Canad. Math. Bull. Vol. **25** (2), 1982

DISCRETE SETS AND DISCRETE MAPS

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ABSTRACT. A subset of a topological space is called discrete iff every point in the space has a neighborhood which meets the set in at most one point. Discrete sets are useful for decomposing the images of certain maps and for generalizing closed maps. All discrete sets are closed iff the space is T_1. As a result of characterizing discrete and countably discrete maps, theorems due to Vaĭnšteĭn and Engelking are extended to these maps.

In [A] it is proved that certain image spaces Y from a closed map f can be decomposed $Y=\bigcup_{n=0}^{\infty} Y_n$ where Y_n is discrete in Y for all $n\geq 1$ and $f^{-1}(y)$ is compact for all y in Y_0. Tani [T] showed that the same decomposition holds for discrete maps which are a generalization of closed maps. We extend Engelking's technique [E] for characterizing closed maps to obtain characterizations of discrete and countably discrete maps. Also, we prove that the set of points for which a map is discrete or countably discrete is a G_δ*-set.*

An arbitrary (countable) set in a topological space is called *discrete* (*countably discrete*) iff every point in the space has a neighborhood which meets the set in at most one point.

Since $\{x\}$ is discrete, it follows that a topological space is T_1 iff each discrete set has no limit points. Likewise, a space is T_1 iff each of its discrete sets is closed. Since each subset of a discrete set is discrete, in a T_1-space a set is discrete iff all of its subsets are closed. Thus, in a T_1-space a set is discrete iff it is closed and the relative topology on it is the discrete topology. Furthermore, a set is closed and discrete iff all of its subsets are closed. Therefore, a set is closed and discrete iff all of the supersets of its complement are open.

Since this last concept will be used to characterize discrete maps and determine when discrete maps are closed, we call a set in a topological space *superopen* iff all of its supersets are open; i.e., if W is a set in a space X, then W is superopen iff whenever $W\subset V\subset X$, it follows that V is open in the space. A set is called ω*-superopen* iff it is superopen and its complement is countable.

A function $f:X\to Y$ is called a *discrete* (*countably discrete*) map iff for every discrete (countably discrete) closed set A in X, $f(A)$ is closed in Y. We do not assume that a discrete map is continuous or onto, as Tani has. It is easily

Received by the editors, June 11, 1980 and in revised form, October 25, 1980.
AMS Subj. Class.: Primary 54C10, 54A25.
Key Words: discrete set, countably discrete set, generalizations of closed map.

https://doi.org/10.4153/CMB-1982-035-2 Published online by Cambridge University Press

verified that if X is a T_1-space and if $f: X \to Y$ is a countably discrete map onto Y, then Y is a T_1-space. If X is a T_1-space, $f: X \to Y$ is a discrete map and A is a discrete (countably discrete) subset of X, then $f(A)$ is a discrete (countably discrete) subset of Y.

Let $f: X \to Y$ be a function. Denote by $C(f)$ ($D(f), E(f)$) the subset of Y consisting of all points y such that for every open (superopen, ω-superopen) $W \subset X$ containing $f^{-1}(y)$, there is a neighborhood $N_y \subset Y$ satisfying $f^{-1}(N_y) \subset W$. Engelking [E] showed that f is closed iff $C(f) = Y$. Clearly $C(f) \subset D(f) \subset E(f)$. We note that if f maps a space X into a space Y, then f is a discrete (countably discrete) map iff $D(f) = Y$ ($E(f) = Y$). Also, if f is a continuous countably discrete map of a T_1-space into a sequential T_2-space, then f is a closed map and $C(f) = D(f) = E(f)$. Since a first countable space is a sequential space, it follows that we are able to state two theorems of Vaĭnšteĭn (see [E]) in terms of countable discrete maps.

THEOREM 1. *For every continuous mapping $f: X \to Y$ of a complete metric space X into a first countable T_2-space Y and for any set $A \subset X$ such that $f \mid A : A \to f(A)$ is countably discrete, there exists a G_δ-set $B \subset X$ such that $A \subset B$ and $f \mid B : B \to f(B)$ is closed.*

THEOREM 2. *The image of a complete metric space under a continuous countably discrete map has a complete metric if it is a first countable T_2-space.*

For the remainder of this paper we assume that $f: X \to Y$ is a continuous mapping of a metric space X with metric ρ onto a first countably T_2-space Y.

THEOREM 3. *For every $y \in E(f)$, the boundary of $f^{-1}(y)$ is compact.*

Proof. Denote the boundary of $f^{-1}(y)$ by F. Let $\{V_n : n = 1, 2, \ldots\}$ be a base at y and let $A = \{x_1, x_2, \ldots\}$ be a countably infinite subset of F. Since $x_n \in F$, it follows that $x_n \in \overline{X - f^{-1}(y)}$, but $x_n \notin X - f^{-1}(y)$ as $f^{-1}(y)$ is closed and thus contains F. So for every n we can choose a point $z_n \in f^{-1}(V_n) - f^{-1}(y)$ such that $\rho(x_n, z_n) < 1/n$. Set $Z = \{z_1, z_2, \ldots\}$. As $y \in E(f)$, the set $X - Z \supset f^{-1}(y)$ is not superopen. Hence there is a set $B \subset Z$ which is not closed. Therefore $\emptyset \neq B' \subset Z'$, and so $A' \neq \emptyset$. Thus F is compact.

COROLLARY. *For every $y \in D(f)$, the boundary of $f^{-1}(y)$ is compact.*

Let $W_i(f)$ denote for $i = 1, 2, \ldots$ the subset of Y consisting of all points $y \in Y$ which have a neighborhood $N_y \subset Y$ such that every set $K \subset f^{-1}(N_y)$ satisfying the conditions

(1) $\rho(s, t) \geq 1/i$ and $f(s) \neq f(t)$ for distinct $s, t \in K$

is finite. Clearly, the sets $W_i(f)$ are open.

LEMMA 1. *$E(f) \subset W_i(f)$ for $i = 1, 2, \ldots$.*

https://doi.org/10.4153/CMB-1982-035-2 Published online by Cambridge University Press

Proof. If $a \in X$, then we designate $B(a, r) = \{x \in X : \rho(a, x) < r\}$. Suppose that there exists a point $y \in E(f) - W_i(f)$. Let $\{V_n\}_{n=1}^{\infty}$ be a base at y. Choose for $n = 1, 2, \ldots$ an infinite subset K_n of $f^{-1}(V_n)$ satisfying conditions (1). By the second part of (1), there exists an infinite set $L_n \subset K_n$ such that $f(x) \neq y$ for all $x \in L_n$ for $n = 1, 2, \ldots$. We will select a point $x_n \in L_n$ for $n = 1, 2, \ldots$ such that $A = \{x_1, x_2, \ldots\}$ is discrete. Choose $x_1 \in L_1$. Having chosen $x_k \in L_k$ satisfying $\rho(x_j, x_k) \geq 1/2i$ for $1 \leq j < k$, then every open ball $B(x_j, 1/2i)$ for $1 \leq j \leq k$ contains at most one member of L_{k+1}. Therefore we can choose $x_{k+1} \in L_{k+1}$ such that $\rho(x_j, x_{k+1}) \geq 1/2i$ for $1 \leq j \leq k$. Thus for every $x \in X$, the open ball $B(x, 1/4i)$ contains at most one member of A and so A is discrete. It follows that $W = X\text{-}A$ is ω-superopen and contains $f^{-1}(y)$. As $y \in E(f)$, we have $f^{-1}(V_n) \subset W$ for some n. But this is impossible, because $x_n \in A \cap f^{-1}(V_n)$.

Engelking [E, Lemma 3] proved that:

Lemma 2. *If the metric ρ is complete, then $\bigcap_{i=1}^{\infty} W_i(f) \subset C(f)$.*

Theorem 4. *If f is a continuous mapping of a complete metric space X onto a first countable T_2-space Y, then $C(f) = D(f) = E(f)$ and is a G_δ-set in Y.*

Proof. This result is a consequence of Lemmas 1 and 2.

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