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ON CERTAIN POLYNOMIALS OF GAUSSIAN TYPE

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Introduction. We shall consider functions of the form

$$f(t) = \prod_{i=1}^{m} (t^{r_i} - 1)/(t^{s_i} - 1),$$

where $\{r_i\}$ and $\{s_i\}$ are sets of positive integers. Such functions were studied by E. Grosswald in [**2**], who took $\{s_i\}$ to be pairwise relatively prime, and asked the following two questions:

(a) When is $f(t)$ a polynomial?

(b) When does $f(t)$ have positive coefficients?

These questions arise naturally from the work of Allday and Halperin, who show in [**1**] that under suitable circumstance $f(t)$ will be the Poincare polynomial of the orbit space of a certain Lie group action. Grosswald gives a complete answer to (a), but (b) is a much harder question, and a complete answer is provided only for the case $m = 2$. His treatment involves the representation of the coefficients of $f(t)$ by partition functions, and uses a classical description by Sylvester of the semigroup generated by $\{s_i\}$.

In a more general vein, Halperin has shown the following: let $X = \prod_{i=1}^{m} K(Q, 2s_i)$, $Y = \prod_{i=1}^{m} K(Q, 2r_i)$, and let F be the homotopy theoretic fibre of a continuous map $X \to Y$. Suppose F has finite cohomological dimension; then (denoting the Poincare polynomial by $P(\ )$),

$$P(F) = P(X)/P(Y) = f(t^2)$$

is a polynomial with positive coefficients. It is not known which polynomials f so occur.

In this connection, Halperin now asks if the following hold:

(1) $\deg f \geqq m$

(2) $f(1) \geqq 2^m$.

With the above interpretation, these would be lower bounds on the cohomological dimension and the Euler characteristic of F, respectively.

The present paper is in effect a sequel to [**2**]. In section (1) we give an affirmative answer to Halperin's first question. The proof is based on a slightly strengthened version of Grosswald's polynomial criterion, and does not require the coefficients of f to be positive. Moreover, it is shown that, usually, the degree of f is actually of order m^2.

In section (2) we show that estimate (2) is valid in some special cases. Again, the proof depends on slightly strengthened results of Grosswald on

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positivity. We also give an elementary proof of the result for $m = 2$, using the approach taken by Nijenhuis and Wilf in their paper on representation of integers by linear forms [**3**]. A general answer to (2) must await a more inclusive result on positivity.

I would like to thank Steve Halperin for asking these questions, and Emil Grosswald and Jim Stasheff for discussing them.

1. Gaussian polynomials. For any pair of sequences of positive integers of length m

$$R = \{r_1, \ldots, r_m\}, S = \{s_1, \ldots, s_m\}$$

define a rational function

$$f_{R,S}(t) = \prod_{i=1}^{m} (t^{r_i} - 1) \Big/ \prod_{i=1}^{m} (t^{s_i} - 1).$$

(We will write $f(t)$ when there is no danger of ambiguity.)

The first proposition gives a necessary and sufficient condition for $f(t)$ to be a polynomial (thus generalizing the result in [**2**]). For any positive integer l, and any set T of positive integers, let $N_l(T)$ be the number of multiples of l in T.

PROPOSITION 1. *$f(t)$ is a polynomial if and only if $N_l(S) \leqq N_l(R)$ for all l.*

Proof. Let ζ be a primitive l^{th} root of unity. ζ is a root of $t^k - 1$ precisely when $l \mid k$, and has multiplicity one. Thus the order of $f(t)$ at ζ is $N_l(R) - N_l(S)$. Since the only possible poles of f are roots of unity, f is a polynomial if and only if $N_l(R) - N_l(S) \geqq 0$ for all l.

Examples.

1) The case treated by Grosswald is S pairwise relatively prime; ie, $N_l(S) \leqq 1$ for all l. Here the polynomial condition reduces to:

$$\text{for all } l, i;\ l \mid s_i \Rightarrow l \mid r_j, \text{ for some } j.$$

This can be simply restated as:

$$\text{for all } i;\ s_i \mid r_j \text{ for some } j.$$

2) The case treated by Franklin (see [**4**]) is

$$S = \{1, \ldots, m\}, R = \{k + 1, \ldots, k + m\}.$$

Here the polynomial condition is satisfied:

$$N_l(S) = [m/l]$$
$$N_l(R) = [(k + m)/l] - [k/l] \geqq [m/l]$$

3) An interesting case (suggested by J. Stasheff in the context of homogeneous spaces of Lie groups) is that of consecutive odd numbers,

$$S = \{1, 3, \ldots, 2m - 1\}$$
$$R = \{2k + 1, 2k + 3, \ldots, 2(k + m) - 1\}$$

For a fixed $l \in S$, let λ_r denote $r/l - [r/l]$, the fractional part of r/l.
Note that

$$[(a + b)/l] = [a/l] + [b/l] + [\lambda_a + \lambda_b].$$

Thus,

$$[2a/l] = \begin{cases} 2[a/l] & \text{if } \lambda_a < 1/2 \\ 2[a/l] + 1 & \text{if } \lambda_a \geqq 1/2. \end{cases}$$

That is, the parity of $[2a/l]$ corresponds to the size of λ_a. So, for example,

$$N_l(S) = [2m/l] - [m/l] = \begin{cases} [m/l] & \text{if } \lambda_m < 1/2 \\ [m/l] + 1 & \text{if } \lambda_m \geqq 1/2 \end{cases}$$

We also have

$$N_l(R) = [2(k + m)/l] - [(k + m)/] - \{[2k/l] - [k/l]\}.$$

According to the proposition, $f(t)$ is a polynomial if and only if for all $l \in S$,

$$\begin{aligned} 0 &\leqq N_l(R) - N_l(S) \\ &= \{[2(k + m)/l] - [2k/l] - [2m/l]\} - \{[(k + m)/l] - [k/l) - [m/l]\} \\ &= [\lambda_{2k} + \lambda_{2m}] - [\lambda_k + \lambda_m] \\ &= A_l - B_l \end{aligned}$$

Here A_l, $B_l = 0$ or 1, depending only on the values of λ_m, λ_k and λ_{m+k}, and we see that f is a polynomial unless for some $l \in S$, $A_l = 0$ and $B_l = 1$. By constructing a table of the eight possibilities λ_m, λ_k, $\lambda_{m+k} < 1/2$ or $\geqq 1/2$, and the corresponding values of A_l and B_l, we find that $A_l = 0$, $B_l = 1$ only in the case λ_k, $\lambda_m \geqq 1/2$ and $\lambda_{k+m} < 1/2$; ie, $[2m/l]$, $[2k/l]$ odd and $[2(k + m)/l]$ even.

Thus the result is:

$f(t)$ is a polynomial $\Leftrightarrow$ for all odd $l < 2m$,
$[2m/l]\cdot[2k/l]$ odd $\Rightarrow [2(k + m)/l]$ odd.

This leaves open the question as to whether there is a relevant "nice" condition on k and m alone.

When $f_{R,S}$ is a polynomial, there is a lower bound on its degree (as suggested by Halperin):

PROPOSITION 2. *Let* $f(t) = \prod_{i=1}^{m}(t^{r_i} - 1)/\prod_{i=1}^{m}(t^{s_i} - 1)$.
Suppose (1) *f is a polynomial*
(2) $r_i \neq s_j$ *all* i, j.

Then $\deg f \geqq m$.

Proof. The result is trivial if all $s_i = 1$; thus we may assume, after applying proposition 1 and renumbering R and S, the following:

$$s_1 > 1,\ s_1 \mid r_1,\ r_1' = r_1/s_1 > 1 \quad \text{and}$$
$$r_1' = \text{Max}\ \{r_i/s_j;\ s_j \mid r_i \text{ and } s_j \neq 1\}.$$

If $r_1' = 2$, we see (after renumbering) that $r_i = e_i s_i$, where $e_i = 2$ if $s_i \neq 1$, and so the result follows. Thus we may assume $r_1' \geqq 3$.

Now consider $f'(t) = f_{R',S'}(t)$, where $R' = \{r_1', r_2, \ldots, r_m\}$, $S' = \{1, s_2, \ldots, s_m\}$. Note that for $l \nmid s_1$,

$$N_l(R') = N_l(R) \geqq N_l(S) = N_l(S')$$

while for $l \mid s_1$,

$$N_l(R') \geqq N_l(R) - 1 \geqq N_l(S) - 1 = N_l(S').$$

Thus, according to Proposition 1, $f'(t)$ is a polynomial. We have

(A) $\deg f = \deg f' + (r'_1 - 1)(s_1 - 1)$.

After cancellation of the new terms $t - 1, t^{r_1'} - 1$ in $f'(t)$ if they are duplicated, f' satisfies condition (2) of the proposition, and so by induction on $\sum s_i$,

$$\deg f' \geqq m - 2.$$

Now apply (A), $r_1' \geqq 3$ and $s_1 > 1$ to obtain the result.

This estimate can be considerably improved when there are no repetitions among the factors.

PROPOSITION 3. *With hypotheses* (1), (2) *of Proposition* 2, *suppose also* (3): $\{r_i\}$, $\{s_i\}$ *all distinct.*
Then (a) $\deg f \geqq \sum s_i - 3$
(b) *If* $s_i \neq 2$ *all* i, $\deg f \geqq \sum s_i - 1$
(c) *If* $s_i \neq 2, 3$ *all* i, $\deg f \geqq \sum s_i$.

Proof. Proceeding as in the proof of Proposition 2, if $r_1' = 2$, the current strongest estimate is easily seen to hold. Consider now $r_1' \geqq 3$.

Proof of (c): After cancelling $t - 1$ and $r^{r_1'} - 1$ in $f'(t)$ (if duplicated), we have by induction

(B) $\deg f' \geqq \sum s_i - s_1 - r_1'$.

Combining (A) and (B) we obtain

(C) $\deg f \geqq \sum s_i + (r_1' - 2)(s_1 - 2) - 3$.

Thus the result holds if $(r_1' - 2)(s_1 - 2) \geqq 3$. But in case (c), $s_1 - 2 \geqq 2$, so we are done if $r_1' \geqq 4$. Now if $r_1' = 3$, $s_i \neq r_i'$ for all i, so $t^{r_1'} - 1$ will not be duplicated in the denominator of f', and will not be cancelled. The induction assumption (B) may then be replaced by the stronger

(B_1) $\deg f' \geqq \sum s_i - s_1$

and combining (B_1) with (A),

$$\deg f \geqq \sum s_i - s_1 + 2(s_1 - 1) > \sum s_i.$$

Proof of (b): According to (B) we are done if $r_1' \geqq 4$ and $s_1 \geqq 4$. Suppose $s_1 = 3$, $r_1' \geqq 4$ or $s_1 \geqq 4$, $r_1' = 3$. Since the $\{r_i\}$, $\{s_i\}$ are all distinct, after the cancelling step f' will fall under case (c), and so we may use the inductive step (B). Since then $(r_1' - 2)(s_1 - 2) - 3 \geqq -1$, application of (C) yields the result.

The remaining possibility is $s_1 = r_1' = 3$. In this case we may similarly use (B_1) of case (c). This concludes the proof of (b).

Note that if no $r_i = 1$, then $t - 1$ does not disappear from f', and the estimate becomes again $\deg f \geqq \sum s_i$.

Proof of (a): We may assume some $s_i = 2$. Suppose $s_1 = 2$. After cancelling, $t - 1$ does not occur in the numerator of f', and we may apply the above note to f', and thus may use (C):

$$\deg f \geqq \sum s_i - 3.$$

Now suppose $s_1 > 2$; then (say) $s_2 = 2$. By (A) and induction, we are done if $(r_1' - 2)(s_1 - 2) \geqq 3$. Now consider the remaining possibilities. Both $s_1 = 3$, $r_1' = 4$ and $s_1 = 4$, $r_1' = 3$ are impossible by the maximality of r_1', since then $r_1/s_2 = 6$. The case $s_1 = r_1' = 3$ is disposed of as in the proof of (b) above.

2. Positive coefficients. Let p, $q > 1$ be positive integers, and denote by $\Gamma(p, q)$ the additive semigroup generated by p, q. The elements of $\Gamma(p, q)$ are said to be representable. If $(p, q) = d$, then $\Gamma(p, q) = d\Gamma(p/d, q/\mathrm{d})$, so one need consider only the case p, q relatively prime. This semigroup was first studied by Sylvester, who showed in [**5**] that $\Gamma(p, q)$ is a cofinite set, and that if we denote by $\Omega(p, q)$ the complement of $\Gamma(p, q)$ (ie, the set of non-representable integers) and by $\kappa(p, q) - 1$ the largest element of Ω, then $\kappa = (p - 1)\ (q - 1)$ and $\#\Omega = \kappa/2$. Here the notation is that of [**3**]. In that paper, Nijenhuis and Wilf reprove this result in the more general setting of representation of integers by linear forms, using a "reversal map" $x \leftrightarrow (\kappa - 1 - x)$ between representable and non-representable integers. Grosswald (in [**2**]) has shown how Sylvester's result can be applied, via a partition theoretic argument, to the determination of when $f_{R,S}$ has positive coefficients, in the case $m = 2$. The result is as follows:

PROPOSITION 4. (Halperin, Grosswald) *Suppose* $f(t) = (t^{r_1} - 1)(t^{r_2} - 1)/(t^{s_1} - 1)(t^{s_2} - 1)$ *is a polynomial. Then* $f(t)$ *has positive coefficients* $\Leftrightarrow$ $r_1, r_2 \in \Gamma(s_1, s_2)$.

The following is a quite elementary proof, avoiding the use of partition functions, based on Nijenhuis and Wilf's reversal map.

LEMMA 1. *Let* $(p, q) = 1$, $\Omega = \Omega(p, q)$, $\kappa = \kappa(p, q)$. *Then*

$$(t^{pq} - 1)/(t^p - 1)(t^q - 1) = \sum_{\omega \in \Omega} t^{\omega} + 1/(t - 1).$$

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Proof. Let $g(t) = (t^{pq} - 1)/(t^p - 1)(t^q - 1)$. For any $n \geqq m$ we may write

$$t^n/(t^m - 1) = \sum_{j=1}^{[n/m]} t^{n-jm} + \rho(t)/(t^m - 1)$$

where $\deg \rho < m$.
Apply this repeatedly to $g(t)$, with $m = p, q$, to obtain

$$\text{(D)} \quad g(t) = \sum t^{pq-(ab+bq)} + \sigma(t)/(t^p - 1)(t^q - 1),$$

where the sum is over $\{(a, b); a, b \geqq 1, ap + bq \leqq pq\}$ and $\deg \sigma < p + q$.

On the other hand, we know that $f(t) = (t - 1)g(t)$ is a polynomial (according to Proposition 1) and $f(1) = 1$. Thus

$$\sigma(t)/(t^p - 1)(t^q - 1) = 1/(t - 1).$$

Now in (D) substitute $r = a - 1$, $s = b - 1$ and we have

$$g(t) = \sum t^{pq-p-q-(rp+sq)} + 1/(t - 1),$$

where the sum is over $\{(r, s); r, s \geqq 0, rp + sq \leqq pq - p - q\}$. But $pq - p - q = \kappa - 1$, and $rp + sq$ represents distinct elements of $\Gamma(p, q)$, and thus $\kappa - 1 - (rp + sq)$ represents each element of Ω exactly once (this is the reversal map of [**3**]). This completes the proof of Lemma 1.

Proof of Proposition 4: Since $f(t)$ is a polynomial, according to Proposition 1 either $s_1 \mid r_1$, $s_2 \mid r_2$ or $s_1, s_2 \mid r_1$ (with suitable numbering). In the former case, both conditions of the conclusion of the proposition clearly hold. In the latter case, let $d = (s_1, s_2)$. Then $N_d(S) = 2$ so we must have $N_d(R) = 2$; i.e., $d \mid r_2$. Thus f is a polynomial in t^d; replacing t^d with t puts f into the following form:

$$f(t) = (t^{apq} - 1)(t^\gamma - 1)/(t^p - 1)(t^q - 1)$$

where $(p, q) = (s_1/d, s_2/d) = 1$. Clearly, it is sufficient to consider only the case $a = 1$. Applying Lemma 1 to this expression, we obtain

$$\text{(E)} \quad f(t) = \sum_{\omega \in \Omega} t^{\gamma+\omega} - \sum_{\omega \in \Omega} t^\omega + \sum_{k=0}^{\gamma-1} t^k.$$

Suppose $\gamma \notin \Gamma$; i.e., $\gamma \in \Omega$. The term $-t^\gamma$ occurs in the second sum in (E), and no other term of degree γ occurs. Thus $f(t)$ does not have positive coefficients. In fact, the negative terms of f are precisely $\{-t^\omega; \omega \in \Omega, \omega - \gamma \in \Gamma\}$.

Now suppose $\gamma \in \Gamma$; then each term $-t^\omega$ of the second sum in (E) is cancelled by another term. For, if $\omega \geqq \gamma$, then $\omega - \gamma \in \Omega$ and thus t^ω occurs in the first sum; if $\omega \leqq \gamma - 1$, then t^ω occurs in the third sum. This completes the proof.

Here one can see that the terms which do not occur in $f(t)$ are precisely all t^α where $\alpha \in \Omega$ or $\alpha - \gamma \in \Gamma$; the latter condition can be restated (by reversal) as $\alpha = \kappa - 1 + \gamma - \omega$ with $\omega \in \Omega$. Noting that $\kappa - 1 + \gamma = \deg f$, we see that the terms which do not occur in f are those of non-representable degree or codegree. This remark appears in [**2**] in this form.

Proposition 4 provides an easy answer to Halperin's second question for $m = 2$; that is, if $f(t)$ is a polynomial with positive coefficients then $f(1) \geqq 4$. For larger values of m, we can use the following rather weak necessary condition for positivity (stated and proved in [**2**] in special cases):

Let $f(t) = f_{R,S}(t)$, and let Γ be the semigroup generated by S.

LEMMA 2. *If f has positive coefficients, then $r_i \in \Gamma$ for all i.*

Proof. Write

$$f(t) = \prod (1 - t^{r_i}) \Big/ \prod (1 - t^{s_i})$$
$$= \left(1 - \sum t^{r_i} + \sum t^{r_i + r_j} - \dots\right) \left(\sum_{\gamma \in \Gamma} c_\gamma t^\gamma\right), \quad \text{where } c_0 = 1.$$

Let r_j be the least non-representable element of R. Then $-t^{r_j}$ occurs as a term in $f(t)$ unless

$$r_j = \gamma_0 + \sum_{i=1}^{2n} r_{k_i},$$

with $\gamma_0 \in \Gamma$ and $r_{k_i} \in R$. But then $r_{k_i} < r_j$, so each r_{k_i} would be representable, and consequently r_j would be representable.

To apply Lemma 2, we call $f_{R,S}$ *elementary* if $(\prod s_i) \mid r_1$.

PROPOSITION 5. *Let f be an elementary polynomial such that $r_i \neq s_j$ all i, j and $s_j \neq 1$ all j. If f has positive coefficients, then $f(1) \geqq 2^m$.*

Proof. $f(1) = \prod r_i / \prod s_i \geqq r_2 \dots r_m$, since f is elementary. Let $s_1 = \operatorname{Min} \{s_j\}$; by Lemma 2, $r_i \geqq 2s_1$, all i. Since $s_1 > 1$, the result follows.

Clearly a much stronger bound on $f(1)$ holds for elementary polynomials, but the example $\prod (t^{2s_i} - 1)/\prod (t^{s_i} - 1)$ shows that the given estimate is the best we can hope for in general.

Finally, we note that in [**2**], Grosswald considers only the case of S pairwise relatively prime. Under this assumption, Halperin's bound is valid.

PROPOSITION 6. *Let $f = f_{R,S}$ with $r_i \neq s_j$ all i, j and $s_j \neq 1$ all j, and with $\{s_j\}$ pairwise relatively prime. If f is a polynomial with positive coefficients, then $f(1) \geqq 2^m$.*

Proof. According to Proposition 1, each s_j divides some r_i. Grouping together all s_j dividing the same r_i (in arbitrary fashion, so as to account for all the s_j), we write (after renumbering)

$$f(t) = \prod_{i=1}^{a} \left[(t^{r_i} - 1) \Big/ \prod_{s_j \mid r_i} (t^{s_j} - 1) \right] \cdot \prod_{i=a+1}^{a+b} [(t^{r_i} - 1)/(t^{s_i} - 1)]$$
$$\cdot \prod_{i=a+b+1}^{m} (t^{r_i} - 1).$$

Here the first a factors are all those with more than one term in the denominator. Now for $i \leqq a$, $r_i \geqq \prod_{s_j | r_i} s_j$, so we have

$$f(1) \geqq \prod_{i=a+1}^{a+b} (r_i/s_i) \cdot \prod_{i=a+b+1}^{m} r_i.$$

As before, by Lemma 2, $r_i \geqq 2 \cdot \text{Min}\{s_i\} \geqq 4$ and $r_i/s_i \geqq 2$, so

$$f(1) \geqq 2^b \cdot 4^{m-a-b} = 2^{2m-(2a+b)}.$$

Now since each factor for $i \leqq a$ involves at least two of the s_j, we know

$$2a + b \leqq m.$$

Putting these two inequalities together, we are done.

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