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A MAP OF A POLYHEDRON ONTO A DISK

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A map $f:X \to Y$ is said to be universal if for every map $g:X \to Y$ there exists an $x \in X$ such that $f(x)=g(x)$. In [2] W. Holsztyński observed that if B is a Boltyanskiĭ continuum (see [1]), then there exists a universal map $f:B \to I^2$ such that the product map $f \times f: B \times B \to I^2 \times I^2$ is not universal. Using this he showed that B can be replaced by a two-dimensional polyhedron. He did not, however, give a concrete example. We exhibit explicitly a two-dimensional polyhedron K and a universal map $f:K \to I^2$ such that $f \times f: K \times K \to I^2 \times I^2$ is not universal.

Consider the annulus $S^1 \times I$, with boundary consisting of the circles $S^1 \times \{0\}$ and $S^1 \times \{1\}$. Identify every four points of the circle $S^1 \times \{0\}$ which divide it into four equal arcs, and identify every two points of the circle $S^1 \times \{1\}$ which divide it into two equal arcs. Let K denote the polyhedron obtained from $S^1 \times I$ with these identifications. (K is called a "leaf of degree one" in [1]).

Define a map $f:K \to I^2$ as follows: f maps the image of $S^1 \times \{0\}$ in K to the centre of I^2, f maps the image of $S^1 \times \{1\}$ in K homeomorphically to the boundary $\dot{I}^2$ of I^2, and f maps the image of a radial line segment from $S^1 \times \{0\}$ to $S^1 \times \{1\}$ in K to a radial line segment from the centre to the boundary of I^2.

Proposition 1. *The map $f:K \to I^2$ is a universal map.*

Proposition 2. *The map $f \times f: K \times K \to I^2 \times I^2$ is not a universal map.*

Lemma 1. *A map $f:X \to I^n$ is not universal if and only if there is an extension F of the map $f \mid f^{-1}(\dot{I}^n): f^{-1}(\dot{I}^n) \to \dot{I}^n$ to all of X.*

Proof. If f is not a universal map then there exists a map $g:X \to I^n$ such that $f(x) \neq g(x)$ for all $x \in X$. Construct a directed line segment from $g(x)$ through $f(x)$, intersecting $\dot{I}^n$ at $F(x)$. Then F is the desired extension. If F is such an extension, let $h:\dot{I}^n \to \dot{I}^n$ be the antipodal map. Then $(h \circ F)(x) \neq f(x)$ for all $x \in X$, and thus f is not universal.

Proof of Proposition 1. Let $A = f^{-1}(\dot{I}^2)$. By Lemma 1 it suffices to show that we cannot extend $f \mid A: A \to \dot{I}^2$ to a map $F:K \to \dot{I}^2$. For if such an extension

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existed, we would have a commutative homology triangle

$$\begin{array}{ccc} H_1(A;Z_4) & \xrightarrow{i_*} & H_1(K;Z_4) \\ & {\scriptstyle (f|A)_*}\searrow & \downarrow {\scriptstyle F_*} \\ & & H_1(\dot{I}^2;Z_4) \end{array}$$

Using the cell structure of K pictured in the proof of the following Lemma 3, $i_*[2e_1^1]=i_*[2e_1^1+4e_2^1]=i_*[\partial e_1^2]=0$. Since $(f\mid A)_*[e_1^1]=[f_1^1]$, this would imply that $2[f_1^1]=0$, a contradiction.

Proposition 2 is an immediate consequence of the following two lemmas. Let $C=(f\times f)^{-1}(\dot{I}^4)$, and let s^* be a generator of $H^3(S^3)$ (we use integral coefficients).

Lemma 2. *$\delta(f\times f\mid C)^*(s^*)=0$ in $H^4(K\times K, C)$ if and only if $f\times f$ is not universal.*

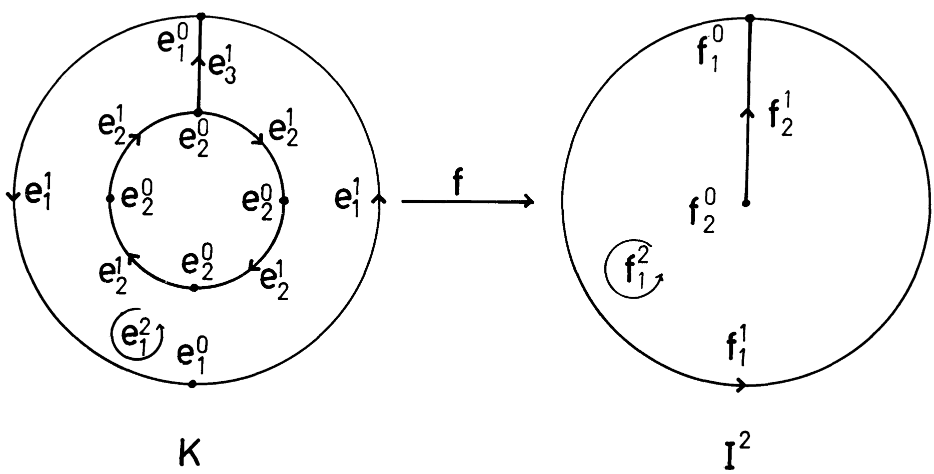
Proof. By the Hopf extension theorem (see Spanier, [4]), $\delta(f\times f\mid C)^*(s^*)=0$ if and only if the map $f\times f\mid C$ can be extended over $K\times K$. The lemma then follows from Lemma 1.

Lemma 3. *$\delta(f\times f\mid C)^*(s^*)=0$ in $H^4(K\times K, C)$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} H^3(K\times K) & \xrightarrow{i^*} & H^3(C) & \longrightarrow & H^4(K\times K, C) \\ & & \uparrow {\scriptstyle (f\times f\mid C)^*} & & \\ & & H^3(S^3) & & \end{array}$$

Since $\operatorname{Ker}\delta=\operatorname{Im} i^*$, it suffices to show that $(f\times f\mid C)^*(s^*)$ is in $\operatorname{Im} i^*$. Give K and I^2 the cell structure indicated below (K is a regular cell complex with identifications and the arrows give the orientations of the cells).



Then $f_*: C_*(K) \to C_*(I^2)$ maps e_1^0 to f_1^0, e_2^0 to f_2^0, e_1^1 to f_1^1, e_2^1 to 0, e_3^1 to f_2^1, and e_1^2 to $2f_1^2$. Choose ordered bases of oriented cells: $\{\alpha_1, \alpha_2\}$ for $C_3(C)$, $\{\beta_1, \ldots, \beta_7\}$ for $C_2(C)$, $\{\gamma_1\}$ for $C_4(K\times K)$, $\{\delta_1, \ldots, \delta_6\}$ for $C_3(K\times K)$, $\{\varepsilon_1, \varepsilon_2\}$ for $C_3(S^3)$, and $\{\phi_1, \ldots, \phi_5\}$ for $C_2(S^3)$. For appropriate choices of these bases we have

$$\partial\alpha_1 = 2\beta_1 \qquad \partial\alpha_2 = 4\beta_2 \qquad \partial\gamma_1 = 2\delta_1 \qquad \partial\varepsilon_1 = \phi_1 \qquad \partial\varepsilon_2 = -\phi_1$$

and

$$i_*(\alpha_1) = \delta_1 - \delta_4 - 2\delta_5 - 2\delta_6 \qquad i_*(\alpha_2) = -\delta_1 + 2\delta_5 + 2\delta_6$$

$$(f\times f)_*(\alpha_1) = 2\varepsilon_1 \qquad (f\times f)_*(\alpha_2) = -2(\varepsilon_1 + \varepsilon_2)$$

Let $(k_1, \ldots, k_n)$ in $\mathrm{Hom}(Z^n, Z)$ denote the homomorphism which multiplies the ith component by k_i. Then a generator s^* of $H^3(S^3)$ is given by $[(1, 0)]$, and hence $(f\times f \mid C)^*(s^*) = [(2, -2)]$. This element is not zero in $H^3(C)$: for $(k_1, \ldots, k_7) = (2k_1, 4k_2) = (2, -2)$ for any choice of $(k_1, \ldots, k_7)$ in $\mathrm{Hom}(C_2(C), Z)$. However, if $(k_1, \ldots, k_6) \in \mathrm{Hom}(C_3(K\times K), Z)$, then $i^*(k_1, \ldots, k_6) = (k_1 - k_4 - 2k_5 - 2k_6, -k_1 + 2k_5 + 2k_6)$. Since $\delta(k_1, \ldots, k_6)(\gamma_1) = 2k_1$, $(k_1, \ldots, k_6)$ represents a cocycle in $\mathrm{Hom}(C_3(K\times K), Z)$ if and only if $k_1 = 0$. Hence $i^*[(k_1, \ldots, k_6)] = [(2, -2)]$ if and only if $k_1 = 0$, $k_4 = 0$, and $k_5 + k_6 = -1$. In particular, $i^*[(0, 0, 0, 0, 0, -1)] = [(2, -2)] = (f\times f \mid C)^*(s^*)$ and hence $(f\times f \mid C)^*(s^*) \in \mathrm{Im}\, i^*$.

Since K can be imbedded in R^4, taking the double of a closed regular neighbourhood of K in R^4 leads to a concrete example of a closed 4-manifold M^4 and a universal map $g: M^4 \to I^2$ such that $g\times g: M^4\times M^4 \to I^2\times I^2$ is not a universal map (compare [3]).

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