}^p A_i$ is (operator) approximately biflat.*
(ii) *If A_1, A_2 are operator approximately biflat quantized Banach algebras and $A = A_1 \oplus A_2$ has an operator space structure such that the projection maps $A \to A_i$ are completely bounded, then A is also operator approximately biflat.*
(iii) *If each A_i is (operator) biflat and $\sup_i BF_{A_i} < \infty$ (respectively $\sup_i BF_{A_i}^{op} < \infty$), then $\bigoplus_{i\in I}^1 A_i$ is (operator) biflat.*

Proof We first prove (ii). Let $\alpha = (F, \Phi, \epsilon)$, where $\epsilon > 0$, and $F \subset A$, $\Phi \subset A^*$ are finite. Let $\theta_i \colon (A_i\widehat{\otimes}_{op}A_i)^* \to A_i^*$ be a completely bounded A_i-bimodule map such that

$$|\langle a_i, \theta_i \circ \pi_{A_i}^*(\phi_i) - \phi_i\rangle| < \epsilon/2 \qquad (i = 1, 2,\ a = (a_1, a_2) \in F,\ \phi \in \Phi),$$

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where $\phi_i = \phi\big|_{A_i}$. Let $E_i \colon A_i \hookrightarrow A$ and $p_i \colon A \to A_i$ be the embedding and projection maps and let $\widetilde{\theta_i} = p_i^* \circ \theta_i \circ (E_i \otimes E_i)^*$ $(i = 1, 2)$. Thus, we have the following commuting diagram:

$$\begin{tikzcd}
(A\widehat{\otimes}_{op}A)^* \arrow[r, "\widetilde{\theta_i}"] \arrow[d, "(E_i\otimes E_i)^*"'] & A^* \\
(A_i\widehat{\otimes}_{op}A_i)^* \arrow[r, "\theta_i"] & A_i^* \arrow[u, "p_i^*"']
\end{tikzcd}$$

Standard arguments show that $\widetilde{\theta} = \widetilde{\theta_1} + \widetilde{\theta_2} \colon (A\widehat{\otimes}_{op}A)^* \to A^*$ is a completely bounded A-bimodule map such that

$$|\langle a, \widetilde{\theta} \circ \pi_A^*(\phi) - \phi\rangle| < \epsilon \qquad (a \in F,\ \phi \in \Phi).$$

This proves (ii). Obviously, the (non-quantized) Banach algebra version of (ii) holds for arbitrary direct sums $A_1 \oplus A_2$. Suppose further that $A = A_1 \oplus^1 A_2$ is the (operator space) ℓ^1-direct sum of A_1 and A_2. If each A_i is (operator) biflat and $\theta_i \circ \pi_{A_i}^* = \mathrm{id}_{A_i^*}$, then observe that $\widetilde{\theta} \circ \pi_A^* = \mathrm{id}_{A^*}$ and $\|\widetilde{\theta}\| \le \max\{\|\theta_1\|, \|\theta_2\|\}$ (respectively, $\|\widetilde{\theta}\|_{cb} \le \max\{\|\theta_1\|_{cb}, \|\theta_2\|_{cb}\}$).

Suppose now that for each $i \in I$, A_i is (operator) approximately biflat. Let $\Gamma = \{\gamma : \gamma \subset I \text{ is finite}\}$ be ordered by inclusion. By induction, the first case shows that $A_\gamma = \bigoplus_{i\in\gamma}^p A_i$ is (operator) approximately biflat. Viewing A_γ as an ideal in $A = \bigoplus_{i\in I}^p A_i$, the natural homomorphic projection maps P_γ of A onto A_γ are (completely) contractive and satisfy $\|P_\gamma a - a\| \to 0$ $(a \in A)$. By Proposition 2.7, A is (operator) approximately biflat. This is statement (i).

Finally, suppose that each A_i is operator biflat with $\sup_{i\in I} BF_{A_i}^{op} < \infty$. As noted above, $A_\gamma = \bigoplus_{i\in\gamma}^1 A_i$ is operator biflat with $BF_{A_\gamma}^{op} \le \max_{i\in\gamma} BF_{A_i}^{op}$, so the biflatness of $\bigoplus_{i\in I}^1 A_i$ follows from Proposition 2.7. This proves the operator space version of (iii). The other case is similar. ∎

3 Approximate Biflatness and Pseudo-Amenability of $S^1(G)$

Throughout this section, $S^1(G)$ will denote an arbitrary Segal algebra in $L^1(G)$, where G is a locally compact group. Observe that because $S^1(G)$ embeds contractively onto a dense subspace of $L^1(G)$, $L^\infty(G)$ in turn embeds contractively into $S^1(G)^*$ via

$$\langle f, \phi\rangle = \int_G f(s)\phi(s)\, ds \qquad (f \in S^1(G), \phi \in L^\infty(G)).$$

Theorem 3.1 *If $S^1(G)$ is pseudo-amenable, then G is amenable.*

Proof Let $(m_\gamma)_{\gamma\in\Gamma} \subset S^1(G)\widehat{\otimes}S^1(G)$ be an approximate diagonal for $S^1(G)$. Let $\iota \colon S^1(G) \hookrightarrow L^1(G)$ be the embedding map, let 1_G be the augmentation character of $L^1(G)$, and put

$$T = \iota \otimes 1_G \colon S^1(G)\widehat{\otimes}S^1(G) \to L^1(G) \colon f \otimes g \mapsto \Big(\int_G g(s)ds\Big) f.$$

By checking with elementary tensors, one can see that T satisfies

$$T(k \cdot m) = k * Tm \quad \text{and} \quad T(m \cdot k) = \Big(\int_G k(s)ds\Big) Tm$$

where $k \in S^1(G),\ m \in S^1(G)\widehat{\otimes}S^1(G)$. Hence, for any $k \in S^1(G)$ with $\int_G k(s)ds = 1$, we have

$$\begin{aligned} \|k * Tm_\gamma - Tm_\gamma\|_{L^1(G)} &= \|T(k \cdot m_\gamma - m_\gamma \cdot k)\|_{L^1(G)} \\ &\leq \|k \cdot m_\gamma - m_\gamma \cdot k\| \to 0. \end{aligned} \tag{3.1}$$

Fix $h \in S^1(G)$ with $\int h = 1$, and for each γ, let $f_\gamma = h * Tm_\gamma$. For each $x \in G$ we then obtain

$$\begin{aligned} \|\delta_x * f_\gamma - f_\gamma\|_{L^1(G)} \leq \|(\delta_x * h) * Tm_\gamma - Tm_\gamma\|_{L^1(G)} \\ + \|Tm_\gamma - h * Tm_\gamma\|_{L^1(G)} \to 0. \end{aligned}$$

When $m = f \otimes g$, note that

$$\langle 1_G, \pi(m)\rangle = \langle 1_G, f * g\rangle = \langle 1_G, f\rangle\langle 1_G, g\rangle = \langle 1_G, \langle 1_G, g\rangle f\rangle = \langle 1_G, Tm\rangle,$$

and so

$$\begin{aligned} 1 = \langle 1_G, h\rangle &= \lim_\gamma \langle 1_G, h * \pi(m_\gamma)\rangle \\ &= \lim_\gamma \langle 1_G, h\rangle\langle 1_G, \pi(m_\gamma)\rangle = \lim_\gamma \langle 1_G, Tm_\gamma\rangle \\ &= \lim_\gamma \langle 1_G, h * Tm_\gamma\rangle = \lim_\gamma \langle 1_G, f_\gamma\rangle. \end{aligned}$$

As $\|f_\gamma\|_{L^1(G)} \geq |\langle 1_G, f_\gamma\rangle|$, we may therefore assume that $\|f_\gamma\|_{L^1(G)} \geq 1/2$ $(\gamma \in \Gamma)$. Defining

$$g_\gamma = \frac{1}{\|f_\gamma\|_{L^1(G)}} |f_\gamma|, \qquad (\gamma \in \Gamma),$$

we obtain a net of positive norm-one functions in $L^1(G)$ which by (3.1) satisfies

$$\|\delta_x * g_\gamma - g_\gamma\|_{L^1(G)} \leq 2\|\delta_x * |f_\gamma| - |f_\gamma|\|_{L^1(G)} \leq 2\|\delta_x * f_\gamma - f_\gamma\|_{L^1(G)} \longrightarrow 0$$

for $x \in G$. This implies that G is amenable [22]—any w^*-limit point of $(g_\gamma)_\gamma$ in $L^\infty(G)^*$ is a left-invariant mean on $L^\infty(G)$. ∎

Corollary 3.2 *Let G be a* [SIN]*-group. Then the following statements are equivalent:*

(i) *G is amenable;*
(ii) *$S^1(G)$ is approximately biflat;*
(iii) *$S^1(G)$ is pseudo-amenable.*

Proof If statement (i) holds, then $L^1(G)$ is amenable and therefore biflat [4, Theorem 2.9.65], and $S^1(G)$ has a central approximate identity $(e_\lambda)_\lambda$ which is bounded in $L^1(G)$ [21]. Hence, $(e_\lambda^2)_\lambda$ is also an approximate identity for $S^1(G)$, so (ii) is a consequence of Proposition 2.5. That (ii) implies (iii) and (iii) implies (i) are special cases of Theorems 2.4 and 3.1 respectively. ■

Proposition 4.4 of [15] states that the converse to Theorem 3.1 holds when $S^1(G)$ has an approximate identity which "approximately commutes with orbits". When G is a [SIN]-group, $S^1(G)$ always has such an approximate identity so, (i) $\Rightarrow$ (iii) of Corollary 3.2 is also a consequence [15, Proposition 4.4].

We do not know whether, in general, the amenability of G implies either approximate biflatness or pseudo-amenability of $S^1(G)$ (see also [15, Question 3, p. 123]). However, as we show below, it is possible to construct a well-behaved approximate diagonal for $S^1(G)$ in $L^1(G)\widehat{\otimes}S^1(G)$ when G is amenable.

We say that $S^1(G)$ has an approximate diagonal in $L^1(G)\widehat{\otimes}S^1(G)$ if there is a net $\{m_\gamma\}_{\gamma\in\Gamma}$ in $L^1(G)\widehat{\otimes}S^1(G)$ such that, for every $f \in S^1(G)$,

$$f \cdot m_\gamma - m_\gamma \cdot f \longrightarrow 0 \text{ as } \gamma \longrightarrow \infty$$

and $\pi(m_\gamma)$ is an approximate identity for $S^1(G)$. If, in addition, the associated left and right multiplication operators $L_\gamma\colon f \mapsto f \cdot m_\gamma$ and $R_\gamma\colon f \mapsto m_\gamma \cdot f$ from $S^1(G)$ into $L^1(G)\widehat{\otimes}S^1(G)$ are uniformly bounded, then we say that $S^1(G)$ has a multiplier-bounded approximate diagonal in $L^1(G)\widehat{\otimes}S^1(G)$. Finally, in either of the above cases, we say that the (multiplier-bounded) approximate diagonal is central if $f\cdot m_\gamma = m_\gamma\cdot f$ for all $\gamma \in \Gamma$ and $f \in S^1(G)$.

Theorem 3.3 *Let G be a locally compact group, and let $S^1(G)$ be a symmetric Segal algebra. Then the following statements are equivalent:*

(i) *G is amenable;*
(ii) *$S^1(G)$ is a flat $L^1(G)$-bimodule;*
(iii) *$S^1(G)$ has an approximate diagonal in $L^1(G)\widehat{\otimes}S^1(G)$;*
(iv) *$S^1(G)$ has a multiplier-bounded approximate diagonal in $L^1(G)\widehat{\otimes}S^1(G)$.*

Proof (i) $\Longrightarrow$ (ii) Since G is amenable, $L^1(G)$ is amenable. Also $S^1(G)$ is an essential Banach $L^1(G)$-bimodule. Hence if π_1 is the convolution multiplication map from $L^1(G)\widehat{\otimes}S^1(G)$ onto $S^1(G)$, then the short exact sequence of $L^1(G)$-bimodules

$$0 \longmapsto S^1(G)^* \xrightarrow{\pi_1^*} (L^1(G)\widehat{\otimes}S^1(G))^* \xrightarrow{\iota^*} (\ker\pi_1)^* \longmapsto 0,$$

is admissible, and therefore splits, [3, Theorem 2.5].

(ii) $\Longrightarrow$ (iv) Let $\theta\colon (L^1(G)\widehat{\otimes}S^1(G))^* \to S^1(G)^*$ be a continuous $L^1(G)$-bimodule morphism such that $\theta\circ\pi^* = \mathrm{id}_{S^1(G)^*}$. Let $\{e_\alpha\}$ be an approximate identity for $S^1(G)$

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with L^1-norm equal to 1. Set $n_\alpha = \theta^*(e_\alpha^2) \in (L^1(G)\widehat{\otimes}S^1(G))^{**}$. Then, for every $f \in S^1(G)$ and every α,

$$\|f \cdot n_\alpha\| = \|\theta^*(f * e_\alpha^2)\| \le \|\theta\|\|f * e_\alpha^2\|_{S^1(G)} \le \|\theta\|\|f\|_{S^1(G)}.$$

Similar to the above, we have $\|n_\alpha \cdot f\| \le \|\theta\|\|f\|_{S^1(G)}$. Also

$$\pi^{**}(n_\alpha) = (\pi^{**} \circ \theta^*)(e_\alpha^2) = e_\alpha^2,$$

which is an approximate identity for $S^1(G)$. Finally, for $f \in S^1(G)$ and $\varphi \in (L^1(G)\widehat{\otimes}S^1(G))^*$, we have

$$\begin{aligned}\langle f \cdot n_\alpha - n_\alpha \cdot f, \varphi\rangle &= \langle \theta^*(e_\alpha^2)\,,\ \varphi \cdot f - f \cdot \varphi\rangle \\ &= \langle \theta(\varphi) \cdot f - f \cdot \theta(\varphi)\,,\ e_\alpha^2\rangle \\ &= \langle \theta(\varphi), f * e_\alpha^2 - e_\alpha^2 * f\rangle.\end{aligned}$$

Hence $\|f \cdot n_\alpha - n_\alpha \cdot f\| \le \|\theta\|\|f * e_\alpha^2 - e_\alpha^2 * f\|_{S^1(G)}$. Therefore $f \cdot n_\alpha - n_\alpha \cdot f \to 0$ as $\alpha \to \infty$. The final result follows from a similar argument to the one made in [15, Proposition 2.3]. (iv) $\Rightarrow$ (iii) is obvious and (iii) $\Rightarrow$ (i) follows the argument found in the proof of Theorem 3.1. ∎

The following is [12, Theorem 3.1]. Here we present an alternative proof using the multiplier-bounded approximate diagonals.

Theorem 3.4 *Let G be a locally compact amenable group, let $S^1(G)$ be a symmetric Segal algebra, and let X be a Banach $L^1(G)$-bimodule. Then for every continuous derivation $D\colon S^1(G) \to X^*$, there is a continuous double centralizer (S, T) such that $D = S - T$.*

Proof Suppose that $D\colon S^1(G) \to X^*$ is a continuous derivation. By applying the argument presented in the first two paragraphs of the proof of [12, Theorem 3.1(ii)], we can assume that X is an essential $L^1(G)$-bimodule. By the proof of Theorem 3.3(iv), we can choose a multiplier-bounded approximate diagonal $\{m_\alpha\}$ for $S^1(G)$ in $L^1(G)\widehat{\otimes}S^1(G)$ such that $\pi(m_\alpha)$ is bounded in L^1-norm.

Let $m_\alpha = \sum_{i=1}^\infty f_i^\alpha \otimes g_i^\alpha$ and define $x_\alpha^* = \sum_{i=1}^\infty f_i^\alpha \cdot D(g_i^\alpha)$. Then, for $f \in S^1(G)$,

$$f \cdot x_\alpha^* - x_\alpha^* \cdot f - \pi(m_\alpha) \cdot D(f) \xrightarrow{\alpha} 0. \tag{3.2}$$

On the other hand, the operators $S_\alpha\colon f \mapsto f \cdot x_\alpha^*$ and $T_\alpha\colon f \mapsto x_\alpha^* \cdot f$ from $S^1(G)$ into X^* are uniformly bounded. Let S be a cluster point of $\{S_\alpha\}$, and let T be a cluster point of $\{T_\alpha\}$ in the weak*-operator topology. Then (S, T) is a double centralizer and for every $f \in S^1(G)$ and $\xi \in X$,

$$\langle \xi, S(f) - T(f) - D(f)\rangle = \lim_\alpha \langle \xi, S_\alpha(f) - T_\alpha(f) - \pi(m_\alpha) \cdot D(f)\rangle = 0,$$

where we have used equation (3.2) and the fact that X is essential. ∎

It is shown in [15, Theorem 4.5] that $S^1(G)$ is pseudo-contractible if G is compact. In the following theorem, we prove the converse of that result and present other equivalent conditions on pseudo-contractiblity of $S^1(G)$ (see also [15, Proposition 3.8]).

Theorem 3.5 *Let G be a locally compact group, and let $S^1(G)$ be a Segal algebra. Then the following statements are equivalent:*

(i) *G is compact;*
(ii) *$S^1(G)$ has a central approximate diagonal in $L^1(G)\widehat{\otimes}S^1(G)$;*
(iii) *$S^1(G)$ is pseudo-contractible.*

If, in addition, $S^1(G)$ is symmetric, then the above statements are equivalent to either of the following statements:

(iv) *$S^1(G)$ is a projective $L^1(G)$-bimodule;*
(v) *$S^1(G)$ has a central, multiplier-bounded, approximate diagonal in $L^1(G)\widehat{\otimes}S^1(G)$.*

Proof (i) $\Longrightarrow$ (iii) This is [15, Theorem 4.5].

(iii) $\Longrightarrow$ (ii) We note that from [33], $S^1(G)$ is (boundedly) approximately complemented in $L^1(G)$. Hence the map $\iota\otimes\mathrm{id}_{S^1(G)}\colon S^1(G)\widehat{\otimes}S^1(G) \longrightarrow L^1(G)\widehat{\otimes}S^1(G)$ is injective [32]. Therefore $\iota \otimes \mathrm{id}_{S^1(G)}$ maps a central approximate diagonal in $S^1(G)\widehat{\otimes}S^1(G)$ into a central approximate diagonal in $L^1(G)\widehat{\otimes}S^1(G)$.

(ii) $\Longrightarrow$ (i) If $S^1(G)$ has a central approximate diagonal in $L^1(G)\widehat{\otimes}S^1(G)$, then a similar argument to the proof of Theorem 3.1 gives a non-zero function $f \in L^1(G)$ such that $\delta_x * f = f$ $(x \in G)$. This implies that f is equal to a non-zero constant almost everywhere, and it follows that G is compact.

(i) $\Longleftrightarrow$ (iv) If G is compact, then $L^1(G)\widehat{\otimes}L^1(G^{op}) = L^1(G \times G^{op})$ is biprojective [16, IV, Theorem 5.13]. Hence $S^1(G)$ is a projective $L^1(G)$-bimodule since it can be regarded as a Banach left $L^1(G)\widehat{\otimes}L^1(G^{op})$-module [16, IV, Theorem 5.3].

Conversely, suppose that $S^1(G)$ is a projective $L^1(G)$-bimodule. Hence there is a continuous $L^1(G)$-bimodule morphism $\rho\colon S^1(G) \longrightarrow L^1(G)\widehat{\otimes}S^1(G)$ such that $\pi\circ\rho = \mathrm{id}_{S^1(G)}$. Let 1_G be the augmentation character of $L^1(G)$, and put

$$T = \iota \otimes 1_G\colon L^1(G)\widehat{\otimes}S^1(G) \to L^1(G)\colon f \otimes g \mapsto \Big(\int_G g(s)ds\Big) f.$$

Now define the operator $\rho_1\colon S^1(G) \longrightarrow L^1(G)$ by $\rho_1 = T \circ \rho$. It is easy to check that ρ_1 is a continuous $L^1(G)$-bimodule morphism. Moreover, for $f \in S^1(G)$ and $g \in I_0 = \ker 1_G \cap S^1(G)$, we have

$$\rho_1(f * g) = \rho_1(f) \cdot g = \rho_1(f)1_G(g) = 0.$$

Hence $\rho_1 = 0$ on I_0 since $S^1(G)I_0$ is dense in I_0. Therefore, ρ_1 induces a left $L^1(G)$-module morphism $\tilde{\rho}\colon S^1(G)/I_0 \longrightarrow L^1(G)$. However, $S^1(G)/I_0$ is isomorphic with $\mathbb{C}$ as a Banach $L^1(G)$-module for the product defined by

$$f \cdot \lambda = \lambda \cdot f = 1_G(f)\lambda \quad (f \in L^1(G), \lambda \in \mathbb{C}).$$

Moreover, with the above identification, $1_G \circ \tilde{\rho} = \mathrm{id}_{\mathbb{C}}$. Thus $\mathbb{C}$ is a projective left $L^1(G)$-bimodule. This implies that G is compact (see, for example, [4, Theorem 3.3.32(ii)].

(i) $\Longleftrightarrow$ (v) If G is compact, then $S^1(G)$ has a central approximate identity $\{e_\alpha\}$ which has L^1-norm equal to 1. On the other hand, from (iv), there is a continuous $S^1(G)$-bimodule morphism $\theta\colon S^1(G) \longrightarrow L^1(G)\widehat{\otimes}S^1(G)$ which is the right inverse to the convolution multiplication $\pi_1\colon L^1(G)\widehat{\otimes}S^1(G) \longrightarrow S^1(G)$. Thus if we put $m_\alpha = \theta(e_\alpha)$, then it is straightforward to show that $\{m_\alpha\}$ is a central, multiplier-bounded, approximate diagonal in $L^1(G)\widehat{\otimes}S^1(G)$ for $S^1(G)$. The converse follows easily because (v) implies (iii). ∎

It is shown in [12] that if G is an amenable group or a SIN group, then every continuous derivation from a symmetric Segal algebra $S^1(G)$ into $S^1(G)^*$ is approximately inner, *i.e.*, $S^1(G)$ is approximately weakly amenable. In an attempt to answer whether or not in general $S^1(G)$ is approximately weakly amenable, we have come up with the result of the following theorem. Here π is the product map from $S^1(G)\widehat{\otimes}S^1(G)$ into $S^1(G)$.

Theorem 3.6 *Let G be a locally compact group, and let $S^1(G)$ be a symmetric Segal algebra. Then for every continuous derivation $D\colon S^1(G) \to S^1(G)^*$, $\pi^* \circ D$ is w^*-approximately inner.*

Proof Let $D\colon S^1(G) \to S^1(G)^*$ be a continuous derivation. Define the operator $\tilde{D}\colon L^1(G) \to (S^1(G)\widehat{\otimes}S^1(G))^*$ by

$$\langle \tilde{D}(f), g\otimes h\rangle = \langle D(f*g) - fD(g), h\rangle \quad (f\in L^1(G), g, h\in S^1(G)).$$

Since D is a derivation, it is straightforward to verify that $\tilde{D}$ is a continuous derivation. Let $\{e_\alpha\}_{\alpha\in I}$ be an approximate identity in $S^1(G)$ having L^1-norm equal to 1. Define the operator $\Lambda_\alpha\colon (S^1(G)\widehat{\otimes}S^1(G))^* \to L^\infty(G\times G)$ by

$$\Lambda_\alpha(T)(f\otimes g) = T(f*e_\alpha\otimes e_\alpha * g),$$

for every $T\in (S^1(G)\widehat{\otimes}S^1(G))^*$ and $f, g\in L^1(G)$. Clearly each Λ_α is a continuous $L^1(G)$-bimodule morphism. Hence $\Lambda_\alpha\circ\tilde{D}$ is a continuous derivation from $L^1(G)$ into $L^\infty(G\times G)$, and so, it is inner ([4, Theorem 5.6.41], in the case where $E = L^1(G\times G)$). This means that there is $\varphi_\alpha \in L^\infty(G\times G)$ such that $\Lambda_\alpha\circ\tilde{D} = ad_{\varphi_\alpha}$ $(\alpha\in I)$. Let $\iota\colon S^1(G)\to L^1(G)$ be the inclusion map and put $\psi_\alpha = (\iota\otimes\iota)^*(\varphi_\alpha)$. Then

$$(\iota\otimes\iota)^*\circ\Lambda_\alpha\circ\tilde{D} = \mathrm{ad}_{\psi_\alpha} \quad (\alpha\in I). \tag{3.3}$$

However, since $\|e_\alpha\|_1 = 1$, it follows that for every $T\in (S^1(G)\widehat{\otimes}S^1(G))^*$ and $g, h\in S^1(G)$

$$\begin{aligned}|\langle(\iota\otimes\iota)^*\circ\Lambda_\alpha(T), g\otimes h\rangle| &= |\langle\Lambda_\alpha(T)\,,\, g\otimes h\rangle| = |\langle T, g*e_\alpha\otimes e_\alpha*h\rangle|\\ &\le \|T\|\|g*e_\alpha\|_{S^1(G)}\|e_\alpha*h\|_{S^1(G)} \le \|T\|\|g\|_{S^1(G)}\|h\|_{S^1(G)}.\end{aligned}$$

Thus $\|(\iota \otimes \iota)^* \circ \Lambda_\alpha\| \leq 1$, and so, $\|(\iota \otimes \iota)^* \circ \Lambda_\alpha \circ \tilde{D}\| \leq 2\|D\|$. Hence there is $\Delta \in B(L^1(G), (S^1(G) \otimes S^1(G))^*)$ such that $(\iota \widehat{\otimes} \iota)^* \circ \Lambda_\alpha \circ \tilde{D} \to \Delta$ in the W^*OT of $B(L^1(G), (S^1(G) \widehat{\otimes} S^1(G))^*)$. Now take $f, g, h \in S^1(G)$. Then

$$\begin{aligned}\langle \Delta(f), g \otimes h\rangle &= \lim_\alpha \langle (\iota \otimes \iota)^* \circ \Lambda_\alpha \circ \tilde{D}(f), g \otimes h\rangle \\ &= \lim_\alpha \langle \tilde{D}(f), g * e_\alpha \otimes e_\alpha * h\rangle \\ &= \langle \tilde{D}(f), g \otimes h\rangle = \langle D(f)\ ,\ g * h\rangle \\ &= \langle \pi^* \circ D(f), g \otimes h\rangle.\end{aligned}$$

Hence $\Delta \circ \iota = \pi^* \circ D$. Therefore, from (3.3), it follows that $\pi^* \circ D = W^*OT - \lim_\alpha \mathrm{ad}_{\psi_\alpha}$. ∎

4 Approximate Biflatness and Pseudo-Amenability of $S^1A(G)$

In the preceding section we saw that the pseudo-amenablity of a Segal algebra $S^1(G)$ in $L^1(G)$, implies that G, and hence $L^1(G)$, is amenable. In this section we prove that (operator) approximate biflatness, and therefore pseudo-amenability, of the (operator) Segal algebra $S^1A(G)$ is much weaker than the (operator) amenability of $A(G)$ (Theorems 4.6 and 4.7). On the other hand, the next theorem shows that the dual version of Theorem 3.5 is true.

If $F(G)$ is any collection of continuous functions on G, we let $F_c(G)$ denote the set of compactly supported functions in $F(G)$.

Lemma 4.1 *Let $SA(G)$ be a Segal algebra in $A(G)$.*

(i) *If $SA(G)$ has an approximate identity, then $SA_c(G)$ is dense in $SA(G)$.*
(ii) *If G is discrete, then δ_g the indicator function at $g \in G$ belongs to $SA(G)$.*

Proof Let $u \in SA(G)$, $\epsilon > 0$. Take $e \in SA(G)$ such that $\|ue - u\|_{SA} < \epsilon/2$. Choosing $e_0 \in A_c(G)$ such that $\|e - e_0\|_A < \epsilon/(2\|u\|_{SA})$, we have $ue_0 \in SA_c(G)$ and

$$\|ue_0 - u\|_{SA} \leq \|ue_0 - ue\|_{SA} + \|ue - u\|_{SA} \leq \|u\|_{SA}\|e_0 - e\|_A + \epsilon/2 < \epsilon.$$

This proves (i). If G is discrete, then for $g \in G$, $\delta_g \in A(G)$, and we can choose $u \in SA(G)$ such that $\|u - \delta_g\|_A < 1/2$. Then $|u(g) - 1| < 1/2$, so $u(g) \neq 0$. Now $\delta_g = \frac{1}{u(g)} u\delta_g \in SA(G)$, proving statement (ii). ∎

Theorem 4.2 *Let $SA(G)$ be an (operator) Segal algebra of $A(G)$. Then the following statements are equivalent:*

(i) *$SA(G)$ has an approximate identity and G is discrete;*
(ii) *$SA(G)$ has an approximate identity and is (operator) approximately biprojective;*
(iii) *$SA(G)$ is (operator) pseudo-contractible.*

Proof We prove the operator space version of the theorem. Suppose that G is discrete and that $SA(G)$ has an approximate identity $(e_\lambda)_{\lambda\in\Lambda}$. By Lemma 4.1, we may assume that each e_λ has compact support E_λ, and we can define $m_\lambda \in SA(G)\widehat{\otimes}_{op}SA(G)$ by

$$m_\lambda = \sum_{x\in E_\lambda} e_\lambda(x)(\delta_x \otimes \delta_x) \qquad (\lambda \in \Lambda).$$

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It is clear that $a \cdot m_\lambda = m_\lambda \cdot a$ ($a \in SA(G)$) and $\pi(m_\lambda) = e_\lambda$ ($\lambda \in \Lambda$), an approximate identity. Hence, $SA(G)$ is operator pseudo-contractible.

Assuming that $SA(G)$ is operator pseudo-contractible, let $(m_\alpha)_\alpha$ be an operator approximate diagonal for $SA(G)$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ ($a \in SA(G)$). Let $T = \mathrm{id}_{SA(G)} \otimes \lambda(e) \colon SA(G)\widehat{\otimes}_{op} SA(G) \to SA(G)$, where $\lambda(e)$ is the (completely) bounded functional on $SA(G)$ defined by $\lambda(e)u = u(e)$. By checking with elementary tensors $m = u \otimes v$, one sees that $T(a \cdot m) = aTm$, $T(m \cdot a) = a(e)Tm$ ($a \in SA(G)$), and $Tm(e) = \pi(m)(e)$. Hence, we can choose $\psi = Tm_\alpha$ such that

$$\psi = \psi a \quad (a \in SA(G),\ a(e) = 1) \quad \text{and} \quad \psi(e) \neq 0.$$

The remainder of the proof is similar to the proof of [25, Proposition 5]. Let $g \in G$ and choose $v \in A(G)$ such that $v(g) = 0$, $v(e) = 1$, and take $a \in SA(G)$ such that $a(e) = 1$. Then $av \in SA(G)$ satisfies $av(e) = 1$, so $0 = av\psi(g) = \psi(g)$. Hence, $\delta_e = \frac{1}{\psi(e)}\psi$, which is a continuous function on G. Hence, G is discrete. The equivalence of statements (ii) and (iii) is a special case of (the operator space version of) [15, Proposition 3.8]. ■

Lemma 4.3 *Let $F \colon S^1A(H) \to S^1A(G)$ be a linear map with a completely bounded extension $F^A \colon A(H) \to A(G)$ and (completely) bounded extension $F^L \colon L^1(H) \to L^1(G)$. Then F is itself completely bounded.*

Proof By definition, $S^1A(G)$ inherits its operator space structure via the embedding

$$S^1A(G) \hookrightarrow A(G) \oplus_1 L^1(G) \colon u \mapsto (u, u)$$

[10, p. 4]. As F^A and F^L are completely bounded, so is

$$F^A \oplus F^L \colon A(H) \oplus_1 L^1(H) \to A(G) \oplus_1 L^1(G)$$

with $\|F^A \oplus F^L\|_{cb} \leq \|F^A\|_{cb} + \|F^L\|_{cb}$. Hence $F = (F^A \oplus F^L)\big|_{S^1A(H)}$ is also completely bounded. ■

The "completely bounded" part of the next lemma will not be needed but may be of independent interest.

Lemma 4.4 *If $A(G)$ has an approximate identity which is bounded in the (completely bounded) multiplier norm, then so does $S^1A(G)$.*

Proof Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for $A(G)$ with bound R in the multiplier norm of $A(G)$; we may further suppose that (e_λ) is contained in $S^1A(G)$. Given $x \in G$, choose $v \in A(G)$ such that $\|v\|_{A(G)} = 1$ and $v(x) = 1$. Then for any λ, $|e_\lambda(x)| \leq \|e_\lambda v\|_\infty \leq \|e_\lambda v\|_{A(G)} \leq R\|v\|_{A(G)} = R$. Hence,

$$\|e_\lambda\|_\infty \leq R \qquad (\lambda \in \Lambda), \tag{4.1}$$

and therefore, for any $v \in S^1A(G)$,

$$\|e_\lambda v\|_{S^1A} = \|e_\lambda v\|_{A(G)} + \|e_\lambda v\|_{L^1} \leq R\|v\|_{A(G)} + R\|v\|_{L^1} = R\|v\|_{S^1A}.$$

Thus, (e_λ) is also bounded in the multiplier norm of $S^1A(G)$. As (e_λ) is an approximate identity for $A(G)$, $e_\lambda \to 1$ in the topology of uniform convergence on compact subsets of G. This, together with equation (4.1), yields

$$\|e_\lambda v - v\|_{L^1} \to 0 \qquad (v \in S^1A(G)).$$

Consequently, (e_λ) is an approximate identity for $S^1A(G)$.

Suppose now that (e_λ) is bounded, again by R, in the completely bounded multiplier norm in $A(G)$. Again, we can suppose without loss of generality that (e_λ) is contained in $S^1A(G)$. From equation (4.1) we know that the maps

$$L^1(G) \longrightarrow L^1(G)\colon a \mapsto e_\lambda a$$

are bounded by R. It follows from Lemma 4.3 that (e_λ) is bounded in the completely bounded multiplier norm taken with respect to $S^1A(G)$. ■

If u is a function defined on a subgroup H of G, we let

$$u^\circ(x) = \begin{cases} u(x) & \text{if } x \in H \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.5 *Let G be a locally compact group such that $S^1A(G)$ has an approximate identity. If H is an open subgroup of G, and $S^1A(H)$ is (operator) approximately biflat, then so is $S^1A(G)$.*

Proof Let C be a transversal for left cosets of H in G, and assume that $e \in C$. Order the collection Γ of finite subsets of C by inclusion, and for each $\gamma \in \Gamma$, let $E_\gamma = \cup_{x\in\gamma} xH$. Let $S^1A_\gamma = \{u \in S^1A(G) : u = u1_{E_\gamma}\}$, $S^1A_x = S^1A_{\{x\}}$.

Assuming that Haar measure on H is the restriction to H of the Haar measure on G, the map $u \mapsto u^\circ$ defines a (completely) isometric isomorphism of $A(H)$ onto $A_e = \{v \in A(G) : v = v1_H\}$ [31, Proposition 4.2] and of $L^1(H)$ into $L^1(G)$. On $A(H)$, the inverse of this map is the restriction of $r\colon A(G) \to A(H)\colon u \to u|_H$ to A_e, which by [31, Proposition 4.3] is a complete contraction. It follows that $u \mapsto u^\circ$ is an isometric isomorphism of $S^1A(H)$ onto S^1A_e which, by Lemma 4.3, is a complete isomorphism. Similarly, for each $x \in G$ left translation by x^{-1} is a complete isomorphism of S^1A_e onto S^1A_x [8, Lemma 4.4]. Hence, the (operator) approximate biflatness of $S^1A(H)$ implies that of S^1A_x. Lemma 4.3 also implies that the projection maps $S^1A(G) \to S^1A_x : u \mapsto u1_{xH}$ are completely bounded, so $S^1A_\gamma = \bigoplus_{x\in\gamma} S^1A_x$ $(\gamma \in \Gamma)$ is (operator) approximately biflat by Proposition 2.8(b). Let $(e_\lambda)_\lambda$ be an approximate identity for $S^1A(G)$. As noted on [10, p. 10], $A_c(G)$ is dense in $S^1A(G)$, so we may assume that each e_λ has compact support so that $(e_\lambda)_\lambda \subset \cup_\gamma S^1A_\gamma$. If we define projections P_γ of $S^1A(G)$ onto S^1A_γ by $P_\gamma u = u1_{E_\gamma}$, the (operator) approximate biflatness of $S^1A(G)$ follows from Proposition 2.7. ■

Theorem 4.6 *Let G be a locally compact group such that $S^1A(G)$ has an approximate identity. If G contains an abelian open subgroup, then $S^1A(G)$ is approximately biflat and therefore pseudo-amenable.*

Proof Let H be an open abelian subgroup of G. Then $A(H)$ is amenable and therefore biflat by [4, Theorem 2.9.65]. By Lemma 4.4, $S^1A(H)$ has an approximate identity $(e_\lambda)_\lambda$ which is bounded in the multiplier norm of $S^1A(H)$; hence, $(e_\lambda^2)_\lambda$ is also an approximate identity for $S^1A(H)$. By first applying Proposition 2.5 then Theorem 4.5, we can conclude that $S^1A(G)$ is approximately biflat. Pseudo-amenability of $S^1A(G)$ follows from Theorem 2.4. ∎

Theorem 4.7 *Let G be a locally compact group such that $S^1A(G)$ has an approximate identity, and suppose that G contains an open subgroup H such that $A(H)$ has an approximate identity which is multiplier-norm bounded. If Δ_H has a bounded approximate indicator, then $S^1A(G)$ is operator approximately biflat and operator pseudo-amenable.*

Proof By [1, Proposition 2.3], $A(H)$ is operator biflat. As with the proof of Theorem 4.6, Lemma 4.4, Proposition 2.5, and Theorem 4.5 yield the operator approximate biflatness of $S^1A(G)$. ∎

It is shown in [1] that Δ_H has a bounded approximate indicator whenever H can be continuously embedded in a [QSIN]-group. Every amenable group and every [SIN]-group is a [QSIN]-group. When G_e, the principle component of G, is amenable, the proof of [14, Proposition 5.2] shows that G contains an amenable open subgroup. Hence we have the following corollary to Theorem 4.7.

Corollary 4.8 *Let G be a locally compact group such that $S^1A(G)$ has an approximate identity. If G_e is amenable, then $S^1A(G)$ is operator approximately biflat and operator pseudo-amenable.*

Remark 4.9 The same arguments show that under the hypotheses of Theorems 4.6 and 4.7, $A(G)$ is, respectively, approximately biflat and operator approximately biflat. By choosing G to be any amenable group which contains an open abelian subgroup but which is not a finite extension of an abelian group (such as the integer Heisenberg group), $A(G)$ provides an example of a Banach algebra which is approximately biflat, but not biflat. Indeed, in this case $A(G)$ has a bounded approximate identity, so if $A(G)$ were biflat, it would be amenable (see [4, Theorem 2.9.65]) in contradiction to the main result of [9].

5 Feichtinger's Segal algebra

Let us recall the definition of $S_0(G)$. Let K be a compact subset of G with nonempty interior and $A_K(G) = \{u \in A(G)\colon \operatorname{supp} u \subset K\}$. We let

$$q_K\colon \ell^1(G)\widehat{\otimes}_{op}A_K(G) \to A(G) \quad q_K(\delta_s \otimes v) = s{*}v,$$

where $s{*}v(t) = v(s^{-1}t)$ and $\widehat{\otimes}_{op}$ denotes the operator projective tensor norm, which in this case is the same as the projective tensor norm $\widehat{\otimes}$. Then we set $S_0(G) = \operatorname{ran} q_K$ and assign $S_0(G)$ the operator space structure (hence Banach space structure) it inherits as a quotient of $\ell^1(G)\widehat{\otimes}A_K(G)$. We recall that this operator space structure is completely isomorphically, though not completely isometrically, independent of the choice of the set K. We do not know a tractable formula for the norm of a matrix $[v_{ij}]$ in $M_n(S_0(G))$. However, if we consider a dual formulation, and consider matrices with a "trace-class" norm, $T_n(S_0(G)) \cong T_n \widehat{\otimes} S_0(G)$, we obtain for any $n \times n$ matrix $[v_{ij}]$ with entries in $S_0(G)$

$$\|[v_{ij}]\|_{T_n(\operatorname{ran} q_K)} = \inf\left\{ \sum_{k=1}^{\infty} \left\| [v_{ij}^{(k)}] \right\|_{T_n(A)} : \begin{array}{c} [v_{ij}] = \sum_{k=1}^{\infty} [s_k * v_{ij}^{(k)}], \text{ where each} \\ s_k \in G \text{ and } [v_{ij}^{(k)}] \in T_n(A_K(G)) \end{array} \right\}.$$

We recall that, for any operator space $\mathcal{V}$, a linear map $S\colon \mathcal{V} \to \mathcal{V}$ is completely bounded if and only if the sequence of maps

$$T_n(S)\colon T_n(\mathcal{V}) \to T_n(\mathcal{V}), \quad T_n(S)[v_{ij}] = [Sv_{ij}]$$

are uniformly bounded, and we have $\|S\|_{cb} = \sup_{n\in\mathbb{N}} \|T_n(S)\|$.

We let the *multiplier algebra* of $S_0(G)$ be given by

$$MS_0(G) = \{u\colon G \to \mathbb{C}\colon uS_0(G) \subset S_0(G)\}.$$

The usual closed graph theorem argument tells us that for each u in $MS_0(G)$, the operator $v \mapsto uv$ is bounded. We further define the *completely bounded multiplier algebra* of $S_0(G)$ by

$$M_{cb}S_0(G) = \{u \in MS_0(G)\colon v \mapsto uv : S_0(G) \to S_0(G) \text{ is c.b}\}.$$

We thus obtain the following modest description of the multipliers and the completely bounded multipliers.

Proposition 5.1 *Let $u\colon G \to \mathbb{C}$.*

(i) *$u \in MS_0(G)$ if and only if for any compact subset K of G with nonempty interior we have $uA_K(G) \subset A_K(G)$ and*

$$\|u\|_{M \operatorname{ran} q_K} = \sup\{\|u\, s * v\|_A : s \in G, v \in A_K(G), \|v\|_A \le 1\} < \infty.$$

(ii) *$u \in M_{cb}S_0(G)$ if and only if for any compact subset K of G with nonempty interior we have $uA_K(G) \subset A_K(G)$ and*

$$\|u\|_{M_{cb} \operatorname{ran} q_K} = \sup\left\{ \|[u\, s * v_{ij}]\|_{T_n(A)} : \begin{array}{c} s \in G, [v_{ij}] \in T_n(A_K(G)) \\ \|[v_{ij}]\|_{T_n(A)} \le 1 \end{array} \right\} < \infty.$$

We note that by regularity of $A(G)$, the condition $uA_K(G) \subset A_K(G)$, for any K as above, is equivalent to saying that u is locally an element of $A(G)$.

Proof We will show only (ii), the proof of (i) being similar.

If $u \in M_{cb}S_0(G)$, let $m_u\colon S_0(G) \to S_0(G)$ be given by $m_u v = uv$. Note that for any s in G, compact $K \subset G$ with nonempty interior and $[v_{ij}]$ in $T_n(A_K(G))$, we have $[s * v_{ij}] \in T_n(S_0(G))$ with

$$\|[s * v_{ij}]\|_{T_n(\operatorname{ran} q_K)} = \|[s * v_{ij}]\|_{T_n(A)}.$$

Since $A_K(G) \subset S_0(G)$, it is clear that $uA_K(G) \subset A_K(G)$. Moreover, since $S_0(G)$ is closed under translations, it follows that $u(s*A_K(G)) \subset s*A_K(G)$ too. Hence, for $s, [v_{ij}]$, as above with $\|[v_{ij}]\|_{T_n(A)} \leq 1$, we have

$$\begin{aligned}\|[u\, s*v_{ij}]\|_{T_n(A)} &= \|[u\, s*v_{ij}]\|_{T_n(A)}\\ &= \|T_n(m_u)\| \leq \|m_u\|_{\mathcal{CB}(\operatorname{ran} q_K)}.\end{aligned}$$

Conversely, if the latter conditions hold, we let $[v_{ij}] \in T_n(S_0(G))$, $\varepsilon > 0$, and find elements s_k in G and matrices $[v_{ij}^{(k)}]$ in $T_n(A_K(G))$ such that

$$[v_{ij}] = \sum_{k=1}^{\infty}[s_k*v_{ij}^{(k)}] \text{ and } \sum_{k=1}^{\infty}\|[v_{ij}^{(k)}]\|_{T_n(A)} < \|[v_{ij}]\|_{T_n(\operatorname{ran} q_K)} + \varepsilon.$$

Then we have

$$\begin{aligned}\|T_n(m_u)[v_{ij}]\|_{T_n(\operatorname{ran} q_K)} &= \|[uv_{ij}]\|_{T_n(\operatorname{ran} q_K)}\\ &\leq \sum_{k=1}^{\infty}\|[u\, s_k*v_{ij}^{(k)}]\|_{T_n(A)}\\ &\leq \sum_{k=1}^{\infty}\|u\|_{M_{cb}\operatorname{ran} q_K}\|[s_k*v_{ij}^{(k)}]\|_{T_n(A)}\\ &< \|u\|_{M_{cb}\operatorname{ran} q_K}\left(\|[v_{ij}]\|_{T_n(\operatorname{ran} q_K)} + \varepsilon\right).\end{aligned}$$

Hence for each n, $\|T_n(m_u)\| \leq \|u\|_{M_{cb}\operatorname{ran} q_K} < \infty$, and thus $u \in M_{cb}S_0(G)$. ■

We let $MA(G)$ and $M_{cb}A(G)$ denote the algebras of multipliers and completely bounded multipliers of $A(G)$. The following is immediate from the proposition above.

Corollary 5.2 (i) *$MA(G) \subset MS_0(G)$ with $\|u\|_{M\operatorname{ran} q_K} \leq \|u\|_{MA}$ for any $u \in MA(G)$ and K as above.*

(ii) *$M_{cb}A(G) \subset M_{cb}S_0(G)$ with $\|u\|_{M_{cb}\operatorname{ran} q_K} \leq \|u\|_{M_{cb}A}$ for any $u \in M_{cb}A(G)$ and K as above. In particular, $S_0(G)$ is a completely contractive $B(G)$-module.*

Proof The only thing which does not follow directly from the proposition above is that $S_0(G)$ is a completely contractive $B(G)$-module. This can be seen by a straightforward modification of the proof of the fact that $S_0(G)$ is a completely contractive $A(G)$-module in [29]. ■

We are now ready to state the main result of this section.

Theorem 5.3 *Let G be a locally compact group, and let H be an open subgroup of G such that H is weakly amenable and Δ_H has a bounded approximate indicator in $B(H \times H)$. Then $S_0(G)$ is operator biflat. In particular, $S_0(G)$ is operator pseudo-amenable.*

Proof We first prove that $S_0(H)$ is operator biflat. Let $\{f_\alpha\}_{\alpha\in I}$ be a bounded approximate indicator for Δ_H. For each $\alpha \in I$, define the operator $\rho_\alpha\colon S_0(H\times H) \to S_0(H\times H)$ by $\rho_\alpha(u) = uf_\alpha \quad (\alpha \in I)$. By the preceding corollary, each ρ_α is a completely bounded $B(H\times H)$-bimodule morphism. Moreover, $\|\rho_\alpha\|_{cb} \le \|f_\alpha\|_{B(H\times H)} \le M$, where $M = \sup\{\|f_\alpha\|_{B(H\times H)} \mid \alpha \in I\}$. Let $\rho\colon S_0(H\times H) \to S_0(H\times H)^{**}$ be a cluster-point of ρ_α in the W^*OT of $\mathcal{CB}(S_0(H\times H), S_0(H\times H)^{**})$. Clearly ρ is a $B(H\times H)$-bimodule morphism. Let

$$I(\Delta_H) = \{u \in S_0(H\times H) \mid u = 0 \text{ on } \Delta_H\};$$

and

$$I_0(\Delta_H) = \{u \in S_0(H\times H) \mid u \text{ has a compact support disjoint from } \Delta_H\}.$$

It is easy to see that, for each $u \in I_0(\Delta_H)$, $uf_\alpha \to 0$ as $\alpha \to \infty$. On the other hand, from Proposition 5.1 and [29, Theorem 3.1], $S_0(H\times H)$ has an approximate identity bounded in its completely bounded multiplier norm. Hence, from the fact that Δ_H is a set of synthesis for $A(H\times H)$ [30, Theorem 3], it follows that $I_0(\Delta_H)$ is dense in $I(\Delta_H)$. Thus, for $u \in I(\Delta_H)$ and $\epsilon > 0$, there is $u_\epsilon \in I_0(\Delta_H)$ such that $\|u - u_\epsilon\| < \epsilon$. Hence,

$$\begin{aligned}\|uf_\alpha\| &\le \|(u-u_\epsilon)f_\alpha\| + \|u_\epsilon f_\alpha\|\\ &\le \|u-u_\epsilon\|M + \|u_\epsilon f_\alpha\|\\ &\le \epsilon M + \|u_\epsilon f_\alpha\|\\ &\to \epsilon M,\end{aligned}$$

as $\alpha \to \infty$. Thus $uf_\alpha \to 0$ as $\alpha \to \infty$. This implies that $\rho = 0$ on $I(\Delta_H)$. Hence

$$\tilde\rho\colon \frac{S_0(H\times H)}{I(\Delta_H)} \to S_0(H\times H)^{**}$$

is well defined. Using the identification $S_0(H\times H)/I(\Delta_H) = S_0(H)$ (see [29, Theorem 3.3]), we can assume that $\tilde\rho$ is defined on $S_0(H)$. It is clear that $\tilde\rho$ is a continuous $B(H)$-bimodule morphism, and so, it is a $S_0(H)$-bimodule morphism. Moreover, if $\pi\colon S_0(H\times H) \to S_0(H)$ is the multiplication map, then $\pi^{**}\circ\tilde\rho$ is the canonical embedding of $S_0(H)$ into $S_0(H)^{**}$. Hence $S_0(H)$ is operator biflat.

Now by [29, Corollary 2.6], there is a natural, completely-bounded, algebra homomorphism from $S_0(G)$ onto $\ell^1(T)\widehat{\otimes}S_0(H)$, where T is a transversal for left cosets of H and $\ell^1(T)$ has pointwise multiplication. Hence, by Proposition 2.8(iii), $S_0(G)$ is operator biflat. Moreover, from Proposition 5.1, $S_0(H)$ has an approximate identity bounded in its completely bounded multiplier norm. Since the same is true for $\ell^1(T)$, it follows that $S_0(G)$ has an approximate identity bounded in its completely bounded multiplier norm. Hence, from Theorem 2.4, $S_0(G)$ is operator pseudo-amenable. ■

It is shown in [1] that $A(G)$ is operator biflat whenever G is either an amenable or a [SIN]-group. Since every [IN]-group contains an open amenable subgroup, Theorem 5.3 shows that $S_0(G)$ is operator biflat whenever G is either an amenable or an [IN]-group.

References

[1] O. Y. Aristov, V. Runde, and N. Spronk, *Operator biflatness of the Fourier algebra and approximate indicators for subgroups.* J. Funct. Anal. **209**(2004), no. 2, 367–387. doi:10.1016/S0022-1236(03)00169-1

[2] J. T. Burnham, *Closed ideals in subalgebras of Banach algebras. I.* Proc. Amer. Math. Soc. **32**(1972), 551–555. doi:10.2307/2037857

[3] P. C. Curtis, R. J. Loy, *The structure of amenable Banach algebras.* J. London Math. Soc. (2) **40**(1989), no. 1, 89–104. doi:10.1112/jlms/s2-40.1.89

[4] H. G. Dales, *Banach algebras and automatic continuity.* London Mathematical Society Monographs, 24, The Clarendon Press, Oxford University Press, New York, 2000.

[5] H. G. Dales, F. Ghahramani, and A. Ya. Helemskii, *The amenability of measure algebras.* J. London Math. Soc. (2) **66**(2002), no. 1, 213–226. doi:10.1112/S0024610702003381

[6] E. G. Effros and Z.-J. Ruan, *Operator spaces.* London Mathematical Society Monographs, 23, The Clarendon Press, Oxford University Press, New York, 2000.

[7] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact.* Bull. Soc. Math. France **92**(1964), 181–236.

[8] B. Forrest and P. Wood, *Cohomology and the operator space structure of the Fourier algebra and its second dual.* Indiana Univ. Math. J. **50**(2001), no. 3, 1217–1240. doi=10.1512/iumj.2001.50.1835

[9] B. E. Forrest and V. Runde, *Amenability and weak amenability of the Fourier algebra.* Math. Z. **250**(2005), 731–744. doi:10.1007/s00209-005-0772-2

[10] B. E. Forrest, N. Spronk, and P. J. Wood, *Operator Segal algebras in Fourier algebras.* Studia Math. **179**(2007), no. 3, 277–295. doi:10.4064/sm179-3-5

[11] F. Ghahramani and A. T.-M. Lau, *Weak amenability of certain classes of Banach algebras without bounded approximate identities.* Math. Proc. Cambridge Philos. Soc. **133**(2002), no. 2, 357–371. doi=10.1017/S0305004102005960

[12] ———, *Approximate weak amenability, derivations and Arens regularity of Segal algebras.* Studia Math.**169**(2005), no. 2, 189–205. doi:10.4064/sm169-2-6

[13] F. Ghahramani and R. J. Loy, *Generalized notions of amenability.* J. Funct. Anal. **208**(2004), no. 1, 229–260. doi:10.1016/S0022-1236(03)00214-3

[14] F. Ghahramani and R. Stokke, *Approximate and pseudo-amenability of the Fourier algebra.* Indiana Univ. Math. J. **56**(2007), no. 2, 909–930. doi:10.1512/iumj.2007.56.2951

[15] F. Ghahramani and Y. Zhang, *Pseudo-amenable and pseudo-contractible Banach algebras.* Math. Proc. Cambridge Philos. Soc. **142**(2007), no. 1, 111–123. doi:10.1017/S0305004106009649

[16] A. Ya. Helemskii, *Banach and locally convex algebras.* The Clarendon Press, Oxford University Press, New York, 1993.

[17] B. E. Johnson, *Cohomology in Banach algebras.* Memoirs of the American Mathematical Society, 127, American Mathematical Society, Providence, RI, 1972.

[18] ———, *Approximate diagonals and cohomology of certain annihilator Banach algebras.* Amer. J. Math. **94**(1972), 685–698. doi:10.2307/2373751

[19] ———, *Non-amenability of the Fourier algebra of a compact group.* J. London Math. Soc. (2) **50**(1994), no. 2, 361–374.

[20] J. L. Kelley, *General topology.* D. Van Nostrand Company, Inc., Toronto–New York–London, 1955.

[21] E. Kotzmann and H. Rindler, *Segal algebras on non-abelian groups.* Trans. Amer. Math. Soc. **237**(1978), 271–281. doi:10.2307/1997622

[22] A. L. T. Paterson, *Amenability.* Mathematical Surveys and Monographs, 29, American Mathematical Society, Providence, RI, 1988.

[23] A. Yu. Pirkovskii, *Approximate characterizations of projectivity and injectivity for Banach modules.* Math. Proc. Cambridge Philos. Soc. **143**(2007), no. 2, 375–385. doi=10.1017/S0305004107000163

[24] G. Pisier, *Introduction to operator space theory.* London Mathematical Society Lecture Note Series, 294, Cambridge University Press, Cambridge, 2003.

[25] P. F. Renaud, *Invariant means on a class of von Neumann algebras.* Trans. Amer. Math. Soc. **170**(1972), 285–291. doi:10.2307/1996311

[26] H. Reiter and J. D. Stegeman, *Classical Harmonic Analysis and locally compact groups.* Second ed., London Mathematical Society Monographs, 22, The Clarendon Press, Oxford University Press, New York, 2000.

[27] Z.-J. Ruan, *The operator amenability of $A(G)$.* Amer. J. Math. **117**(1995), no. 6, 1449–1474. doi:10.2307/2375026

[28] ———, *Connes-amenability and normal, virtual diagonals for measure algebras. I.* J. London Math. Soc. (2) **67**(2003), no. 3, 643–656. doi:10.1112/S0024610703004125

[29] N. Spronk, *Operator space structure on Feichtinger's Segal algebra.* J. Funct. Anal. **248**(2007), no. 1, 152–174. doi:10.1016/j.jfa.2007.03.028

[30] M. Takesaki and N. Tatsuuma, *Duality and subgroups. II.* J. Funct. Anal. **11**(1972), 184–190. doi:10.1016/0022-1236(72)90087-0

[31] P. J. Wood, *The operator biprojectivity of the Fourier algebra.* Canad. J. Math. **54**(2002), no. 5, 1100–1120.

[32] Y. Zhang, *Nilpotent ideals in a class of Banach algebras.* Proc. Amer. Math. Soc. **127**(1999), no. 11, 3237–3242. doi:10.1090/S0002-9939-99-04896-0

[33] ———, *Approximate identities for ideals of Segal algebras on a compact group.* J. Funct. Anal. **191**(2002), no. 1, 123–131. doi:10.1006/jfan.2001.3872

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