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KNOT SINGULARITIES OF HARMONIC MORPHISMS

PAUL BAIRD

Département de Mathématiques, Université de Bretagne Occidentale, 6 Avenue Le Gorgeu, BP 452, 29275 Brest Cedex, France
(Paul.Baird@univ-brest.fr)

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Abstract A harmonic morphism defined on $\mathbb{R}^3$ with values in a Riemann surface is characterized in terms of a complex analytic curve in the complex surface of straight lines. We show how, to a certain family of complex curves, the singular set of the corresponding harmonic morphism has an isolated component consisting of a continuously embedded knot.

Keywords: knot; knot singularity; harmonic morphism; algebraic curve

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Secondary 57M12; 58E20

1. Introduction

In complex variable theory the polynomial equation

$$P(w,z) \equiv z^2 - w = 0,$$

determines z as a 2-valued analytic function of w. The point $w = 0$ is called a *singular point*. It is a branch point of the function z, i.e. a point where the equation $P = 0$ has a multiple root. More generally, if $P(w,z)$ is analytic in both variables, a point $w \in \mathbb{C}$ is said to be *singular* if there exists $z \in \mathbb{C}$ such that

$$\begin{aligned} P(w,z) &= 0, \\ \frac{\partial P}{\partial z}(w,z) &= 0. \end{aligned}$$

The notion of *harmonic morphism* was introduced in the 1960s as a natural generalization of analytic functions in the plane [**4**, **5**]. In general terms, a harmonic morphism is a mapping that preserves harmonic functions. The study of these mappings in the context of Riemannian manifolds began with the work of Fuglede [**7**] and Ishihara [**10**]. Thus, let $\phi : M \to N$ be a continuous mapping of Riemannian manifolds. Then ϕ is called a *harmonic morphism* if, for every real-valued function f, harmonic on an open subset $V \subset N$ such that $\phi^{-1}(V)$ is non-empty, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V)$

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in M. The fundamental characterization [**7**, **10**] asserts that *$\phi : M \to N$ is a harmonic morphism if and only if ϕ is both horizontally conformal and harmonic.* (For background on harmonic mappings, see [**6**]. A map ϕ is said to be *horizontally conformal* if, for each $x \in M$, where $\mathrm{d}\phi_x \neq 0$, the restriction $\mathrm{d}\phi_x|_{(\ker \mathrm{d}\phi_x)^\perp} : (\ker \mathrm{d}\phi_x)^\perp \to T_{\phi(x)}N$ is conformal and surjective.)

Consequently, if $\phi : M \to N$ is a harmonic morphism between Riemannian manifolds [**7**], then

(i) *ϕ is smooth, i.e. C^∞*; and

(ii) *ϕ is an open mapping* (in particular $\dim M \geqslant \dim N$).

Hence, such mappings possess properties enjoyed by analytic functions.

Harmonic morphisms from domains in $\mathbb{R}^3$ with values in a Riemann surface were characterized by Baird and Wood [**3**] in the following way. Let $\phi : U \subset \mathbb{R}^3 \to N$ be a harmonic morphism to a Riemann surface N. Consider a point $x_0 \in U$, and let $\psi : V \subset N \to \mathbb{C}$ be a local chart about $\phi(x_0) \in N$. Set $z = \psi \circ \phi$. Then, in a neighbourhood of x_0, $z = z(x)$ is determined implicitly by

$$(1 - g(z)^2)x_1 + \mathrm{i}(1 + g(z)^2)x_2 - 2g(z)x_3 = 2h(z) \quad (x = (x_1, x_2, x_3)), \tag{1.1}$$

where $g(z), h(z)$ are meromorphic or anti-meromorphic functions of z. Conversely, any local solution z to (1.1) defined on an open subset $U \subset \mathbb{R}^3$ determines a harmonic morphism $z : U \to \mathbb{C}$. In particular,

(i) *the fibres are always line segments* (this is a consequence of a more general result of [**2**]); and

(ii) *the foliation by line segments extends to critical points of ϕ.*

By changing orientation on the codomain, we can always assume that g and h are *meromorphic* functions of z. If in addition we suppose they are rational functions, then equation (1.1) becomes a polynomial equation in z of the form

$$P(x, z) \equiv a_n(x)z^n + a_{n-1}(x)z^{n-1} + \cdots + a_1(x)z + a_0(x) = 0, \tag{1.2}$$

affine linear in x, which can be thought of as defining a *multivalued* harmonic morphism.

By analogy with analytic function theory, the *singular set* $K \subset \mathbb{R}^3$ is defined to be those points $x \in \mathbb{R}^3$ simultaneously satisfying

$$P = 0, \qquad \frac{\partial P}{\partial z} = 0.$$

Singular points are points where the polynomial P has a multiple root (i.e. the discriminant vanishes). They occur as the envelope points of the congruence of lines determined by (1.2). At such points, a branch z of the solution to (1.2) becomes singular (i.e.

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$|\mathrm{d}z|^2 \to \infty$). They can also occur as the inverse image of a critical point of a weakly conformal transformation of the codomain: since the composition of a harmonic morphism with a weakly conformal mapping is also a harmonic morphism; if, for example, we set $z = (w-a)^p$, $p \geqslant 2$, then

$$\begin{aligned}\frac{\partial P}{\partial w} &= \frac{\partial P}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}w} \\ &= p\frac{\partial P}{\partial z}(w-a)^{p-1}\end{aligned}$$

vanishes when $w = a$, and the fibre over $w = a$ determined by $P(x, w) = 0$ is singular.

Multivalued harmonic morphisms were considered in [**1**] in the compact case (replacing $\mathbb{R}^3$ by S^3, where a similar representation to (1.1) holds, see [**3**]). By taking an r-valued harmonic morphism defined on S^3, branched over two linked circles, it was shown how to construct the Lens spaces $L(r, 1)$ together with a single-valued harmonic morphism $\phi : L(r, 1) \to S^2$, by cutting and gluing in a similar fashion to the procedure for constructing compact Riemann surfaces from multivalued analytic functions [**12**].

This construction was put on a formal footing, at least in the $\mathbb{R}^3$ case (indeed $\mathbb{R}^m$) by Gudmundsson and Wood [**8**]. They showed how an equation of the type (1.2) determines a smooth submanifold $M \subset \mathbb{R}^3 \times \mathbb{C}$, which is a branched covering of $\mathbb{R}^3$, branched over the singular set K, together with a single-valued harmonic morphism $\phi : M \to N$. They proved that the singular set K is real analytic, and, apart from exceptional cases, consists of arcs of curves, joining points where the multiplicity of the roots of (1.2) increases.

A simple example is given by taking $g(z) = z$, $h(z) = \mathrm{i}z$. Then (1.2) becomes the polynomial equation

$$z^2(x_1 - \mathrm{i}x_2) + 2z(x_3 + \mathrm{i}) - (x_1 + \mathrm{i}x_2) = 0.$$

The solution z is a 2-valued harmonic morphism branched along the singular set K which is given by $x_3 = 0$, $x_1^2 + x_2^2 = 1$, i.e. the unit circle in the (x_1, x_2)-plane. The manifold M is diffeomorphic to $S^2 \times \mathbb{R}$, which double covers $\mathbb{R}^3$, branched over the circle K. There are very few other examples where the singular set has been constructed and in all such cases, it has a very simple structure.

Following the well-known topological construction, obtaining 3-manifolds as branched coverings over knots (see [**13**]), it becomes interesting to know whether it is possible to obtain a knot singularity to equation (1.2). In this paper, we prove that such singularities exist. Precisely, we give examples of g and h such that the singular set K has an isolated component consisting of a continuously embedded knotted curve. In fact, we exhibit a family of such parametrized by the odd integers $p = 3, 5, 7, \dots$, beginning with the trefoil knot ($p = 3$).

We consider a particular holomorphic curve $\mathcal{C}$ in $\mathbb{C}^2$ parametrized in the form

$$g = z^p, \qquad h = z^{p+2} + \mathrm{i}\beta z^p,$$

for β a real positive number. By estimating roots of a certain polynomial equation, we are able to demonstrate that a connected component of the singular set in $\mathbb{R}^3$ is a knot.

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2. The singular set of a harmonic morphism

Let $z : U \to \mathbb{C}$, U open in $\mathbb{R}^3$ be implicitly defined by equation (1.1). For ease of exposition we write $q = x_1 + \mathrm{i}x_2$, identifying $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$, so that (1.1) can now be written

$$P(x, z) \equiv g^2\bar{q} + 2gx_3 - q - 2h = 0, \tag{2.1}$$

for meromorphic functions g and h.

The *singular set* of z is defined to be the solution set in $\mathbb{R}^3$ of the simultaneous equations

$$\left.\begin{aligned} P &= 0, \\ \frac{\partial P}{\partial z} &= 0. \end{aligned}\right\} \tag{2.2}$$

Note that this represents four real constraints in five real variables. In general, the solution in x will be a real analytic subset of $\mathbb{R}^3$ of codimension 2 [**8**].

There is also a solution set in the z-plane, which we refer to as the *singular values* of z.

In terms of the representation (2.1), equation (2.2) has the form

$$\begin{aligned} g^2\bar{q} + 2gx_3 - q - 2h &= 0, \\ gg'\bar{q} + g'x_3 - h' &= 0. \end{aligned}$$

Solving for q and x_3 gives

$$\left.\begin{aligned} q &= \frac{2\{\bar{g}'(hg' - gh') - g^2g'(\bar{h}\bar{g}' - \bar{g}\bar{h}')\}}{|g'|^2(1 - |g|^4)}, \\ x_3 &= -g\bar{q} - \frac{h'}{g'}. \end{aligned}\right\} \tag{2.3}$$

The requirement that x_3 be real yields the condition

$$2|g'|^2(g\bar{h} - \bar{g}h) = (1 + |g|^2)(g'\bar{h}' - \bar{g}'h'). \tag{2.4}$$

Equation (2.4) determines the singular values Σ of z. The image of Σ under the mapping (2.3) gives the singular set $K \subset \mathbb{R}^3$.

If we holomorphically reparametrize z by a transformation $z = z(w)$, $z'(w) \neq 0$, then $g'(z) = g'(w)w'(z)$ and we see that equation (2.4) remains invariant as well as the image K determined by (2.3). In particular, the singular set K depends only on the holomorphic curve $\mathcal{C} \subset \mathbb{C} \times \mathbb{C}$ determined by g and h. (More precisely, the space of lines in $\mathbb{R}^3$ is identified with the complex surface TS^2 (cf. [**9**]), and $\mathcal{C}$ should be thought of as lying in TS^2. Then g and h may admit poles. However, for our purposes, it suffices to work in a trivialization: $T(S^2 \setminus \text{point}) \cong \mathbb{C} \times \mathbb{C}$.)

3. Knot examples

Let $g(z) = z^p$, where p is an odd integer $\geqslant 3$ and set $h(z) = z^{p+2} + \mathrm{i}\beta z^p$, where β is a real positive number. Equation (2.4) becomes

$$2p|z|^{4p-2}(\bar{z}^2 - z^2 - 2\mathrm{i}\beta) = (1 + |z|^{2p})|z|^{2p-2}((p + 2)(\bar{z}^2 - z^2) - 2\mathrm{i}p\beta).$$

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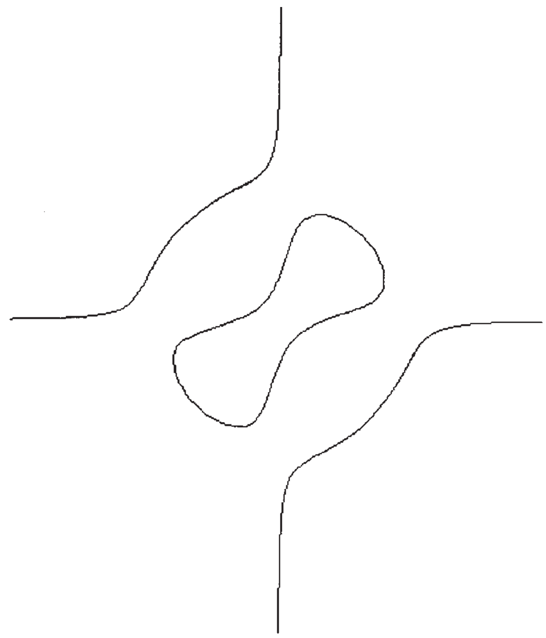

Figure 1. The set of singular values for $(g,h) = (z^p, z^{p+2} + \mathrm{i}\beta z^p)$.

Thus, either $z = 0$, or, incorporating constants into β,

$$(\bar{z}^2 - z^2)((p-2)|z|^{2p} - (p+2)) - \mathrm{i}\beta(|z|^{2p} - 1) = 0. \tag{3.1}$$

We will prove the solution has the form sketched in Figure 1.

Theorem 3.1. *For $\beta > 0$, the solution set to (3.1) inside the circle $|z| = a$, $a = ((p+2)/(p-2))^{1/2p}$, consists of a smoothly embedded closed curve. The curve is symmetric under reflection in the lines $y = \pm x$ and may be parametrized in the form $z = \sqrt{R}\mathrm{e}^{\mathrm{i}\theta}$, where $R = R(\theta)$ is a smooth positive function satisfying $R > 1$ for $\theta \in (0, \pi/2) \cup (\pi, 3\pi/2)$; $R < 1$ for $\theta \in (\pi/2, \pi) \cup (3\pi/2, 2\pi)$; R is monotone increasing on the interval $(-\pi/4, \pi/4)$ and has Taylor expansion about $\theta = 0$ given by*

$$R(\theta) = 1 + \frac{8\theta}{p\beta} - \frac{16(p^2-6)\theta^2}{p^2\beta^2} + \mathcal{O}(\theta^3). \tag{3.2}$$

Remark 3.2. The estimates on R provide essential information in establishing winding numbers, which will confirm that the singular set is a knot.

Note that the point $z = 1$ satisfies (3.1). It is the solution passing through this point that we isolate. By the reflectional symmetry of (3.1), we restrict out attention to the quadrant $-\pi/4 \leqslant \theta \leqslant \pi/4$.

Lemma 3.3. *The solution set to equation (3.1) consists of smoothly embedded curves.*

Proof. This is a simple application of the Implicit Function Theorem. □

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Let $z = r\mathrm{e}^{\mathrm{i}\theta}$ and set $R = r^2$, then (3.1) becomes

$$P_\beta(R,\theta) \equiv R\{(p-2)R^p - (p+2)\}\sin 2\theta - \beta(1-R^p) = 0. \tag{3.3}$$

For $\theta = 0$, this has the unique solution $R = 1$.

Lemma 3.4.

(i) *For $\theta \in (-\pi/4, 0)$, there is precisely one positive root $R = R(\theta)$ to (3.3) satisfying $R^p < (p+2)/(p-2)$. This root lies in the interval*

$$I = \left(0, \min\left\{\frac{\beta}{-(p-2)\sin 2\theta}, 1\right\}\right).$$

(ii) *For $\theta \in (0, \pi/4)$, there is precisely one positive root $R = R(\theta)$ to equation (3.3); this lies in the interval*

$$J = \left(\max\left\{1, \left(\frac{p+2}{p-2} - \frac{4\beta}{(p-2)^2\sin 2\theta}\right)^{1/p}\right\}, \left(\frac{p+2}{p-2}\right)^{1/p}\right).$$

Proof. (i) Define f on the interval

$$(0, a^2), \quad a = \left(\frac{p+2}{p-2}\right)^{1/2p},$$

by

$$f(R) = \frac{\beta(1-R^p)}{R\{(p-2)R^p - (p+2)\}}.$$

Then $P_\beta(R,\theta) = 0$ if and only if $\sin 2\theta = f(R)$. Now,

$$\lim_{R\to 0} f(R) = -\infty \quad \text{and} \quad \lim_{R\to a^2} f(R) = +\infty,$$

so for each θ, there exists $R(\theta) \in (0, a^2)$ such that $f(R(\theta)) = \sin 2\theta$. A calculation of the derivative,

$$f'(R) = \frac{\beta\{2(p-2)R^{2p} + (p+2)R^p + (p+2)\}}{R^2\{(p-2)R^p - (p+2)\}^2} > 0,$$

shows that f is monotone over $(0, a^2)$, and the root $R = R(\theta)$ is unique.

To locate the root more accurately for $\theta \in (-\pi/4, 0)$, we note that

$$P_\beta(0,\theta) = -\beta < 0,$$
$$P_\beta\left(\frac{\beta}{-(p-2)\sin 2\theta}, \theta\right) = 4\beta/(p-2) > 0,$$
$$P_\beta(1,\theta) = -4\sin 2\theta > 0,$$

so the root $R(\theta)$ lies in I.

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(ii) As for part (i), for $\theta \in (0, \pi/4)$, there is a unique root $R = R(\theta)$ lying in the interval $(0, a^2)$. There can be no other root, since $P_\beta(R, \theta) > 0$ for $R \geqslant a^2$.

For $R = ((p+2)/(p-2))^{1/p}$, $P_\beta(R, \theta) = 4\beta/(p-2) > 0$.

For $R = 1$, $P_\beta(R, \theta) = -4 \sin 2\theta < 0$.

For

$$R = \left(\left(\frac{p+2}{p-2} \right) - \frac{4\beta}{(p-2)^2 \sin 2\theta} \right)^{1/p} \quad (\geqslant 1),$$
$$P_\beta(R, \theta) = -\frac{4\beta R}{(p-2)} - \frac{4\beta^2}{(p-2)^2 \sin 2\theta} + \frac{4\beta}{(p-2)} < 0,$$

so the root lies in J. □

For each $\theta \in [0, 2\pi]$, let $R(\theta)$ denote the unique root to $P_\beta(R, \theta) = 0$ lying in the interval $(0, ((p+2)/(p-2))^{1/p})$. Then by Lemma 3.3, $R = R(\theta)$ is a smooth function of θ.

Lemma 3.5. *$R(\theta)$ is monotone increasing on the interval $(-\pi/4, \pi/4)$.*

Proof. Differentiating equation (3.3) implicitly yields

$$R' = \frac{2R\{(p+2) - (p-2)R^p\} \cos 2\theta}{\{(p+1)(p-2)R^p \sin 2\theta - (p+2) \sin 2\theta + p\beta R^{p-1}\}}.$$

Expressing $\sin 2\theta$ in terms of R, again using equation (3.3) yields

$$R' = \frac{2R^2\{(p+2) - (p-2)R^p\}^2 \cos 2\theta}{\{(p-2)R^p + (p+2)\}(1 + R^p)\beta}, \tag{3.4}$$

which is strictly positive on $(-\pi/4, \pi/4)$. □

Setting $\theta = 0$ in (3.4) yields $R'(0) = 8/p\beta$. Differentiating once more establishes $R''(0) = -32(p^2 - 6)/p^2\beta^2$ and the Taylor expansion to order two for $R(\theta)$ about $\theta = 0$ follows. This completes the proof of Theorem 3.1. □

We now consider the map (2.3) into $\mathbb{R}^3$, sending Γ_β to the singular set. We establish the following theorem.

Theorem 3.6. *The singular set of the multivalued harmonic morphism determined by equation (1.1), with $g(z) = z^p$, $h(z) = z^{p+2} + \mathrm{i}\beta z^p$, $p = 3, 5, 7, \dots$, where β is a real positive number, has a compact component K_0^p, consisting of a continuously embedded knotted curve. For each $p = 3, 5, 7, \dots$, all of the knots K_0^p are distinct, i.e. non-isotopic.*

By equation (2.3)

$$\left.\begin{aligned} q(z) &= \frac{4z^{p-2}(-z^4 + |z|^{2p+4})}{p(1 - |z|^{4p})}, \\ x_3(z) &= \frac{4\bar{z}|z|^{2p} + (p-2)z^2|z|^{4p} - (p+2)z^2}{p(1 - |z|^{4p})} - \mathrm{i}\beta. \end{aligned}\right\} \tag{3.5}$$

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Lemma 3.7. *The maps q and x_3 are continuous on the set Γ_β. The image of the point $z=1$ is given by $q(1)=(-2+\mathrm{i}\beta)/p$, $x_3(1)=-1$.*

Proof. By Lemma 3.4 (showing $|z|\neq 1$ for $z\in\Gamma_\beta$, $-\pi/4\leqslant\arg z<0$ and $0<\arg z\leqslant\pi/4$), we need only check continuity at the points $z=\pm1,\pm\mathrm{i}$ and indeed by reflectional symmetry, the point $z=1$ will suffice.

Note that, if $\xi:\mathbb{C}\to\mathbb{C}\times\mathbb{C}$ is the map $\xi(z)=(q(z),x_3(z))$ given by (2.3), then ξ is *not* continuous at $z=1$. Setting $z=1+\rho\mathrm{e}^{\mathrm{i}\alpha}$, we calculate

$$q(z)=-\frac{2}{p}+\frac{4\mathrm{i}\tan\alpha}{p^2},$$

whose limit as $\rho\to0$ depends on α. However, along Γ_β we have continuity.

From equation (3.5),

$$q(\theta)=\frac{4R^{(p+2)/2}}{p(1-R^{2p})}\{-\mathrm{e}^{\mathrm{i}(p+2)\theta}+R^p\mathrm{e}^{\mathrm{i}(p-2)\theta}\},\tag{3.6}$$

$$x_3(\theta)=-\frac{R\{(p-2)R^p+(p+2)\}}{p(1+R^p)}\cos2\theta.\tag{3.7}$$

Substituting the Taylor approximation to first order for $R(\theta)$ about $\theta=0$,

$$R(\theta)=1+\frac{8\theta}{p\beta}+\mathcal{O}(\theta^2),$$

we see that, as $\theta\to0$, $q\to(-2+\mathrm{i}\beta)/p$, $x_3\to-1$. □

Note that q obeys the same reflectional symmetry (now in the planes $x_1=\pm x_2$) as the R-curve, and, as a consequence (or by direct observation), $q(\theta)=-q(\theta+\pi)$. Also, the *height function* x_3 satisfies $x_3(\theta)=x_3(\theta+\pi)$.

A useful picture to have in mind is that of a horizontal rotating rod that moves up and down the x_3-axis. Imagine two points at opposite ends of the rod, equidistant from the x_3-axis, that move in and out. These two points will trace out the curve K_0^p.

Sketches of K_0^p for $p=3$ and $p=5$ are given in Figure 2.

In order to establish this form, we calculate the winding numbers of the inner and outer curves about the x_3-axis. However, to be sure no local knotting, unknotting or self-intersection points occur, we must study the functions $|q(\theta)|$ and $\arg q(\theta)$.

Lemma 3.8. *The argument $\arg q(\theta)$ is a C^1 function of θ such that $(\mathrm{d}/\mathrm{d}\theta)\arg q(\theta)>0$ for all θ. In particular, $\arg q(\theta)$ is a strictly increasing function of θ.*

Note. The continuous function $q(\theta)$ is not even differentiable.

Proof.

$$\begin{aligned}\arg q&=\arg\{-\mathrm{e}^{\mathrm{i}(p+2)\theta}+R^p\mathrm{e}^{\mathrm{i}(p-2)\theta}\}\\&=\arg\{X+\mathrm{i}Y\},\end{aligned}$$

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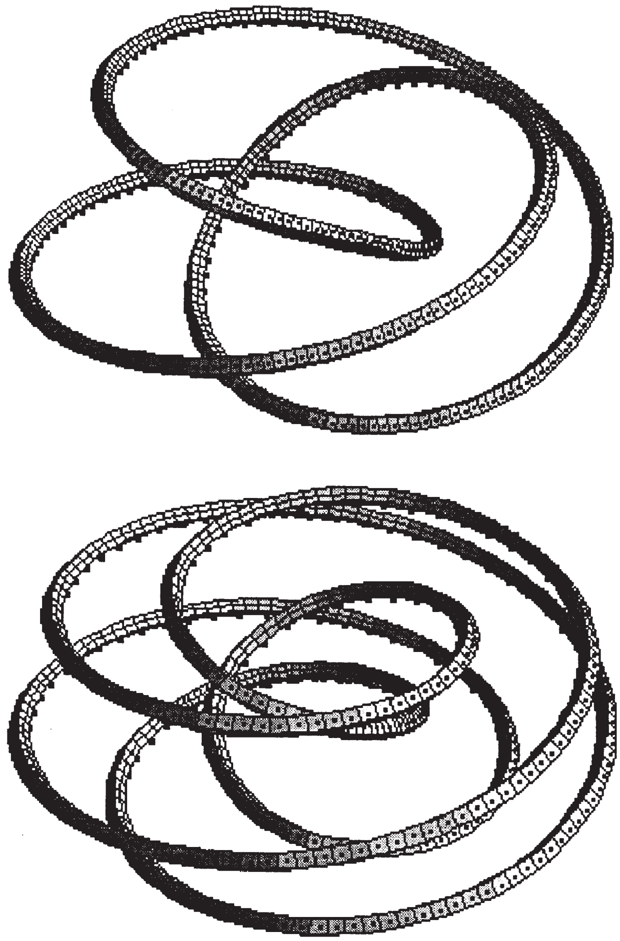

Figure 2. The trefoil and cinquofoil knots arise as the compact components K_0^3 and K_0^5, respectively.

where $X = -\cos(p+2)\theta + R^p \cos(p-2)\theta$ and $Y = -\sin(p+2)\theta + R^p \sin(p-2)\theta$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \arg q = \frac{XY' - YX'}{X^2 + Y^2}.$$

Setting $N(\theta) = XY' - YX'$ and $D(\theta) = X^2 + Y^2$, a lengthy calculation verifies

$$N = p(1 - R^p)^2 + 2pR^2(1 - \cos 4\theta) + 2(1 - R^{2p}) + pR^{p-1}R' \sin 4\theta, \tag{3.8}$$

$$D = (1 - R^p)^2 + 2R^p(1 - \cos 4\theta). \tag{3.9}$$

Clearly $D \geqslant 0$ and $D = 0$ if and only if $R = 1$ and $\theta = 0, \pi/2, \pi, 3\pi/2$.

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Substituting the expression for R' given by equation (3.4) and using (3.3) to eliminate θ, we calculate

$$N = \frac{(1-R^p)^2\{(p+2)-(p-2)R^2\}^2}{(1+R^p)\{(p+2)+(p-2)R^p\}} + \frac{8p\beta^2 R^{p-2}(1-R^p)^2\{(p+2)+(p-2)R^{2p}\}}{(1+R^p)\{(p+2)-(p-2)R^p\}^2\{(p+2)+(p-2)R^p\}}, \tag{3.10}$$

which again is $\geqslant 0$ and $= 0$ if and only if $R = 1$ (and so $\theta = 0, \pi/2, \pi, 3\pi/2$).

In order to evaluate the limit, $\lim_{\theta\to 0} N(\theta)/D(\theta)$, we evaluate the Taylor expansions to order 2 of N and D, using the expansion for R given by (3.2). This is most simply done by substituting into equations (3.8) and (3.9) and we find

$$N(\theta) = \frac{16}{p\beta^2}(16 + p^2\beta^2)\theta^2 + \mathcal{O}(\theta^3),$$
$$D(\theta) = \frac{16}{p^2\beta^2}(4 + p^2\beta^2)\theta^2 + \mathcal{O}(\theta^3),$$

so that $(\mathrm{d}/\mathrm{d}\theta)\arg q$ is continuous and positive for all θ with

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\arg q|_{\theta=0} = \frac{p(16+p^2\beta^2)}{4+p^2\beta^2}.$$

□

Thus, there can be no 'local' knotting or self-intersection, i.e. for θ in a sufficiently small interval. Knotting or self-intersection can only occur as a consequence of winding about the x_3-axis.

Lemma 3.9. *The function $|q|^2$ is monotone increasing on the interval $-\pi/4 < \theta < \pi/4$.*

Proof. Now

$$|q|^2 = \frac{16R^{p+2}}{p^2(1-R^{2p})^2}\{1 + R^{2p} - 2R^p\cos 4\theta\},$$

which, after setting $\cos 4\theta = 1 - 2\sin^2 2\theta$ and substituting for $\sin 2\theta$ from equation (3.3), becomes

$$|q|^2 = \frac{16R^{p+2}}{p^2(1+R^p)^2}\left\{1 + \frac{4R^{p-2}\beta^2}{\{(p-2)R^p-(p+2)\}^2}\right\}.$$

Differentiating with respect to θ yields

$$\frac{\mathrm{d}}{\mathrm{d}\theta}|q|^2 = \frac{16R^{p+1}}{p^2(1+R^p)}\left\{\frac{\{(p+2)-(p-2)R^2\}}{(1+R^p)^2}\left(1 + \frac{4\beta^2R^{p-2}}{\{(p-2)R^p-(p+2)\}^2}\right) + \frac{4(p-2)(p+2)R^{p-2}\beta^2}{\{(p+2)-(p-2)R^p\}^3}\right\}R'.$$

The coefficient of R' in this expression is always > 0, so that the sign of $(\mathrm{d}/\mathrm{d}\theta)|q|^2$ equals that of R' and $|q|^2$ increases and decreases with R. The result now follows from Theorem 3.1. □

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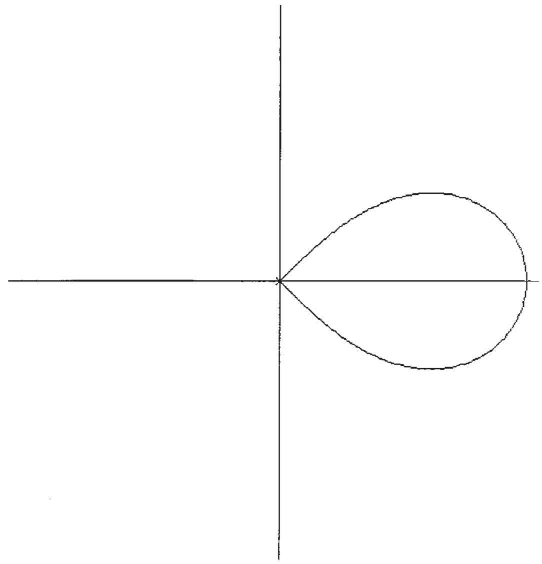

Figure 3. The tear drop curve $\alpha(\theta) = 1 - R^p e^{-4i\theta}$.

Lemma 3.10.

(i) *As θ varies from $-\pi/2$ to 0 (the inner curve), $q(\theta)$ rotates about the origin through an angle $(p+2)\pi/2 - 2\tan^{-1}(\beta/2)$, beginning at the point $(-1)^{(p-1)/2}(\beta - 2i)/p$ and ending at the point $(-2 + i\beta)/p$.*

(ii) *As θ varies from 0 to $\pi/2$ (the outer curve), $q(\theta)$ rotates about the origin through an angle $(p-2)\pi/2 + 2\tan^{-1}(\beta/2)$, beginning at the point $(-2+i\beta)/p$ and ending at the point $(-1)^{(p-1)/2}(-\beta + 2i)/p$.*

Remark 3.11. Because of reflectional symmetry of q in the lines $\theta = \pm\pi/4$, the above lemma describes the rotation of q completely. For example, as θ varies from $-\pi/4$ to $\pi/4$, q rotates through $(p+2)\pi/4 - \tan^{-1}(\beta/2) + (p-2)\pi/4 + \tan^{-1}(\beta/2) = p\pi/2$ about the origin.

Proof. The index about the origin,

$$\mathrm{Ind}_\gamma(0) = \int \frac{\gamma'}{\gamma}\,d\theta,$$

of a curve γ not passing through 0, measures the change in argument about 0. Noting that $\mathrm{Ind}_{\gamma\mu}(0) = \mathrm{Ind}_\gamma(0) + \mathrm{Ind}_\mu(0)$, we deduce

$$\mathrm{Ind}_{(\gamma+\mu)}(0) = \mathrm{Ind}_{\gamma(1+(\mu/\gamma))}(0) = \mathrm{Ind}_\gamma(0) + \mathrm{Ind}_{(1+(\mu/\gamma))}(0). \tag{3.11}$$

Suppose $-\pi/2 < \theta < 0$. Let $\gamma(\theta) = -e^{i(p+2)\theta}$, $\mu(\theta) = R^p e^{i(p-2)\theta}$. Then $\mu/\gamma = -R^p e^{-4i\theta}$. Noting that $R < 1$ in the interval $(-\pi/2, 0)$, we apply equation (3.11) and

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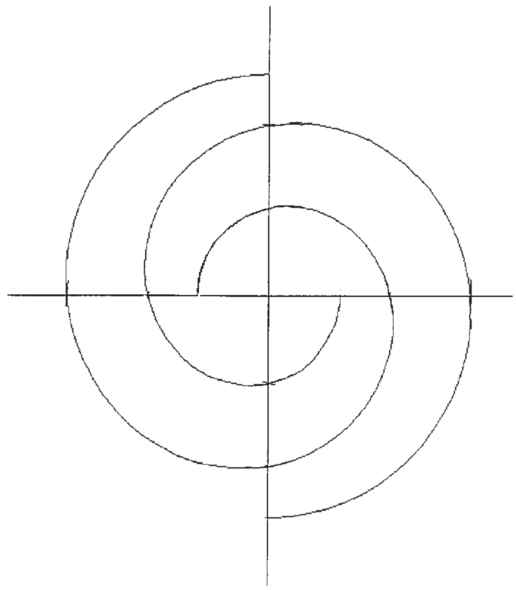

Figure 4. Two interlacing spirals corresponding to the shadows of two arcs of K_0^5.

study the change in argument of the curve $\alpha(\theta) = 1 - R^p e^{-4i\theta}$, which begins and terminates at 0 and has the tear drop shape indicated in Figure 3.

Now,

$$\arg\alpha = \tan^{-1}\left(\frac{R^p \sin 4\theta}{1 - R^p \cos 4\theta}\right).$$

We evaluate $\lim_{\theta\to 0} \tan\arg\alpha$. From Theorem 3.1, the Taylor expansion to first order of $R(\theta)$ is given by

$$R(\theta) \sim 1 + \frac{8\theta}{p\beta},$$

so that

$$R^p \sin 4\theta \sim 4\theta\left(1 + \frac{8\theta}{\beta}\right)$$

and

$$1 - R^p \sin 4\theta \sim -8\theta/\beta.$$

Thus

$$\lim_{\theta\to 0} \tan\arg\alpha = -\beta/2.$$

On the other hand, $\mathrm{Ind}_\gamma(0) = (p+2)\pi/2$. The result now follows from (3.11) and Lemma 3.7. Similar arguments give part (ii). □

Finally, we note the behaviour of the height function x_3.

Lemma 3.12. *The sign of x_3 is equal to the sign of $-\cos 2\theta$.*

Proof. This is a simple consequence of equation (3.7). □

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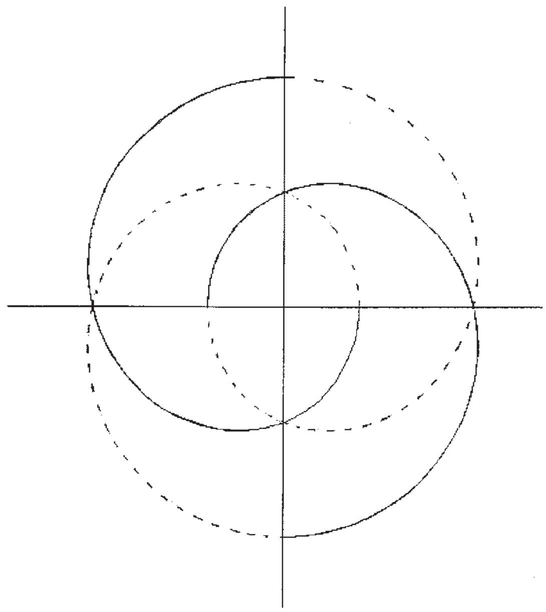

Figure 5. The shadow of the trefoil knot with $x_3 < 0$ depicted by the continuous curve and $x_3 > 0$ depicted by the dashed curve.

Proof of Theorem 3.6. As θ varies from $-\pi/4$ to $\pi/4$ and from $3\pi/4$ to $5\pi/4$ (x_3 negative), the shadows in the q-plane of the corresponding arcs of K_0^p trace out two interlacing spirals, which, by Remark 3.11, rotate through $p\pi/2$ as indicated in Figure 4.

These spirals can never intersect on account of Lemma 3.9, since $|q|$ is strictly increasing over these intervals.

A similar pair of interlacing spirals occur on the complementary arcs $(\pi/4, 3\pi/4)$ and $(5\pi/4, 7\pi/4)$, now with x_3 positive. In $\mathbb{R}^3$, the curves are joined at the points $(q(\theta_k), 0)$, $\theta_k = (2k+1)\pi/4$, $k = 0, 1, 2, 3, \dots$, giving a simple continuous closed curve K_0^p with no self-intersection points, i.e. a knot. This is sketched in Figure 5, where the continuous curve corresponds to $x_3 < 0$ and the broken curve to $x_3 > 0$.

To establish the topological nature of the knot K_0^p, we can 'pull back' the outer curve ($0 \leqslant \theta \leqslant \pi/2$ and $\pi \leqslant \theta \leqslant 3\pi/2$), like winding a spring and increase the rotation of the inner curve by $(p-2)\frac{1}{2}\pi + 2\tan^{-1}(\frac{1}{2}\beta)$. Then each 'inner curve' now rotates through $p\pi$.

It is well known (cf. [**11**]) that these knots all have distinct Alexander Polynomials, and, hence, are non-isotopic. This establishes Theorem 3.6. □

4. Comments and further problems

Similar considerations apply to the holomorphic curve $g = z^p$, $h = z^{p+1} + \mathrm{i}\beta z^p$, for $p = 2, 3, 4, \dots$, which yields a closed loop of singular values as sketched in Figure 6. However, the corresponding closed loop singularity in $\mathbb{R}^3$ is *never* knotted.

For more general holomorphic curves, the singularity becomes difficult to compute,

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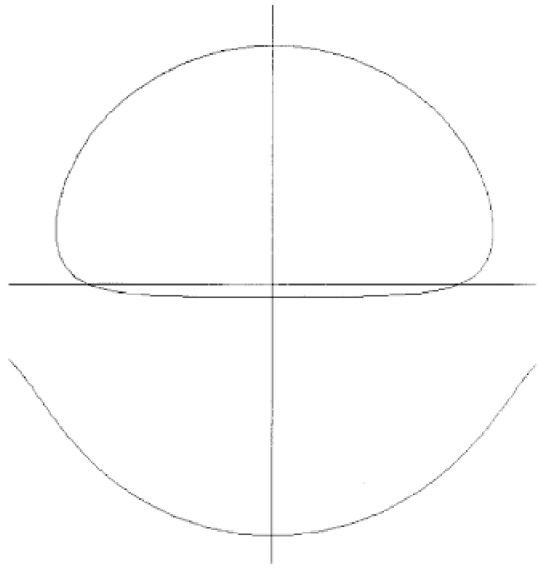

Figure 6. The set of singular values for $(g, h) = (z^p, z^{p+1} + \mathrm{i}\beta z^p)$.

however, it is to be expected that knots and links of various kinds should occur. Also, more general deformations of the type

$$g = z^p, \qquad h = z^q + \varepsilon_1 z^{r_1} + \varepsilon_2 z^{r_2} + \cdots,$$

following a Puiseux expansion, should yield interesting behaviour. It would be useful to have an effective computer method to sketch the singular set in $\mathbb{R}^3$.

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