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Double covers of graphs

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A projection morphism $\rho : G_1 \to G_2$ of finite graphs maps the vertex-set of G_1 onto the vertex-set of G_2, and preserves adjacency. As an example, if each vertex v of the dodecahedron graph D is identified with its unique antipodal vertex $\bar{v}$ (which has distance 5 from v) then this induces an identification of antipodal pairs of edges, and gives a (2:1)-projection $p : D \to P$ where P is the Petersen graph.

In this paper a category-theoretical approach to graphs is used to define and study such double cover projections. An upper bound is found for the number of distinct double covers $\rho : G_1 \to G_2$ for a given graph G_2. A classification theorem for double cover projections is obtained, and it is shown that the n-dimensional octahedron graph $K_{2,2,\ldots,2}$ plays the role of universal object.

1. Introduction

Related to any graph G (finite, undirected, without loops or multiple edges) is a set $D(G)$ of graphs each having twice as many vertices and edges as G, and each having a two-fold projection onto G which 'preserves local structure'. Such a 'double cover concept' is considered by Biggs [1, §19] (using group-actions) and Massey [7, VI.7] (using topological spaces).

For a simple way of making the idea precise, and leading to a classification theorem, it is convenient to work in the category *Graph*

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whose *objects* are graphs and whose *morphisms* are adjacency-preserving maps between graphs. Thus $f : G \to H$ is a morphism if $f(v)$ is adjacent to $f(w)$ in H whenever v is adjacent to w (denoted $v \sim w$) in G. It follows that a morphism f sends an edge $[v, w]$ of G to an edge $[f(v), f(w)]$ of H. A category-theoretical approach to graph theory enables us to define, study and classify double covers of graphs.

2. Localisation of Kronecker products of graphs

Our first type of double cover of G can now be dealt with using the fact that the category *Graph* has *products* (see for example Farzan and Waller [3] and the references therein). The *product* $G_1 \wedge G_2$ of two graphs G_1 and G_2 (often known as their *Kronecker product*) has vertex-set $V(G_1 \wedge G_2)$ equal to the cartesian product $V(G_1) \times V(G_2)$ of the vertex-sets of the given graphs, with adjacency in $G_1 \wedge G_2$ given by

$$(v_1, v_2) \sim (w_1, w_2) \text{ if (and only if)}$$
$$v_1 \sim w_1 \text{ in } G_1 \text{ and } v_2 \sim w_2 \text{ in } G_2 .$$

In particular, taking the complete graph K_2 with vertices denoted 0 and 1 to play the role of G_2, we can associate to any graph G the graph $G \wedge K_2$. This bears the required relationship to G, and we shall call it the *Kronecker double cover* of G. The projection morphism $p : G \wedge K_2 \to G$ is defined by

$$\left.\begin{array}{l}(v, 0)\\ \\ (v, 1)\end{array}\right\} \mapsto v$$

on vertices, and this induces a (2:1) map on edges:

$$\left.\begin{array}{l}[(v, 0), (w, 1)]\\ \\ [(v, 1), (w, 0)]\end{array}\right\} \mapsto [v, w] .$$

The Kronecker double cover of an odd circuit C_{2n+1} is the circuit C_{4n+2} An illustrative example is K_4, whose Kronecker double cover is

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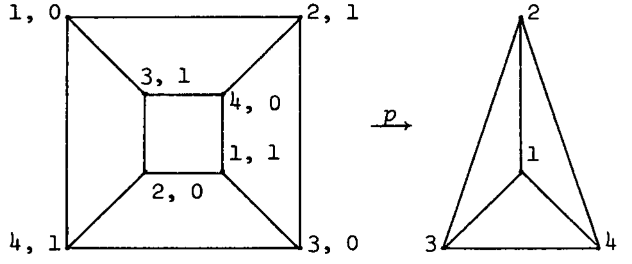
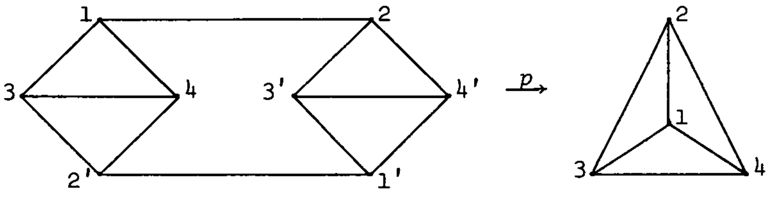
the 3-cube graph Q_3 :



A second form of 'double cover concept' is the 'trivial' case of 2-fold projections $G \amalg G \to G$ involving the *disjoint union* of two copies of G .

Thirdly, in between these two extremes, we must also allow for 'hybrids' such as



(where the projection morphism is given by $v \mapsto v$, $v' \mapsto v$).

The Kronecker double cover and trivial double cover of G are isomorphic if and only if G is bipartite. In particular for star-graphs $K_{1,d}$ we have $K_{1,d} \wedge K_2 \cong K_{1;d} \amalg K_{1,d}$.

In order to coordinate these examples, and also to include (and generalise) the 'antipodal double covers' of Smith [8] and Biggs [1, p. 151], one can 'localise' the definition of Kronecker double cover:

DEFINITION. A *double cover* of a graph G consists of a graph D together with a morphism $p : D \to G$ satisfying

(i) for each $v \in V(G)$, $p^{-1}(v) = \{v^1, v^2\}$,

(ii) if $v \sim w$ in G , then in D , *either*

(a) $v^1 \sim w^1$ and $v^2 \sim w^2$, *or*

(b) $v^1 \sim w^2$ and $v^2 \sim w^1$.

NOTE. The definition ensures that the *double cover projection* p is a (2:1) epimorphism not only on vertices but also on edges. Where there is no danger of confusion we shall abbreviate $p : D \to G$ by either p or D as appropriate.

One can regard work on the automorphism group of G as the study of (1:1) morphisms onto G ; here we are studying (2:1) morphisms onto G .

PROPOSITION 2.1. *If $p : D \to G$ is a double cover projection, then the vertex v has degree d in G if and only if the two associated vertices v^1 and v^2 in D both have degree d .*

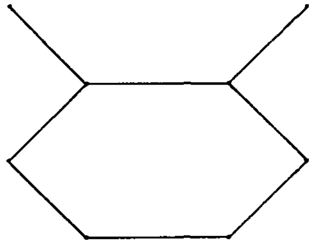
Proof. v is adjacent to the vertices $v_1, \ldots, v_d$ in G if and only if in D the vertices v^1 and v^2 are each adjacent to exactly one vertex v_i^1 or v_i^2 corresponding to each v_i , $i = 1, \ldots, d$, and the result follows. □

COROLLARY 2.2. *If G is regular with n vertices of degree d , then every double cover D of G is regular with $2n$ vertices of degree d .*

Thus, considering the star-graph $K_{1,d}$ as a closed neighbourhood of the vertex v in G , we have in each double cover D of G a disjoint union $p^{-1}(K_{1,d}) \cong K_{1,d} \amalg K_{1,d}$, as the subgraph of D which projects onto the subgraph $K_{1,d}$ of G .

Thus *locally*, any double cover is isomorphic to a Kronecker double cover. (In the category of topological spaces, one also meets spaces which are *local* products, and the concept of *fibre bundle* is an appropriate form of localisation of that product (see [6]).) In our analogous graph-theoretical construction, the 'fibre' $p^{-1}(v)$ over the vertex v consists of two vertices.

PROBLEM. Characterise those graphs D for which there exists a graph G such that D is a double cover of G . Clearly D must have an even number of vertices and of edges; this is not sufficient, since the graph



is not a double cover of any graph G .

3. Antipodal double covers

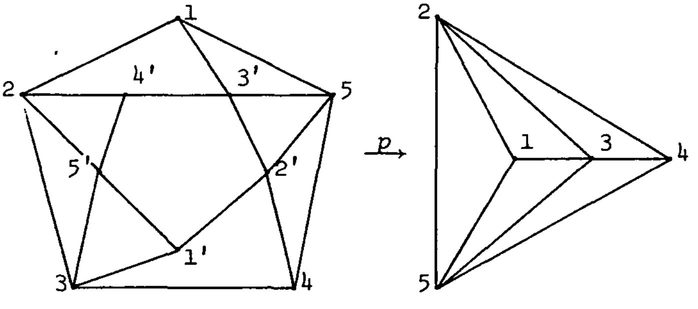
An important class of examples of double covers is given by the antipodal double covers:

DEFINITIONS. (i) The *distance* $\partial(v, w)$ of v from w in G is the number of edges in a shortest path from v to w .

(ii) The *diameter* $d = \operatorname{diam} G$ of G is $\max\limits_{v,w} \partial(v, w)$.

(iii) A double cover projection $p : D \to G$ is called *antipodal* if the *fibres* $p^{-1}v$ in D consist of antipodal pairs of vertices; that is, $\partial\left(v^1, v^2\right) = \operatorname{diam} G$ if and only if $p\left(v^1\right) = p\left(v^2\right)$.

This concept (and more generally n-fold antipodal covers) was first studied in the context of distance-transitive graphs by Smith [8], and has been extended by Gardiner [4]. They refer to G in such cases as the *derived graph* of D . We do not restrict attention to regular graphs. In fact some irregular graphs have an antipodal double cover in $D(G)$, for example:



It should also be noted that $D(G)$ may contain non-isomorphic antipodal graphs. (The reader is invited to construct *three* different antipodal double covers of Petersen's graph.)

As an aid to the construction of antipodal double covers, recall (Harary [5], p. 22) the *cartesian product* $G_1 \times G_2$ of the graphs G_1 and G_2 has vertex-set $V(G_1) \times V(G_2)$, with $(u_1, u_2) \sim (v_1, v_2)$ if and only if $[u_1 = v_1$ and $u_2 \sim v_2]$ or $[u_1 \sim v_1$ and $u_2 = v_2]$. The set of all antipodal double covers is closed under cartesian product.

THEOREM 3.1. *If the graph* D_i *is an antipodal double cover of* G_i *with* $\bar{v}_i$ *antipodal to* v_i *in* D_i $(i = 1, 2)$, *then there exists a graph* $G_1 \tilde{\times} G_2$ *over which* $D_1 \times D_2$ *is an antipodal double cover, with the "product vertex"* (v_1, v_2) *antipodal to* $(\bar{v}_1, \bar{v}_2)$.

Proof. Now we have $\operatorname{diam}(D_1 \times D_2) = \operatorname{diam} D_1 + \operatorname{diam} D_2$. It is clear that the product-projection maps

$$p_i : D_1 \times D_2 \to D_i \quad (i = 1, 2)$$

given by

$$(v_1, v_2) \mapsto v_i$$

are not morphisms. However, they do project paths in $D_1 \times D_2$ to paths in D_i $(i = 1, 2)$. What is more p_i projects a minimal path from (v_1, v_2) to $(\bar{v}_1, \bar{v}_2)$ in $D_1 \times D_2$ to a minimal path from v_i to $\bar{v}_i$ in D_i. It follows that the length of a minimal path from (v_1, v_2) to $(\bar{v}_1, \bar{v}_2)$ in $D_1 \times D_2$ is equal to the sum of the lengths of the projected paths in D_1 and D_2, which is equal to $\operatorname{diam}(D_1 \times D_2)$.

It is also clear that no vertex other than $(\bar{v}_1, \bar{v}_2)$ is so far from (v_1, v_2) in $D_1 \times D_2$ (for if such existed, then projection by p_i would contradict the antipodality of $\bar{v}_i$ to v_i in D_i). Thus each vertex of $D_1 \times D_2$ has a unique antipodal vertex, given by $\overline{(v_1, v_2)} = (\bar{v}_1, \bar{v}_2)$.

Finally, the graph $G_1 \tilde{\times} G_2$ of which $D_1 \times D_2$ is an antipodal double

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cover is derived in the obvious way. We have

$$\begin{aligned}\overline{(v_1, v_2)} \sim \overline{(w_1, w_2)} &\Leftrightarrow \bar{v}_1 \sim_{D_1} \bar{w}_1 \text{ and } \bar{v}_2 \sim_{D_2} \bar{w}_2 \\ &\Leftrightarrow v_1 \sim_{D_1} w_1 \text{ and } v_2 \sim_{D_2} w_2 \\ &\Leftrightarrow (v_1, v_2) \sim (w_1, w_2) .\end{aligned}$$

Thus $G_1 \tilde{\times} G_2$ has a vertex for each antipodal pair of vertices in $D_1 \times D_2$, and an edge corresponding to each antipodal pair of edges in $D_1 \times D_2$. □

4. Pullbacks of double covers

DEFINITION. Given any two graph-morphisms:

$$\begin{tikzcd} & D \arrow[d, "p"] \\ H \arrow[r, "\alpha"] & G \end{tikzcd}$$

the *graph* D_α *induced by* α *from* D has vertex-set

$$V(D_\alpha) = \{(h, d) \in V(H) \times V(D) : \alpha(h) = p(d)\}$$

and adjacency

$$(h, d) \sim (h', d') \Leftrightarrow h \sim_H h' \text{ and } d \sim_D d' .$$

The *morphism* $p_\alpha : D_\alpha \to H$ *induced by* α *from* p is given by $(h, d) \mapsto h$. Thus we have a (commutative) *pullback diagram* in the category *Graph*:

$$\begin{tikzcd} D_\alpha \arrow[r, "\alpha_p"] \arrow[d, "p_\alpha"] & D \arrow[d, "p"] \\ H \arrow[r, "\alpha"'] & G \end{tikzcd}$$

where α_p is given by $(h, d) \mapsto d$.

PROPOSITION 4.1. *If* $p : D \to G$ *is a double cover projection, then so is* $p_\alpha : D_\alpha \to H$.

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Proof. For each $h \in V(H)$, we have that $p^{-1}(\alpha(h))$ consists of two vertices, $\alpha(h)^1$ and $\alpha(h)^2$. Thus $p_\alpha^{-1}(h) = \{(h, \alpha(h)^1), (h, \alpha(h)^2)\}$. These two points constituting the fibre over h are not adjacent, since $\alpha(h)^1$ and $\alpha(h)^2$ are not adjacent in D . Axiom (ii) for p_α to be a double cover follows from the corresponding condition for p . □

EXAMPLE. If $\alpha : H \subset G$ is a subgraph-inclusion, then the double cover $p_\alpha : D_\alpha \to G$ is *isomorphic* to the restriction $p|p^{-1}H : p^{-1}H \to H$.

If $p : D \to G$ is a Kronecker double cover then so is $p_\alpha : D_\alpha \to H$ (see [3]).

The concept of isomorphism of double covers which is needed here is defined as follows (see Djoković [2]).

DEFINITIONS. (i) A *morphism* f *of double covers* over G is a commutative diagram

$$\begin{tikzcd} D_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & D_2 \arrow[dl, "p_2"] \\ & G & \end{tikzcd}$$

(ii) f is called an *isomorphism of double covers* if also there is a morphism $g : D_2 \to D_1$ such that $g.f = 1_{D_1}$ and $f.g = 1_{D_2}$.

Isomorphism is an equivalence relation (denoted $D_1 \cong D_2$); let $[p]$ denote the isomorphism class represented by $p : D \to G$, and let $\mathcal{D}(G)$ denote the set of isomorphism classes of double covers of G .

Let $\mathcal{S}et$ denote the category of sets. Proposition 4.1 can now be applied to give our first structure-theorem.

THEOREM 4.2. *There is a contravariant functor* $\mathcal{D} : \mathcal{G}raph \to \mathcal{S}et$.

Proof. First we observe that a graph-morphism $\alpha : H \to G$ induces a set-function

$$\mathcal{D}(\alpha) : \mathcal{D}(G) \to \mathcal{D}(H)$$

given by

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$$\begin{array}{ccc} D & & D_\alpha \\ \downarrow p & \mapsto & \downarrow p_\alpha \\ G & & G \end{array}$$

We define $\mathcal{D}(\alpha)(p) = p_\alpha$ (using the above pullback construction).

The conditions for $\mathcal{D}$ to be a contravariant functor are now easily verified:

(i) if α is the identity morphism, then the induced double cover is isomorphic to p itself. Thus $\mathcal{D}(1_G) = 1_{\mathcal{D}(G)}$;

(ii) $\mathcal{D}$ respects (but reverses) composite morphisms.

If $K \xrightarrow{\beta} H \xrightarrow{\alpha} G$, then it is easily verified that $\mathcal{D}(\alpha\beta)(p) = p_{\alpha\beta} = \mathcal{D}(\beta)\mathcal{D}(\alpha)(p)$. □

Clearly for all G , $\mathcal{D}(G)$ contains the *trivial double cover* which we shall denote by $G \amalg G$ or p_0 . In the case of a tree, this is the only possibility.

PROPOSITION 4.3. *If T is a tree then $\mathcal{D}(T)$ consists of $[p_0]$ alone.*

Proof. Since T is bipartite, there is a morphism $\alpha : T \to K_2$. The graph K_2 only has the trivial double cover, and the absence of circuits from T implies that the pullback diagram

$$\begin{tikzcd} T \amalg T \arrow[r] \arrow[d, "p_0"] & K_2 \amalg K_2 \arrow[d, "p_0"] \\ T \arrow[r, "\alpha"'] & K_2 \end{tikzcd}$$

gives the only possibility over T (with $(K_2 \amalg K_2)_\alpha \cong T \amalg T$). □

In particular, this applies to spanning trees, and we can use Proposition 4.3 to study non-trivial double covers. In the following, let G be a fixed arbitrary, connected graph, and let $T \subset G$ be a spanning tree. Of course, a choice is involved here, and this will affect our labelling, but the main results will be seen to be independent of this choice of spanning tree.

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Suppose G has n vertices and m edges, whence the cutset rank $\kappa(G)$ is equal to the number $n-1$ of edges in (any) spanning tree T. The circuit rank $\gamma(G)$ is the number $m-n+1$ of edges in $G-T$.

NOTATION 4.4. Throughout §4, we choose a spanning tree T for G, and consider the pullback with respect to the inclusion $T \subset G$. Then $T \amalg T$ is a subgraph of any double cover D of G. If $V(G) = v_1, \ldots, v_n$, then label the vertices of one copy T_1 of T as $v_1^1, \ldots, v_n^1$, and those of the other copy T_2 of T as $v_1^2, \ldots, v_n^2$.

Subject to this labelling, we shall call an edge $[v_i, v_j]$ of G *trivial* in $p : D \to G$ if $v_i^1 \sim v_j^1$ and $v_i^2 \sim v_j^2$ (in the definition of double cover) in D, and *non-trivial* if $v_i^1 \sim v_j^2$ and $v_i^2 \sim v_j^1$ in D.

LEMMA 4.5. *If $p : D \to G$ is any double cover projection, then D is disconnected if and only if p is isomorphic to the trivial projection p_0. In this case $\kappa(D) = 2(n-1)$ and $\gamma(D) = 2(m-n+1)$. In all other cases, D is connected, with $\kappa(D) = 2n-1$ and $\gamma(D) = 2m-2n+1$.*

Proof. Consider the spanning forest $T \amalg T$ for D as in 4.4. Clearly D is disconnected (and $D = G \amalg G$) if and only if every edge of $G-T$ is trivial (in which case the two copies of T are never linked). The rank-computations then follow immediately. □

PROBLEM. How many (non-isomorphic) double covers does a graph G have?

PROPOSITION 4.6. $1 \le |\mathcal{D}(G)| \le 2^{\gamma(G)}$.

Proof. Again consider the pullback and notation of 4.4. Every double cover D of G contains $T_1 \amalg T_2$ as a spanning forest. To reconstruct D we must add two edges corresponding to each edge $e \in G-T$. For each of these $\gamma(G)$ edges there are exactly two possibilities: either we make it trivial or non-trivial in p, and the result follows. □

NOTE. These $2^{\gamma(G)}$ double covers are not necessarily distinct (hence the inequality in Proposition 4.6). For example $\gamma(K_4) = 3$, but the

$2^3 = 8$ double covers can be partitioned into exactly *three* isomorphism-classes, namely those given in §2.

LEMMA 4.7. *The n-circuit* C_n *has exactly two non-isomorphic double covers.*

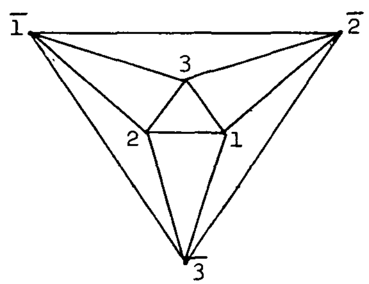
Proof. The (only) non-trivial one is given by the morphism $C_{2n} \to C_n$. □

THEOREM 4.8. *If* $\gamma(G) = 1$, *then* $|\mathcal{D}(G)| = 2$.

Proof. Here the 4.4 construction leaves one edge $e \in G-T$. There is one isomorphism class of connected double covers, corresponding to e being a non-trivial edge for p . □

5. Classification of double covers

The *n-dimensional octahedron* $K_{n(2)} = K_{\underbrace{2,\ldots,2}_{n \text{ times}}}$ is the complete n-partite graph which generalises the graph $K_{3(2)} = K_{2,2,2}$ of the Platonic solid octahedron:



For each natural number n , $K_{n(2)}$ is an antipodal graph but it is (understandably) excluded from study by Smith and Biggs because its small diameter of 2 causes its 'derived graph' to have the apparently undesirable feature of being a *multigraph* K_n^2 . (For any graph G , we denote by G^2 the associated *multigraph* with the same vertex-set as G , and each edge duplicated.)

We shall show that not only can this feature be turned to our

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advantage, but also that $K_{n(2)}$ is the most important double cover of all! (Of course K_n^2 is not in the category of graphs. However morphisms and pullbacks involving K_n^2 are defined as for graphs.)

We begin by labelling the $2n$ vertices of $K_{n(2)}$ as $\{v_1^1, \ldots, v_n^1, v_1^2, \ldots, v_n^2\}$ with v_i^2 antipodal to v_i^1. The $2n(n-1)$ edges of $K_{n(2)}$ are then of two types:

Type 1 of the form $[v_i^1, v_j^1]$ or $[v_i^2, v_j^2]$;

Type 2 of the form $[v_i^1, v_j^2]$.

There are $n(n-1)$ edges of each type. This provides a natural expression of $K_{n(2)}$ as an antipodal double cover of the multigraph K_n^2 which has a 'type 1 edge' $[v_i, v_j]$ derived from the two type 1 edges $[v_i^1, v_j^1]$ and $[v_i^2, v_j^2]$ and also a 'type 2 edge' $[v_i, v_j]$ derived from the two type 2 edges $[v_i^1, v_j^2]$ and $[v_i^2, v_j^1]$.

This classifies the two edges between any two vertices of K_n^2 in a useful way. If G and H have the *same* vertex-set, we denote by $G \cup H$ their *edge-disjoint union* which has again the same vertex-set, with all edges of both G and H . In particular, $K_n^2 = K_n \cup K_n$. This helps clarify the octahedron double cover.

THEOREM 5.1. $K_{n(2)} \cong (K_n \perp\!\!\!\perp K_n) \cup (K_n \wedge K_2)$.

Proof. The graphs $K_n \perp\!\!\!\perp K_n$ and $K_n \wedge K_2$ each have $2n$ vertices. They are respectively the trivial and Kronecker double covers of K_n . The double cover $K_{n(2)} \to K_n^2$ is the edge disjoint union of these two double covers of K_n , with the type 1 edges of $K_{n(2)}$ projecting (2:1) to type

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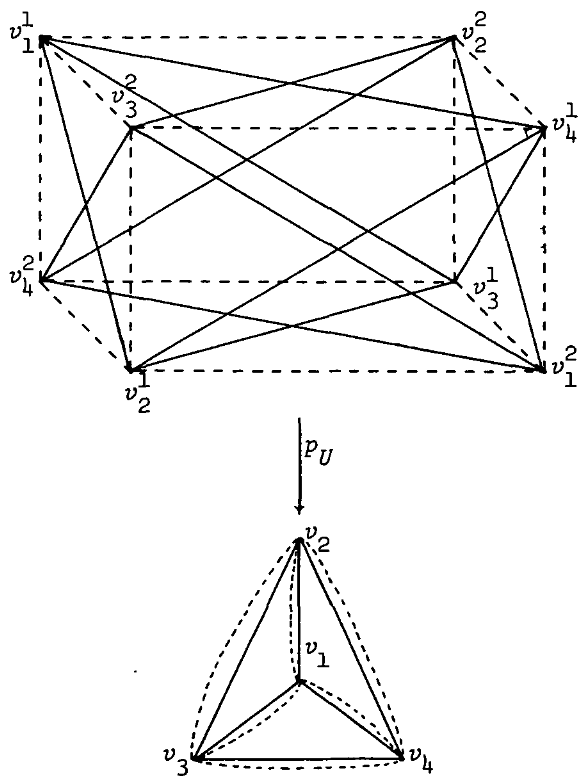
1 edges of K_n^2 as the trivial double cover, and the type 2 edges projecting (2:1) to the type 2 edges of K_n^2, as the Kronecker double cover $K_n \wedge K \to K_n$. □

EXAMPLE. The $n = 4$ case expresses the 4-dimensional octahedron $K_{4(2)}$ as the edge disjoint union of a cube Q_3 and two tetrahedra K_4, as illustrated by the antipodal double cover projection:



We now show that this double cover $p_U : K_{n(2)} \to K_n^2$ is 'universal' for all n-colourable graphs G in the sense that all double covers of G can be expressed as pullbacks (as in §4) of p_U with respect to a certain morphism.

LEMMA 5.2. *A graph* G *is* n*-colourable if and only if there is a morphism* $\alpha : G \to K_n$.

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Proof. A morphism $\alpha : G \to K_n$ sends adjacent vertices to adjacent vertices. If we give a vertex in G the same colour as $\alpha(v)$ has in some (fixed) n-colouring of K_n, this gives a valid n-colouring of G.

Conversely any given n-colouring of G defines a morphism α to K_n : simply colour K_n with the same n colours and let α preserve colours. □

Denoting by $\chi(G)$ the chromatic number of G, we have

PROPOSITION 5.3. *Every double cover D of a graph G satisfies $\chi(D) \leq \chi(G)$.*

Proof. Application of Lemma 5.2 to the composite $D \xrightarrow{p} G \xrightarrow{\alpha} K_n$ allows us to 'lift' an n-colouring of G to an n-colouring of D. □

COROLLARY 5.4. *If G is bipartite, then so is every double cover D of G.* □

We can now show that all double covers can be expressed as pullbacks of an octahedron.

THEOREM 5.5. *Let $p : D \to G$ be a double cover, with G n-colourable. Then there is a morphism $\alpha' : G \to K_n^2$ such that p is isomorphic to the double cover*

$$(p_U)_{\alpha'} : (K_{n(2)})_{\alpha'} \to G$$

induced by α' from the 'universal' double cover p_U.

Proof. Let $\alpha : G \to K_n$ be an n-colouring morphism for G, as in Lemma 5.2.

Define a morphism $\alpha' : G \to K_n^2$ by:

$$\alpha'(v) = \alpha(v) , \quad v \in V(G) ;$$

$$\alpha'(e) = \begin{cases} \alpha(e)^1 & \text{if } e \text{ is a trivial edge of } p , \\ \alpha(e)^2 & \text{if } e \text{ is a non-trivial edge of } p . \end{cases}$$

The pullback $(p_U)_{\alpha'}$ has the required property. □

There is a strong analogy between this use of the n-octahedron as a 'universal double cover' and the universal bundle (due to Milnor) for the contravariant fibre bundle functor (see for example Husemoller ([6], 4.11.1). Milnor's construction involves the *join* of n copies of the 0-sphere S^0, and $K_{n(2)}$ is indeed the graph-theoretical analogue of this. Similarly our multigraph K_2^n plays the role of *classifying object* for the functor $\mathcal{D}$.

The inclusion morphism of K_{n-1}^2 in K_n^2 induces inclusions of double covers:

$$\begin{array}{ccccccc} K_{1(2)} & \to & K_{2(2)} & \to & K_{3(2)} & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ K_1^2 & \to & K_2^2 & \to & K_3^2 & \to & \cdots . \end{array}$$

Finally, to give a full classification of double covers of graphs, we must investigate when two morphisms from G to K_n^2 induce isomorphic double covers. A suitable equivalence relation on such morphisms will correspond to that of homotopy in fibre bundle theory.

With Lemma 4.7 in mind, we define two morphisms $\alpha', \beta' : C_n \to K_n^2$ to have the *same parity* if $\alpha' = \beta'$ on vertices and either

(i) they both produce an even number of type 2 edges (in which case they both induce from p_U the trivial double cover $C_n \perp\!\!\!\perp C_n$); or

(ii) they both produce an odd number of type 2 edges (in which case they both induce from p_U the non-trivial double cover C_{2n}).

Then we define two morphisms $\alpha', \beta' : G \to K_n^2$ to be *equivalent* if K_n^2 has an automorphism φ such that β' and $\varphi.\alpha'$ have the same parity on every circuit of G. This is easily seen to be a proper equivalence relation; let $\left[G, K_n^2\right]$ denote the set of equivalence classes of morphisms from G to K_n^2.

THEOREM 5.6. *There is a* 1 : 1 *correspondence* $\mathcal{D}(G) \leftrightarrow \left[G, K_n^2\right]$, *for any* n*-colourable graph* G.

Proof. Double covers induced from the n-octahedron universal double cover $p_U : K_{n(2)} \to K_n^2$ are formed as in Theorem 5.5. The definition of equivalence of morphisms ensures that $(p_U)_{\alpha'}$ and $(p_U)_{\beta'}$ are isomorphic if and only if α' and β' are equivalent. The result then follows immediately. □

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