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ON THE GROUP OF AUTOMORPHISMS OF AN ANALYTIC GROUP

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Let G be an analytic group and let $\operatorname{Aut} G$ denote the group of all topological group automorphisms of G. We investigate when $\operatorname{Aut} G$ is almost algebraic. We provide various conditions, some of which are known, under which $\operatorname{Aut} G$ is almost algebraic. We also provide examples showing that there does not seem to be a clear and concise way to characterise G so that $\operatorname{Aut} G$ is almost algebraic in terms of the maximal central torus in G.

1. INTRODUCTION

Let G be an analytic group, $\mathcal{L}(G)$ be the Lie algebra of G, and $\operatorname{Aut} G$ be the group of all topological group automorphisms of G. This paper will investigate when $\operatorname{Aut} G$ is an almost algebraic subgroup of $GL(\mathcal{L}(G))$, (that is, when $\operatorname{Aut} G$ is a subgroup of finite index in an algebraic subgroup of $GL(\mathcal{L}(G))$).

In [**2**], Dani reproved a result of Wigner [**4, 5**]. The connected component of the identity in $\operatorname{Aut} G$ is an almost algebraic subgroup. Here we give another proof of this result (Theorem 2.2) based upon Dani's proof. Dani also showed that if G has no compact central subgroup of positive dimension, then $\operatorname{Aut} G$ is an almost algebraic subgroup ([**2**]). We include a proof of this fact (Theorem 2.6 (i)).

We also provide various necessary and sufficient conditions for $\operatorname{Aut} G$ to be an almost algebraic subgroup. Specifically, if T is a maximal torus in G or if T is a maximal torus in the radical of G, then $\operatorname{Aut} G$ is almost algebraic if and only if $\operatorname{Aut}_T G$ is almost algebraic, where $\operatorname{Aut}_T G$ consists of all automorphisms τ such that $\tau(T) = T$ (Proposition 2.3 and Proposition 2.10). Another such condition is the following: $\operatorname{Aut} G$ is almost algebraic if and only if the restriction map $\tau \to \tau|_{T_1}$ of $\operatorname{Aut}_T G \to \operatorname{Aut} T_1$ has finite image, where T is a maximal torus of G and $T_1 = T \cap R$, R being the radical of G (Theorem 2.6 (ii)). In particular, if R has a maximal torus of dimension at most 1 or if $\operatorname{Aut} R$ is almost algebraic, then $\operatorname{Aut} G$ is almost algebraic (Corollary 2.7 and Corollary 2.8).

As mentioned above, Dani proved that if G has no compact central subgroup of positive dimension, then $\operatorname{Aut} G$ is almost algebraic. Is there a clear and concise way

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to characterise G in terms of the maximal central torus in G so that $\operatorname{Aut} G$ is almost algebraic? Chen and Wu [**1**] gave such a characterization for faithfully representable analytic groups. We shall provide examples showing that for analytic groups that are not faithfully representable, there does not seem to be a clear and concise way to characterise G in terms of the maximal central torus in G so that $\operatorname{Aut} G$ is almost algebraic.

NOTATION. Let G be a locally compact group. We denote the connected component of G that contains the identity element by G°. $\operatorname{Aut} G$ denotes the group of all topological group automorphisms of G. If $x \in G$, we denote the inner automorphism of G that is determined by x by I_x. If H is a subgroup of G, we denote the subgroup $\{I_x : x \in H\}$ of $\operatorname{Aut} G$ by $\operatorname{Int} H$. $\operatorname{Aut}_H G$ denotes the subgroup of $\operatorname{Aut} G$ which consists of all automorphisms τ such that $\tau(H) = H$. If $\tau \in \operatorname{Aut} G$, we denote the restriction of τ to H by $\tau|_H$. If G is also an analytic group, we denote the Lie algebra of G by $\mathcal{L}(G)$.

Let $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of integers and real numbers, respectively.

Let V be a vector space over $\mathbf{R}$. If F is a subgroup of $GL(V)$, then we denote the smallest algebraic subgroup of $GL(V)$ that contains F by $F^\#$.

2. Main results

We begin our paper by corrrecting Dani's proof of the following lemma [**2**]. Recall that $\operatorname{Aut} G$ can be viewed as a subgroup of $GL(\mathcal{L}(G))$ by associating each automorphism τ of G with its differential on $\mathcal{L}(G)$.

LEMMA 2.1. *Let G be an analytic group, $\mathcal{L}(G)$ be the Lie algebra of G, and $Ad : G \to GL(\mathcal{L}(G))$ be the adjoint representation of G. Then there exists a closed connected normal (characteristic) subgroup H of G such that the following conditions hold for the subgroup* $\operatorname{Int} H$ *of* $\operatorname{Aut} G$*:*

(i) $\operatorname{Int} H$ *is an almost algebraic subgroup of $GL(\mathcal{L}(G))$.*
(ii) *If T is a maximal torus of G, then* $\operatorname{Aut} G = (\operatorname{Int} H)(\operatorname{Aut}_T G)$.

The argument Dani gave in showing that H is closed is incorrect. The fallacy lies in assuming that the semisimple Levi factor is closed. This is true for complex Lie groups, but false for real Lie groups. However, H indeed is closed. This follows by a result of Goto [**3**, Proposition 14]: H is an analytic subgroup of G which contains a maximal compact subgroup of G.

The following theorem is due to Wigner [**4, 5**]. The proof we give is based upon Dani's proof of the theorem [**2**]. We have simplified Dani's proof and clarified some of the details.

THEOREM 2.2. *Let G be an analytic group and let $\mathcal{L}(G)$ denote the Lie algebra of G. Then the connected component of the identity in* $\operatorname{Aut} G$ *is an almost algebraic* *subgroup of* $GL(\mathcal{L}(G))$.

PROOF: Let $\eta : \widetilde{G} \to G$ be a group covering with $\widetilde{G}$ simply connected. Let $K = \ker \eta$. Then K is a discrete normal, hence central, subgroup of $\widetilde{G}$. Let H be a closed connected normal subgroup of G as in Lemma 2.1. Let T be a maximal torus of G. Then $\operatorname{Aut} G = (\operatorname{Int} H)(\operatorname{Aut}_T G)$ and $\operatorname{Int} H$ is almost algebraic.

CLAIM. To show that $(\operatorname{Aut} G)^\circ$ is almost algebraic, it suffices to show that $(\operatorname{Aut}_T G)^\circ$ is almost algebraic. If $(\operatorname{Aut}_T G)^\circ$ is almost algebraic, then $(\operatorname{Int} H)(\operatorname{Aut}_T G)^\circ$ is almost algebraic, and therefore $(\operatorname{Int} H)(\operatorname{Aut}_T G)^\circ$ is closed in the Euclidean topology. Also, $(\operatorname{Aut}_T G)/(\operatorname{Aut}_T G)^\circ$ is countable since $(\operatorname{Aut}_T G)$ is a Lie group. Thus, $\operatorname{Aut} G = (\operatorname{Int} H)(\operatorname{Aut}_T G) \supseteq (\operatorname{Int} H)(\operatorname{Aut}_T G)^\circ$ and $(\operatorname{Aut} G)/(\operatorname{Int} H)(\operatorname{Aut}_T G)^\circ$ is countable, so $(\operatorname{Int} H)(\operatorname{Aut}_T G)^\circ$ is open. Therefore, $(\operatorname{Int} H)(\operatorname{Aut}_T G)^\circ$ is an open and closed subgroup which implies that $(\operatorname{Aut} G)^\circ = (\operatorname{Int} H)(\operatorname{Aut}_T G)^\circ$. Thus, $(\operatorname{Aut} G)^\circ$ is almost algebraic.

Let $S = \eta^{-1}(T)$. We claim S is a closed connected Abelian subgroup of $\widetilde{G}$. G/T is simply connected and the induced map $\widetilde{G}/\eta^{-1}(T) \to G/T$ is a homeomorphism, so $\widetilde{G}/\eta^{-1}(T)$ is simply connected. Also, the induced map $\widetilde{G}/\eta^{-1}(T)^\circ \to \widetilde{G}/\eta^{-1}(T)$ is a covering of the simply connected space $\widetilde{G}/\eta^{-1}(T)$, so $\widetilde{G}/\eta^{-1}(T)^\circ$ is homeomorphic with $\widetilde{G}/\eta^{-1}(T)$. Thus $\eta^{-1}(T)^\circ = \eta^{-1}(T)$, so S is connected. S is Abelian because $\mathcal{L}(S) \cong \mathcal{L}(T)$, and $\mathcal{L}(T)$ is Abelian.

If we view $\operatorname{Aut}_T G$ as a subgroup of $\operatorname{Aut} \widetilde{G}$, then $\operatorname{Aut}_T G$ consists of all automorphisms $\tau \in \operatorname{Aut} \widetilde{G}$ such that $\tau(K) = K$ and $\tau(S) = S$. Since $\widetilde{G}$ is simply connected, $\operatorname{Aut} \widetilde{G} \cong \operatorname{Aut} \mathcal{L}(G)$, and therefore $\operatorname{Aut} \widetilde{G}$ is an algebraic subgroup. Let $\Phi = \{\tau \in \operatorname{Aut} \widetilde{G} : \tau(S) = S\}$. Since S is connected, Φ is an algebraic subgroup.

Since S is an Abelian analytic group, it has a unique maximal torus, say F. Since $K \subseteq S$, we can consider the group S/FK. Since S/FK is an Abelian analytic group, it has a unique maximal torus, say L'. Let $\pi : S \to S/FK$ be the canonical map and let $L = \pi^{-1}(L')$. Then L is a closed connected subgroup of S containing FK such that $L/FK \cong L'$ is a torus.

If an automorphism $\tau \in \Phi$ has the property that $\tau|_{FK}$ is the identity on FK, then $\tau|_L$ is the identity on L. Let $\Psi = \{\tau \in \Phi : \tau(x) = x \text{ for all } x \in FK\} = \{\tau \in \Phi : \tau(x) = x \text{ for all } x \in L\}$. Since L is connected, Ψ is an algebraic subgroup.

For any $\tau \in \operatorname{Aut}_T G$, we have $\tau(S) = S$ and $\tau(K) = K$. Thus $\tau(F) = F$. Since F is torus, $\operatorname{Aut} F$ is discrete. Since K is a discrete Abelian group, $\operatorname{Aut} K$ is also discrete. Thus the restriction maps $\tau \to \tau|_F$ and $\tau \to \tau|_K$ of $\operatorname{Aut}_T G$ into $\operatorname{Aut} F$ and $\operatorname{Aut} K$, respectively, are trivial on $(\operatorname{Aut}_T G)^\circ$. Thus, we have $(\operatorname{Aut}_T G)^\circ \subseteq \Psi \subseteq \operatorname{Aut}_T G$. Therefore, $\Psi^\circ \subseteq (\operatorname{Aut}_T G)^\circ \subseteq \Psi$, and since Ψ is algebraic, $(\operatorname{Aut}_T G)^\circ$ is almost algebraic. ∎

Propositions 2.3 and 2.4 will be used in proving Theorem 2.6. With the following propositions, we keep the same notation as in Theorem 2.2.

Proposition 2.3. *Let G be an analytic group, and let T be a maximal torus of G. Then $\operatorname{Aut} G$ is an almost algebraic subgroup if and only if $\operatorname{Aut}_T G$ is an almost algebraic subgroup.*

Proof: Suppose $\operatorname{Aut}_T G$ is almost algebraic. Let H be a closed connected normal subgroup of G as in Lemma 2.1. Then $\operatorname{Int} H$ is almost algebraic and thus $\operatorname{Aut} G = (\operatorname{Int} H)(\operatorname{Aut}_T G)$ is almost algebraic.

Conversely, suppose $\operatorname{Aut} G$ is almost algebraic. Since $\operatorname{Aut}_T G \subseteq \operatorname{Aut} G$, we have $(\operatorname{Aut}_T G)^{\#} \subseteq \operatorname{Aut} G$. Also, viewing $\operatorname{Aut} G$ as a subgroup of $\operatorname{Aut} \widetilde{G}$, we have $\operatorname{Aut}_T G \subseteq \Phi$ and hence $(\operatorname{Aut}_T G)^{\#} \subseteq \Phi$. Thus, if $\tau \in (\operatorname{Aut}_T G)^{\#}$, then $\tau(K) = K$ and $\tau(S) = S$, so $\tau \in \operatorname{Aut}_T G$. Thus, $(\operatorname{Aut}_T G)^{\#} = \operatorname{Aut}_T G$, so $\operatorname{Aut}_T G$ is almost algebraic. ∎

Proposition 2.4. *Let G be an analytic group, and let T be a maximal torus of G. If $\beta : \operatorname{Aut}_T G \to \operatorname{Aut} T$ is the restriction map, then $\operatorname{Aut}_T G$ is an almost algebraic subgroup if and only if $\beta(\operatorname{Aut}_T G)$ is finite.*

Proof: Suppose $\operatorname{Aut}_T G$ is almost algebraic. Then $\operatorname{Aut}_T G$ has finitely many connected components. Since $\operatorname{Aut} T$ is discrete, $\beta(\operatorname{Aut}_T G)$ is finite.

Conversely, suppose that $\beta(\operatorname{Aut}_T G)$ is finite. Then the $\ker \beta = \{\tau \in \operatorname{Aut}_T G : \tau(t) = t \text{ for all } t \in T\}$. Viewing $\operatorname{Aut}_T G$ as a subgroup of $\operatorname{Aut} \widetilde{G}$, $\ker \beta = \Delta = \{\tau \in \operatorname{Aut} \widetilde{G} : \tau(s) = s \text{ for all } s \in S\}$. Then Δ is an algebraic subgroup since S is connected. Since $\operatorname{Aut}_T G/\Delta$ is finite, $\operatorname{Aut}_T G$ is almost algebraic. ∎

Corollary 2.5. *If an analytic group G has a maximal torus of dimension at most 1, then $\operatorname{Aut} G$ is an almost algebraic subgroup.*

In [**2**], Dani proved that if an analytic group G has no compact central subgroup of positive dimension, then $\operatorname{Aut} G$ is an almost algebraic subgroup. For completeness, we include a proof of this fact in the following theorem.

Theorem 2.6. *Let G be an analytic group, and let $\mathcal{L}(G)$ denote the Lie algebra of G.*

(i) *(Dani) If G does not have any compact central subgroups of positive dimension, then $\operatorname{Aut} G$ is an almost algebraic subgroup of $GL(\mathcal{L}(G))$.*
(ii) *$\operatorname{Aut} G$ is an almost algebraic subgroup if and only if the restriction map $\tau \to \tau|_{T_1}$ of $\operatorname{Aut}_T G \to \operatorname{Aut} T_1$ has finite image, where T is a maximal torus of G, and $T_1 = T \cap R$, R being the radical of G.*

Proof: Let G be an analytic group and let $\eta : \widetilde{G} \to G$ be a group covering with $\widetilde{G}$ simply connected. Let $K = \ker \eta$. Then K is a discrete normal, hence central, subgroup of $\widetilde{G}$. Let $\widetilde{Z}$ denote the centre of $\widetilde{G}$. Let T be a maximal torus of G. Let $T_1 = T \cap R$, where R is the radical of G. Then T_1 is a maximal torus of R. Let

$A = \eta^{-1}(T)$. Then A is a closed connected Abelian subgroup of $\widetilde{G}$. Since $\widetilde{G}$ is simply connected, $\widetilde{G} = \widetilde{R}\cdot\widetilde{S}$, where $\widetilde{R}$ is the radical of $\widetilde{G}$ and $\widetilde{S}$ is a semisimple Levi factor of $\widetilde{G}$. Note that $\widetilde{R}$ and $\widetilde{S}$ are both simply connected, and $\widetilde{S}$ is closed. Let $B = \eta^{-1}(T_1)^\circ$. Then $BK = \eta^{-1}(T_1)$, and $B/B\cap K \cong BK/K \cong T_1$. Also B is a simply connected Abelian group (that is, vector group), and $K\cap\widetilde{R} = K\cap B$ is a uniform subgroup of B.

(i) Let $\Phi = \{\tau \in \operatorname{Aut}\widetilde{G} : \tau(A) = A\}$. Then Φ is an algebraic subgroup. Since $\widetilde{G}$ has no compact central subgroup of positive dimension, $\widetilde{Z}\cap A$ is a discrete subgroup. Given any $\tau\in\Phi$, $\tau(\widetilde{Z}\cap A) = \widetilde{Z}\cap A$. In particular, for $\tau\in\Phi^\circ$, we have $\tau(z) = z$ for all $z\in\widetilde{Z}\cap A$. This follows by considering the restriction map $\tau\to\tau|_{\widetilde{Z}\cap A}$ of Φ into $\operatorname{Aut}(\widetilde{Z}\cap A)$, since $\operatorname{Aut}(\widetilde{Z}\cap A)$ is discrete. Since $K\subseteq\widetilde{Z}\cap A$, we have $\tau(k) = k$ for all $k\in K$ and $\tau\in\Phi^\circ$. Thus $\Phi^\circ\subseteq\operatorname{Aut}_T G\subseteq\Phi$. Since Φ is an algebraic subgroup, $\operatorname{Aut}_T G$ is almost algebraic. Thus, $\operatorname{Aut} G$ is almost algebraic by Proposition 2.3.

(ii) Let $\Delta = \{\tau\in\operatorname{Aut}\widetilde{G} : \tau(b) = b \text{ for all } b\in B\}$. Then Δ is an algebraic subgroup, and thus $\Delta\cap\Phi$ is an algebraic subgroup. For each $\tau\in\Delta\cap\Phi$, $\tau(b) = b$ for all $b\in B$. Let $z\in A\cap\widetilde{Z}$. Then $z = bs$, where $b\in B$ and $s\in\widetilde{S}$. Note that s is central in $\widetilde{S}$, though it may not be central in $\widetilde{G}$. We want to show that for all $\tau\in(\Delta\cap\Phi)^\circ$, $\tau(z) = z$ for all $z\in A\cap\widetilde{Z}$. First, for $\tau\in(\Delta\cap\Phi)^\circ$, $\tau(b) = b$ for all $b\in B$. Also, $B(A\cap\widetilde{Z})/B$ is discrete. If we consider the induced map $\delta : \Delta\cap\Phi\to\operatorname{Aut}\big(B(A\cap\widetilde{Z})/B\big)$ then δ is trivial on $(\Delta\cap\Phi)^\circ$. Thus $\delta(\tau)$ fixes each coset of $B(A\cap\widetilde{Z})/B$. This implies that $\tau(z) = \tau(bs) = b'bs$ for some $b'\in B$.

We consider two cases.

Case 1. Assume s is torsion. Then $s^m = 1$ for some $m\in\mathbf{Z}$. Then $\tau\big((bs)^m\big) = \tau(b^m s^m) = \tau(b^m) = b^m$, and $\big(\tau(bs)\big)^m = b'^m b^m s^m = b'^m b^m$. Hence $b'^m = 1$. But B is torsion free, so $b' = 1$.

Case 2. Suppose s is not torsion. Then s^m is central in $\widetilde{G}$ for some integer m. To see this, view $\widetilde{S}$ as a group of automorphisms acting on $\widetilde{R}$ by conjugation; then it has finite centre. Since $z = bs$, we have $b^{-1}z = s$, and thus $s^m\in A\cap\widetilde{Z}$. Now $\tau(s^m)\in\tau(\widetilde{S})$, and since $\tau(\widetilde{S})$ is a semisimple Levi factor, there is $x\in\widetilde{G}$ such that $x\tau(s^m)x^{-1}\in\widetilde{S}$. Since s^m is central, so is $\tau(s^m)$. Therefore $x\tau(s^m)x^{-1} = \tau(s^m)\in\widetilde{S}$. This implies that $\tau(s^m) = s^m$. Since $\tau(bs) = b'bs$, we have $\big(\tau(bs)\big)^m = b'^m b^m s^m$ and $\tau\big((bs)^m\big) = \tau(b^m s^m) = \tau(b^m)\tau(s^m) = b^m s^m$. Hence $b'^m = 1$. But B is torsion free, so $b' = 1$.

Thus we can conclude that $\tau(z) = z$ for all $z\in A\cap\widetilde{Z}$. Therefore $\tau(K) = K$. Now we have $(\Delta\cap\Phi)^\circ\subseteq(\operatorname{Aut}_T G)\cap(\Delta\cap\Phi)\subseteq(\Delta\cap\Phi)$. Hence, $(\operatorname{Aut}_T G)\cap(\Delta\cap\Phi)$ is almost algebraic. By the hypothesis, we have $\operatorname{Aut}_T G/\big((\operatorname{Aut}_T G)\cap(\Delta\cap\Phi)\big)$ is finite, so $\operatorname{Aut}_T G$ is almost algebraic. By Proposition 2.3, we have that $\operatorname{Aut} G$ is almost algebraic.

The converse follows from Propositions 2.3 and 2.4. □

Corollary 2.7. *Let G be an analytic group and suppose that the radical R of G has a maximal torus of dimension at most 1. Then the* $\operatorname{Aut} G$ *is an almost algebraic subgroup.*

Corollary 2.8. *Let G be an analytic group and let R denote the radical of G. If* $\operatorname{Aut} R$ *is an almost algebraic subgroup, then* $\operatorname{Aut} G$ *is an almost algebraic subgroup.*

We conclude this section by proving an analogous result to Proposition 2.3. Here we are considering a maximal torus in the radical of G, rather than a maximal torus of G.

Lemma 2.9. *Let G be an analytic group. Let N be the nilradical of G. Let T be a maximal torus in the radical of G. Then* $\operatorname{Aut} G = (\operatorname{Int} N)(\operatorname{Aut}_T G)$.

Proof: Let $\tau \in \operatorname{Aut}_T G$. Then $\tau(T) = nTn^{-1}$ for some $n \in N$. Then $I_n^{-1} \circ \tau(T) = T$, so $I_n^{-1} \circ \tau \in \operatorname{Aut}_T G$ and $\tau = I_n \circ I_n^{-1} \circ \tau$. Thus $\operatorname{Aut} G = (\operatorname{Int} N)(\operatorname{Aut}_T G)$. □

Proposition 2.10. *Let G be an analytic group and let T be a maximal torus in the radical of G. Then* $\operatorname{Aut} G$ *is an almost algebraic subgroup if and only if* $\operatorname{Aut}_T G$ *is an almost algebraic subgroup.*

Proof: Suppose $\operatorname{Aut}_T G$ is almost algebraic. Let N be the nilradical of G. Then $\operatorname{Aut} G = (\operatorname{Int} N)(\operatorname{Aut}_T G)$. Since N is the nilradical, $\operatorname{Int} N$ is almost algebraic, and therefore $\operatorname{Aut} G = (\operatorname{Int} N)(\operatorname{Aut}_T G)$ is almost algebraic.

Conversely, suppose that $\operatorname{Aut} G$ is almost algebraic. Let $\eta : \widetilde{G} \to G$ be a group covering with $\widetilde{G}$ simply connected. Let $S = \eta^{-1}(T)$ and $S^\circ = \eta^{-1}(T)^\circ$. Let $\Phi_1 = \{\tau \in \operatorname{Aut} \widetilde{G} : \tau(S^\circ) = S^\circ\}$. Since S° is connected, Φ_1 is an algebraic subgroup. Since $\operatorname{Aut} G$ is almost algebraic, $(\operatorname{Aut}_T G)^\# \subseteq \operatorname{Aut} G$. Also, viewing $\operatorname{Aut} G$ as a subgroup of $\operatorname{Aut} \widetilde{G}$, we have $(\operatorname{Aut}_T G) \subseteq \Phi_1$, and hence $(\operatorname{Aut}_T G)^\# \subseteq \Phi_1$. Thus, $(\operatorname{Aut}_T G)^\# = \operatorname{Aut}_T G$, so $\operatorname{Aut}_T G$ is almost algebraic. □

3. Examples

Let G be an analytic group. Is there a clear and concise way of characterising G in terms of the maximal central torus in G so that $\operatorname{Aut} G$ is almost algebraic? Chen and Wu gave such a characterisation for faithfully representable analytic groups. Specifically, suppose G has a faithful representation and let T be the maximal central torus in G. Then $\operatorname{Aut} G$ is almost algebraic if and only if T is trivial, or the dimension of T is 1 and T is exactly the maximal torus in the radical of G ([1]). Here we provide examples showing that for analytic groups that are not faithfully representable, there does not seem to be a clear and concise way to characterise G in terms of the maximal central torus in G so that $\operatorname{Aut} G$ is almost algebraic.

The following example, due to Dani [**2**], is of an analytic group whose maximal central torus is of dimension n and whose automorphism group is almost algebraic. Dani omitted the proof for $n \geqslant 2$. Here we sketch a proof of the case $n = 2$. The general case follows in a similar manner.

EXAMPLE. Let H be the three-dimensional Heisenberg group and let D be a uniform subgroup contained in the centre of H. Let S be a subgroup of $\operatorname{Aut}(H/D)$ isomorphic to $SL(2, \mathbf{R})$. Let G be the semidirect product of S and H/D. Then the centre of G^n is an n-dimensional torus and $\operatorname{Aut} G^n$ is an almost algebraic subgroup.

Let

$$\widetilde{G} = \big(H_1 \cdot SL(2,\mathbf{R})\big) \times \big(H_2 \cdot SL(2,\mathbf{R})\big) = (H_1 \times H_2) \cdot \big(SL(2,\mathbf{R}) \times SL(2,\mathbf{R})\big).$$

Let θ be an automorphism of $\widetilde{G}$. By composing with a suitable inner automorphism, we can assume that

$$\theta\big(SL(2,\mathbf{R}) \times SL(2,\mathbf{R})\big) = SL(2,\mathbf{R}) \times SL(2,\mathbf{R}).$$

Let $B = SL(2,\mathbf{R})$. Then $\theta(B_1 \times B_2) = B_1 \times B_2$. Hence $\theta(B_1) = B_1$ and $\theta(B_2) = B_2$, or $\theta(B_1) = B_2$ and $\theta(B_2) = B_1$. We consider the case where $\theta(B_1) = B_2$ and $\theta(B_2) = B_1$. Also we have $\theta(H_1 \times H_2) = H_1 \times H_2$.

Each element $h \in H$, where

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{R} \right\}$$

will be denoted by $h = (x, y, z)$. Consider $(h, 1) \in H_1 \times H_2$; thus $(h,1) = h_1 = \big((x,y,z),(0,0,0)\big)$. Then $\theta(h_1) = \big((x',y',z'),(x'',y'',z'')\big) = (h_1', h_1'')$. If $(x,y) \neq (0,0)$, we claim that $x' = 0$ and $y' = 0$. To see this, let $b_2 \in B_2$. Then $\theta(b_2 h_1) = \theta(b_2) h_1' h_1'' = \theta(b_2) h_1' \theta(b_2)^{-1} \theta(b_2) h_1'' = \theta(b_2) h_1' \theta(b_2)^{-1} h_1'' \theta(b_2)$ and $\theta(h_1 b_2) = \theta(h_1)\theta(b_2) = h_1' h_1'' \theta(b_2)$. Thus $h_1' = \theta(b_2) h_1' \theta(b_2)^{-1}$ for all $b_2 \in B_2$. When $(x', y') \neq (0,0)$, there exists $\widetilde{b}_1 \in B_1 = SL(2,\mathbf{R})$, hence $\widetilde{b}_2 \in B_2$ with $\theta(\widetilde{b}_2) = \widetilde{b}_1$, such that (x',y') is not fixed under the action of $\widetilde{b}_1 = \theta(\widetilde{b}_2) \in SL(2,\mathbf{R})$. Hence $(x',y') = (0,0)$. Similarly, if we consider $(1,h) = h_2 = \big((0,0,0),(x,y,z)\big)$, then $\theta(h_2) = \big((x',y',z'),(x'',y'',z'')\big)$ where $(x'',y'') = (0,0)$.

By composing θ with an automorphism ϕ, $\phi \circ \theta$ will map $H_1 \times B_1$ into $H_2 \times B_2$ and $H_2 \times B_2$ into $H_1 \times B_1$. After considering the case where $\theta(B_1) = B_1$ and $\theta(B_2) = B_2$, it follows that

$$\operatorname{Aut} \widetilde{G} \cong \Big(\operatorname{Aut}\big(H_1 \cdot SL(2,\mathbf{R})\big) \times \operatorname{Aut}\big(H_2 \cdot SL(2,\mathbf{R})\big)\Big) \cdot \{\phi\} \cdot \Psi,$$

where Ψ is the group generated by the automorphism ψ, where $\psi(g_1, g_2) = (g_2, g_1)$, with $g_1 \in H_1 \cdot SL(2,\mathbf{R})$ and $g_2 \in H_2 \cdot SL(2,\mathbf{R})$. From this, it follows that $\operatorname{Aut} G^2$ is almost algebraic.

EXAMPLE. Let H be the three-dimensional Heisenberg group and let D be a uniform subgroup contained in the centre of H. Then the centre of $(H/D)^n$ is an n-dimensional torus and $\operatorname{Aut}\big((H/D)^n\big)$ is almost algebraic. Here we provide a proof of the case where $n = 2$. The general case follows in a similar manner.

Let $\mathcal{L}(H)$ be the Lie algebra of H. Then $\mathcal{L}(H)$ is generated by e, f, and g with $[e,f] = g$, $[e,g] = 0$, and $[f,g] = 0$. Consider $H_1 \times H_2$. Then $\mathcal{L}(H_1 \times H_2) = \mathcal{L}(H_1) \times \mathcal{L}(H_2)$. Here the basis for $\mathcal{L}(H_i)$ is e_i, f_i, g_i, $i = 1, 2$. Let $\mathcal{D}$ denote the Lie algebra of all derivations of $\mathcal{L}(H_1 \times H_2)$. Also, let $\mathcal{F}_i$ denote the Lie algebra of all derivations of $\mathcal{L}(H_i)$, $i = 1, 2$. Let $D_1 \in \mathcal{D}$. Then the matrix of D_1 relative to the basis e_1, f_1, g_1, e_2, f_2, g_2 is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & & & a_{16} \\ a_{21} & a_{22} & a_{23} & & & a_{26} \\ & & (a_{11}+a_{22}) & & & \\ & & a_{43} & a_{44} & a_{45} & a_{46} \\ & & a_{53} & a_{54} & a_{55} & a_{56} \\ & & & & & (a_{44}+a_{55}) \end{bmatrix}$$

Consider the derivation $D_{(\alpha,\beta,\gamma,\delta)}$ of $\mathcal{L}(H_1 \times H_2)$ given by

$$\begin{aligned} D_{(\alpha,\beta,\gamma,\delta)}(e_1) &= \alpha g_2, & D_{(\alpha,\beta,\gamma,\delta)}(e_2) &= \gamma g_1 \\ D_{(\alpha,\beta,\gamma,\delta)}(f_1) &= \beta g_2, & D_{(\alpha,\beta,\gamma,\delta)}(f_2) &= \delta g_1 \\ D_{(\alpha,\beta,\gamma,\delta)}(g_1) &= 0, & D_{(\alpha,\beta,\gamma,\delta)}(g_2) &= 0, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers.

Let $\mathcal{E} = \{D_{(\alpha,\beta,\gamma,\delta)} : \alpha, \beta, \gamma, \delta \text{ are real numbers}\}$. Then $\mathcal{E}$ is an ideal of $\mathcal{D}$ and $\mathcal{D} = \mathcal{F}_1 \times \mathcal{F}_2 + \mathcal{E}$. To show that $\mathcal{E}$ is an ideal, let $D_1 \in \mathcal{D}$ and let $D_{(\alpha,\beta,\gamma,\delta)} \in \mathcal{E}$. Show that $[D_1, D_{(\alpha,\beta,\gamma,\delta)}] = D_1 D_{(\alpha,\beta,\gamma,\delta)} - D_{(\alpha,\beta,\gamma,\delta)} D_1 \in \mathcal{E}$. Then $D_1 D_{(\alpha,\beta,\gamma,\delta)}(e_1) = D_1(\alpha g_2) = \alpha(a_{44} + a_{55}) g_2$, while $D_{(\alpha,\beta,\gamma,\delta)} D_1(e_1) = D_{(\alpha,\beta,\gamma,\delta)}(a_{11}e_1 + a_{12}f_1 + a_{13}g_1 + a_{16}g_2) = (\alpha a_{11} + \beta a_{12}) g_2$. Thus $[D_1, D_{(\alpha,\beta,\gamma,\delta)}](e_1)$ is a scalar multiple of g_2. Similar computations hold for f_1, e_2, and f_2.

Thus,

$$\operatorname{Aut}\big(\mathcal{L}(H_1 \times H_2)\big)^\circ = \operatorname{Aut}\big(\mathcal{L}(H_1)\big)^\circ \times \operatorname{Aut}\big(\mathcal{L}(H_2)\big)^\circ \cdot \exp(\mathcal{E}),$$

where $\exp : \mathcal{D} \to \operatorname{Aut}\big(\mathcal{L}(H_1 \times H_2)\big)^\circ$ is the exponential map. Since H is simply connected, we have $\operatorname{Aut}(H) \cong \operatorname{Aut}\big(\mathcal{L}(H)\big)$. Thus

$$\operatorname{Aut}(H_1 \times H_2)^\circ = \operatorname{Aut}(H_1)^\circ \times \operatorname{Aut}(H_2)^\circ \cdot E,$$

where E is a normal subgroup of $\operatorname{Aut}(H_1 \times H_2)^\circ$ consisting of all automorphisms θ of the form

$$\theta\big((x_1, y_1, z_1),\ (x_2, y_2, z_2)\big) = \big((x_1, y_1, z_1 + \gamma x_2 + \delta y_2),\ (x_2, y_2, z_2 + \alpha x_1 + \beta y_1)\big),$$

where $\alpha, \beta, \gamma, \delta$ are real numbers.

Since $\operatorname{Aut}(H \times H)$ has finitely many connected components, it follows that $\operatorname{Aut}(H/D \times H/D)$ is almost algebraic. Similarly, $\operatorname{Aut}\big((H/D)^n\big)$ is almost algebraic.

REMARK. Using Corollary 2.8, this example shows that $\operatorname{Aut}\big((H/D \cdot SL(2, \mathbf{R}))^n\big)$ is almost algebraic.

The following example is of a nilpotent group which, if we factor out a uniform subgroup of the centre, we obtain a group whose maximal central torus is of positive dimension and whose automorphism group is not almost algebraic.

EXAMPLE. Consider $\mathbf{R} \times V \times W$, where V and W are vector spaces over $\mathbf{R}$ of dimension n. Let $v_1, \dots, v_n$ be a basis of V and $w_1, \dots, w_n$ be a basis of W. Define the group operation

$$\begin{aligned}\langle r, v, w\rangle \cdot \langle r', v', w'\rangle &= \Big\langle r, \sum \alpha_i v_i,\ \sum \beta_i w_i \Big\rangle \cdot \Big\langle r', \sum \alpha'_i v_i,\ \sum \beta'_i w_i \Big\rangle \\ &= \Big\langle r + r', \sum (\alpha_i + \alpha'_i) v_i,\ \sum (\beta_i + \beta'_i) w_i + \sum r \alpha'_i w_i \Big\rangle.\end{aligned}$$

Then $G = \mathbf{R} \times V \times W$ is a nilpotent goup with centre $\langle 0, 0, W\rangle$.

For each non-singular linear transformation $L : W \to W$, $L(w_i) = \sum \gamma_{ij} w_j$, we have a non-singular linear transformation $L^* : V \to V$, $L^*(v_i) = \sum \gamma_{ij} v_j$. Now for each L, define $\theta : G \to G$ by $\theta\langle r, v, w\rangle = \big\langle r,\ L^*(v),\ L(w)\big\rangle$. We claim that θ is an automorphism. Clearly, θ is one-to-one and onto. To show θ is a homomorphism, let g, $g' \in G$, where $g = \Big\langle r, \sum \alpha_i v_i,\ \sum \beta_i w_i \Big\rangle$ and $g' = \Big\langle r', \sum \alpha'_i v_i,\ \sum \beta'_i w_i \Big\rangle$. Then

$$\begin{aligned}\theta(g) \cdot \theta(g') &= \Big\langle r, \sum \alpha_i L^*(v_i),\ \sum \beta_i L(w_i) \Big\rangle \cdot \Big\langle r', \sum \alpha'_i L^*(v_i),\ \sum \beta'_i L(w_i) \Big\rangle \\ &= \Big\langle r + r', \sum (\alpha_i + \alpha'_i) L^*(v_i),\ \sum (\beta_i + \beta'_i) L(w_i) + \sum r \alpha'_i L(w_i) \Big\rangle \\ &= \theta \Big\langle r + r', \sum (\alpha_i + \alpha'_i) v_i,\ \sum (\beta_i + \beta'_i) w_i + \sum r \alpha'_i w_i \Big\rangle \\ &= \theta(g \cdot g').\end{aligned}$$

Let Z be a uniform subgroup of W such that W/Z is an n-dimensional torus. Consider the group $G/Z = \mathbf{R} \times V \times W/Z$. Then the restriction map $\operatorname{Aut}(G/Z) \to \operatorname{Aut}(W/Z)$ is onto. If $\dim W = n \geqslant 2$, $\operatorname{Aut}(W/Z)$ has infinitely many connected components. Thus $\operatorname{Aut}(G/Z)$ has infinitely many connected components, and is therefore not almost algebraic.

Example. Let H be the three-dimensional Heisenberg group and let D be a uniform subgroup contained in the centre of H. Let T be a 1-dimensional torus. Consider the group $H/D \times T$. Then $\operatorname{Aut}(H/D \times T)$ has infinitely many connected components. This follows from the following proposition.

Proposition 3.1. *Let H be an analytic group and suppose that the maximal central torus in H is of dimension 1. Let T be a 1-dimensional torus, and let $G = H \times T$. Then $\operatorname{Aut} G$ has infinitely many connected components. Hence $\operatorname{Aut} G$ is not almost algebraic.*

Proof: Let $T_1 \subseteq H$ be the central torus. Consider the automorphisms of the 2-dimensional torus $T_1 \times T$. Then $\pi : \mathbf{R} \times \mathbf{R} \to T_1 \times T$ is the simply connected group covering, where π is the quotient morphism. Thus $\operatorname{Aut}(T_1 \times T)$ may be identified with the subgroup of $\operatorname{Aut}(\mathbf{R} \times \mathbf{R})$ consisting of all $\mathbf{R}$-linear automorphisms $\theta : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ with $\theta(\mathbf{Z} \times \mathbf{Z}) = \mathbf{Z} \times \mathbf{Z}$. If we identify $\operatorname{Aut}(\mathbf{R} \times \mathbf{R})$ with $GL(2, \mathbf{R})$, then $\operatorname{Aut}(T_1 \times T)$ is $GL(2, \mathbf{Z})$, the group of all 2×2 integral matrices with determinant ± 1.

Let $B \in \mathbf{Z}$, and define $\theta_B : G \to G$ by $\theta_B\left(h, e^{2\pi i t}\right) = \left(h e^{2\pi i B t}, e^{2\pi i t}\right)$. Then θ_B is an automorphism of G. Since $\theta_B(T_1 \times T) = T_1 \times T$, we can consider the restriction map $\operatorname{Aut} G \to \operatorname{Aut} T_1 \times T$. Let $B \in \mathbf{Z}$, and consider the automorphism of $T_1 \times T$ given by $\widetilde{\theta}_B\left(e^{2\pi i s}, e^{2\pi i t}\right) = \left(e^{2\pi i (s + Bt)}, e^{2\pi i t}\right)$. Then $\widetilde{\theta}_B$ extends to the automorphism $\theta_B : G \to G$. Thus the image of $\operatorname{Aut} G$ under the restriction map contains $\left\{\widetilde{\theta}_B : B \in \mathbf{Z}\right\}$. Thus, $\operatorname{Aut} G$ has infinitely many connected components. ∎

To conclude, we consider the following result of Dani [**2**]. If G is an analytic group with no compact central subgroup of positive dimension, then the group of automorphisms that leave a central element fixed is almost algebraic. For such a G, $\operatorname{Aut} G$ is almost algebraic. It is natural to ask the following question: if G is an analytic group whose automorphism group is almost algebraic, is the group of automorphisms that leave a central element fixed necessarily almost algebraic? Here we provide an example to show that this is not the case in general.

Example. Let H be the three-dimensional Heisenberg group, and let D be a uniform subgroup contained in the centre of H. Let $G = H/D \times \mathbf{R}$. Since the maximal torus T_1 of G is of dimension 1, $\operatorname{Aut} G$ is almost algebraic. For $B \in \mathbf{Z}$ consider the automorphism θ_B of G given by $\theta_B(h, r) = \left(h e^{2\pi i B r}, r\right)$. Then $\theta_B(1,1) = (1,1)$. We claim $\Phi = \left\{\tau \in \operatorname{Aut} G : \tau(1,1) = (1,1)\right\}$ is not almost algebraic. Let E be the central subgroup generated by $(1,1) : E = \left\{(1, m) : m \in \mathbf{Z}\right\}$. Then $G/E \cong H/D \times T$, where T is a 1-dimensional torus. The centre of G/E is the 2-dimensional torus $T_1 \times T$. By viewing Φ as a subgroup of $\operatorname{Aut}(G/E)$, we can consider the restriction map $\Phi \to \Phi|_{T_1 \times T}$. The image of Φ under this restriction map is infinite, so Φ is not almost algebraic.

References

[1] P.B. Chen and T.S. Wu, 'On the component group of the automorphism group of a Lie group', *J. Austral. Math. Soc. Ser. A* **57** (1994), 365–385.

[2] S.G. Dani, 'On automorphism groups of connected Lie groups', *Manuscripta Math.* **74** (1992), 445–452.

[3] M. Goto, 'Dense imbeddings of locally compact connected groups', *Ann. of Math.* **61** (1955), 154–169.

[4] D. Wigner, 'On the automorphism group of a Lie group', *Proc. Amer. Math. Soc.* **45** (1974), 140–143.

[5] D. Wigner, 'Erratum to "On the automorphism group of a Lie group"', *Proc. Amer. Math. Soc.* **60** (1976), 376–376.

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