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Nagoya Math. J.
Vol. 94 (1984), 171-191

ON A DUAL RELATION FOR ADDITION FORMULAS OF ADDITIVE GROUPS, I

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Introduction

This is the first in a series of papers concerned with a relation between a representation in the polynomial ring of additive groups and its translation invariant operators. The present study is to observe several properties of a polynomial sequence $p_\alpha(x)$ satisfying the binomial identity:

$$p_\alpha(x + y) = \sum_{\alpha=\beta+\gamma} p_\beta(x)p_\gamma(y) , \tag{1}$$

by means of some translation invariant operators. For example, to take the very simple case, that is, $x^\alpha/\alpha!$, the set of translation invariant operators is $\{\partial/\partial x_1, \cdots, \partial/\partial x_n\}$. This technique is the so called "umbral calculus" or "symbol calculus" widely used in the past century (cf. [2], [7]). This gives an effective technique for expressing a set of polynomials in terms of another.

In this series, we call it the dual relation for addition formulas of the additive groups. In the case of a polynomial sequence of one variable, G.C. Rota etc. [9] deals with the dual relation. In this series, we investigate the case of generic n variables.

Let us give a brief description of the contents of this paper.

Section 1 deals with one to one correspondence between a polynomial sequence with the binomial identity (1) and a set of n translation invariant operators.

Section 2 deals with the expansion theorem by a polynomial sequence with the binomial identity (1). Then as corollaries of this theorem, we obtain a characterization of the polynomial sequence by a numerical sequence, and a generating function of the polynomial sequence.

Section 3, that is a main result, deals with an analogy of the classical

Received May 10, 1983.

https://doi.org/10.1017/S0027763000020894 Published online by Cambridge University Press

Rodrigues' formula. Also, we obtain the so called "transfer formula" to connect $x^\alpha/\alpha!$ with the other polynomial.

Section 4 gives some nontrivial examples.

Let us enumerate symbols and notations used in this paper. The symbol Z_+^n is the subset in the n dimensional integral lattice Z^n, in which each point has all nonnegative entries. Let the Greek letters $\alpha, \beta, \cdots$ be vectors in Z^n and, for example, the components of α be written in the form

$$\alpha = (\alpha_1, \cdots, \alpha_n) .$$

Let $\{e_1, \cdots, e_n\}$ be a unit coordinate system, that is,

$$e_1 = (1, 0, \cdots, 0), \cdots, e_n = (0, \cdots, 0, 1) .$$

For $\alpha \in Z_+^n$ and the variable $x = (x_1, \cdots, x_n)$, the length $\alpha_1 + \cdots + \alpha_n$ of α is denoted by $|\alpha|$, and the polynomial $x^\alpha/\alpha!$ is $x_1^{\alpha_1}/\alpha_1! \cdots x_n^{\alpha_n}/\alpha_n!$. Instead of the partial differential operators $\partial/\partial x_1, \cdots, \partial/\partial x_n$, we use the symbol $\partial_1, \cdots, \partial_n$. Also the vector $(\partial/\partial x_1, \cdots, \partial/\partial x_n)$ is denoted by the symbol ∂. Let $P_1, \cdots, P_n$ be translation invariant operators. Then the multiple $P_1^{\alpha_1} \cdots P_n^{\alpha_n}$ is denoted by P^α. The notation $\delta_\alpha{}^\beta$ is a generalization of the Kronecker's delta symbol δ_{ij} such that

$$\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta , \\ 0 & \text{if } \alpha \neq \beta . \end{cases}$$

Chapter 1. Basic Polynomials

§1. Fundamental properties

We shall be concerned with the algebra (over a field K of characteristic zero) of all polynomials $p(x)$ in n variables to be denoted $\boldsymbol{P}$.

By a *polynomial sequence* we shall denote a sequence of polynomials $p_\alpha(x)$, where the parameter α is always in Z_+^n. A polynomial sequence $p_\alpha(x)$ is called to be of *binomial type* if it satisfies the followings:

(i) the degree of $p_\alpha(x)$ is $|\alpha|$;

(ii) setting $p_{e_i}(x) = \sum_{k=1}^n a_{ik}x_k + b_i$, $i = 1, \cdots, n$, the determinant $|a_{ij}|_{i,j=1,\cdots,n}$ does not vanish;

(iii) (*Binomial Identity*) $p_\alpha(x + y) = \sum_{\alpha=\beta+\gamma} p_\beta(x)p_\gamma(y)$.

The simplest sequence of binomial type is, of course, $x^\alpha/\alpha!$, but we give some nontrivial examples in Section 4.

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PROPOSITION 1.1.1. *The polynomial sequence $p_\alpha(x)$ of binomial type has the following properties*:

(i) $p_0(x) = 1, \qquad 0 = (0, \cdots, 0)$;

(ii) $p_\alpha(0) = 0$ *whenever* $\alpha \in Z^n_+ - 0$;

(iii) *the sequence $p_\alpha(x)$ generates the set of all polynomials.*

Proof. Setting $y = 0$ in (iii) of the binomial properties, we obtain

$$p_{e_i}(x) = p_{e_i}(x)p_0(0) + p_{e_i}(0)p_0(x), \qquad i = 1, \cdots, n .$$

Since each $p_{e_i}(x)$ is exactly of degree 1, it follows that

$$p_0(0) = 1 \text{ and ,hence, } p_0(x) = 1 \text{ and } p_{e_i}(0) = 0, \text{ for } i = 1, \cdots, n .$$

Hence, using (ii) of the binomial properties, and changing the coordinate, we get

(1.1.1) $$p_{e_i}(x) = x_i, \qquad i = 1, \cdots, n .$$

Fix the coordinate to satisfy (1.1.1). Let $p_\alpha(x)$ be written in the form

$$p_\alpha(x) = \sum_\beta a(\alpha; \beta)x^\beta/\beta! .$$

Iterating (iii) of the binomial properties, we obtain for $\beta = \beta_1 + \cdots + \beta_k$,

$$a(\alpha; \beta) = \sum_{\alpha = \alpha_1 + \cdots + \alpha_k} a(\alpha_1; \beta_1) \cdots a(\alpha_k; \beta_k) .$$

Hence, by (1.1.1), $a(\alpha; \beta)$ in the highest degree $|\alpha|$ of $p_\alpha(x)$ is equal to $\delta_{\alpha\beta}$. So, we obtain (iii). Also, setting $y = 0$ in (iii) of the binomial properties, and comparing the monomial of the highest degree of $p_\beta(x)$, we have (ii).
Q.E.D.

We introduce the another algebra $\sum$ to be the *algebra of translation invariant operators*. All operators we consider, are assumed to be *linear* and act the algebra of polynomials. Setting the *translation operators* E_a, that is,

(1.1.2) $$E_a p(x) = p(x + a) ,$$

an operator T to commute with E_a is said to be *translation invariant*.

The following proposition gives an isomorphism between the algebra of translation invariant operators and the algebra of all formal power series in the partial derivation ∂_i, $i = 1, \cdots, n$.

PROPOSITION 1.1.2. *If T is a translation invariant operator, then T is* *written in the form of a formal power series in the partial derivation* ∂_i, $i = 1, \cdots, n$.

Proof. Set the operator T in the following:

$$Tx^\alpha/\alpha! = \sum_\beta a^\alpha_\beta x^\beta/\beta!\ .$$

Since T is translation invariant, we have

$$TE_y x^\alpha/\alpha! = E_y Tx^\alpha/\alpha!\ .$$

By the linearity of T and the binomial property of $x^\alpha/\alpha!$,

$$TE_y x^\alpha/\alpha! = T(x+y)^\alpha/\alpha! = \sum_{\alpha=\beta+\delta} a^\beta_\gamma x^\gamma/\gamma!\ y^\delta/\delta!\ .$$

On the other hand, we have

$$E_y Tx^\alpha/\alpha! = E_y \sum_\beta a^\alpha_\beta x^\beta/\beta! = \sum_{\beta=\gamma+\delta} a^\alpha_\beta x^\gamma/\gamma!\ y^\delta/\delta!\ .$$

Hence, comparing the coefficients of the both sides, we get

$$a^{\alpha-\delta}_\gamma = a^\alpha_{\gamma+\delta}\ .$$

Setting $\gamma = 0$, we have

$$a^{\alpha-\delta}_0 = a^\alpha_\delta\ .$$

Here, we note that if $\alpha - \delta$ does not belong to $\boldsymbol{Z}^n_+$,

$$a^\alpha_\delta = 0\ .$$

Hence we obtain

$$Tx^\alpha/\alpha! = \sum_\beta a^{\alpha-\beta}_0 x^\beta/\beta! = \sum_\beta a^\beta_0 \partial^\beta x^\alpha/\alpha!\ .$$

Q.E.D.

Let $P_1, \cdots, P_n$ be translation invariant operators. Proposition 1.1.2 gives differential expressions $p_i(\partial)$ of P_i, i $= 1, \cdots, n$, respectively. A set of the translation invariant operators $\{P_1, \cdots, P_n\}$ is called the *delta set* if

(i) $p_i(0) = 0$, $i = 1, \cdots, n$;

(ii) the Jacobian at the origin

$$\begin{vmatrix} \partial p_1(0)/\partial\xi_1, & \cdots, & \partial p_n(0)/\partial\xi_1 \\ \vdots & & \vdots \\ \partial p_1(0)/\partial\xi_n, & \cdots, & \partial p_n(0)/\partial\xi_n \end{vmatrix} \neq 0\ .$$

Remark. The symbol $\{p_1(\xi), \cdots, p_n(\xi)\}$ of the delta set $\{P_1, \cdots, P_n\}$ is a regular homeomorphism fixing the origin. Hence, there exists the unique delta set $\{Q_1, \cdots, Q_n\}$ such that the symbol $\{q_1(\xi), \cdots, q_n(\xi)\}$ satisfies

$$q_i(p_1(\xi), \cdots, p_n(\xi)) = \xi_i, \qquad i = 1, \cdots, n.$$

The delta set $\{P_1, \cdots, P_n\}$ to satisfy

$$\partial p_i(0)/\partial \xi_j = \delta_{ij}$$

is said to be *normal*.

Let $\{P_1, \cdots, P_n\}$ be a delta set. A polynomial sequence $p_\alpha(x)$ is called the *sequence of the basic polynomials* for $\{P_1, \cdots, P_n\}$ if:

(i) $p_0(x) = 1, \qquad 0 = (0, \cdots, 0);$

(ii) $p_\alpha(0) = 0 \qquad$ whenever $\alpha \in \boldsymbol{Z}_+^n - 0;$

(iii) for each $i = 1, \cdots, n$

$$P_i p_\alpha(x) = \begin{cases} p_{\alpha-e_i}(x), & \text{if } \alpha - e_i \in \boldsymbol{Z}_+^n, \\ 0 & \text{if } \alpha - e_i \notin \boldsymbol{Z}_+^n. \end{cases}$$

PROPOSITION 1.1.3. *The followings are equivalent:*

(i) *the delta set* $\{P_1, \cdots, P_n\}$ *is normal;*

(ii) *the basic polynomials* $p_\alpha(x)$ *for* $\{P_1, \cdots, P_n\}$ *satisfy*

$$p_{e_i}(x) = x_i, \qquad i = 1, \cdots, n;$$

(iii) *the coefficients* $a(\alpha; \beta)$ *of the highest degree of the basic polynomial* $p_\alpha(x)$ *for* $\{P_1, \cdots, P_n\}$ *is equal to* $\delta_{\alpha\beta}$.

Proof. We shall prove (i) $\Rightarrow$ (iii). Inducing on the degree of $p_\alpha(x)$, we assume that (iii) holds for $p_\alpha(x)$ of the degree less than l. Set $p_\alpha(x)$ in the form

$$p_\alpha(x) = \sum_{|\beta| \neq 0} a(\alpha; \beta) x^\beta / \beta!,$$

and the differential expression $\{p_1(\partial), \cdots, p_n(\partial)\}$ for $\{P_1, \cdots, P_n\}$ as the following

$$P_i(\partial) = \partial_i + \sum_{|\alpha|>1} c_{i,\alpha} \partial^\alpha.$$

Hence, by (iii) of the basic polynomials, $a(\alpha; \beta)$ of the highest degree satisfy

$$a(\alpha; \beta) = a(\alpha - e_i; \beta - e_i), \qquad i = 1, \cdots, n,$$

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if $\alpha - e_i$ and $\beta - e_i$ belong to Z^n_+. But the right side of the above relation is a coefficient of the highest degree of $p_{\alpha - e_i}(x)$. Therefore we obtain (iii). It is clear (iii) $\Rightarrow$ (ii). Also, (i) and (iii) of the basic polynomial give (ii) $\Rightarrow$ (i). Q.E.D.

COROLLARY. *The monomial of the highest degree of the basic polynomial* $p_\alpha(x)$ *for the normal delta set is equal to* $x^\alpha/\alpha!$.

The basic polynomial $p_\alpha(x)$ for the normal delta set $\{P_1, \cdots, P_n\}$ is called the *normal basic polynomial.*

PROPOSITION 1.1.4. *Every delta set has a unique sequence of basic polynomials.*

Proof. By a change of the coordinate, we may consider the normal delta set. Inducing on the degree of polynomials, we assume that, for $|\alpha| < l$, $p_\alpha(x)$ satisfies the conditions of the basic polynomial. We show that, for $|\alpha| = l$, $p_\alpha(x)$ also exists and is unique. Let $p_\alpha(x)$ ($|\alpha| = l$) be written in the form

$$p_\alpha(x) = x^\alpha/\alpha! + \sum_{|\beta|<l} c_\beta p_\beta(x) .$$

Now, for $i = 1, \cdots, n$, we have

$$p_i(\partial)p_\alpha(x) = p_i(\partial)x^\alpha/\alpha! + \sum_{|\beta|<l} c_\beta p_{\beta - e_i}(x) ,$$

where, for $\beta - e_i \notin Z^n_+$, $p_{\beta - e_i}(x)$ is regarded as vanishing. Each $p_i(\partial)x^\alpha/\alpha!$ is at most of degree $l - 1$. Hence, Proposition 1.1.3 and (iii) of the basic polynomial give a unique choice of the constants c_β. Q.E.D.

The typical example of a basic polynomial sequence is $x^\alpha/\alpha!$, basic for $\{\partial_1, \cdots, \partial_n\}$. Several properties of the polynomial sequence $x^\alpha/\alpha!$ can be generalized to an arbitrary sequence of basic polynomials. A basic property of $x^\alpha/\alpha!$ is of binomial type. This turns out to be true for every sequence of basic polynomials.

THEOREM 1.1.5. (i) *If* $p_\alpha(x)$ *is a basic sequence for some delta set* $\{P_1, \cdots, P_n\}$, *then it is a sequence of polynomials of binomial type.*

(ii) *If* $p_\alpha(x)$ *is a sequence of polynomials of binomial type, then it is a basic sequence for some delta set.*

Proof. (i) By Proposition 1.1.3, the proof of (i) and (ii) of the

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binomial property is clear. We prove the binomial identity (iii). Iterating the property (iii) of basic polynomials, we see that

$$P^{\beta}p_{\alpha}(x) = p_{\alpha-\beta}(x) ,$$

where for $\alpha - \beta \notin Z_{+}^{n}$, we regard it as vanishing. Hence, for $\alpha = \beta$,

$$[P^{\alpha}p_{\alpha}(x)]_{x=0} = 1$$

while, for, $\alpha \neq \beta$,

$$[P^{\beta}p_{\alpha}(x)]_{x=0} = 0 .$$

Thus, we may trivially express $p_{\alpha}(x)$ in the form

$$p_{\alpha}(x) = \sum_{\beta} p_{\beta}(x)[P^{\beta}p_{\alpha}(x)]_{x=0} .$$

By Corollary of Proposition 1.1.3, this expression also holds for all polynomials $p(x)$, that is,

$$p(x) = \sum_{\beta} p_{\beta}(x)[P^{\beta}p(x)]_{x=0} .$$

Now suppose $p(x)$ is the polynomial $p_{\alpha}(x + y)$ for fixed y. Then

$$p_{\alpha}(x + y) = \sum_{\beta} p_{\beta}(x)[P^{\beta}p_{\alpha}(x + y)]_{x=0} .$$

But by the translation invariant property of P^{β}, we see

$$p_{\alpha}(x + y) = \sum_{\alpha=\beta+\gamma} p_{\beta}(x)p_{\gamma}(y) .$$

Hence, the sequence $p_{\alpha}(x)$ is of binomial type.

(ii) Suppose now $p_{\alpha}(x)$ is a sequence of binomial type. By Proposition 1.1.1, we have only to define a delta set for which such a sequence $p_{\alpha}(x)$ is the sequence of basic polynomials. Let P_i, $i = 1, \cdots, n$, be the operator defined by the following:

$$P_i p_{\alpha}(x) = \begin{cases} p_{\alpha-e_i}(x) & \text{if } \alpha - e_i \in Z_{+}^{n} , \\ 0 & \text{if } \alpha - e_i \notin Z_{+}^{n} . \end{cases}$$

We may trivially write (iii) of binomial type in the form

$$p_{\alpha}(x + y) = \sum_{\beta} p_{\beta}(x)P^{\beta}p_{\alpha}(y) ,$$

and, using (iii) in Proposition 1.1.1, this may be extended to all polynomials:

$$p(x + y) = \sum_{\beta} p_{\beta}(x)P^{\beta}p(y) .$$

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Now replace p by $P_i p$, and we obtain

$$(P_i p)(x+y) = \sum_\beta p_\beta(x) P^{\beta+e_i} p(y)\,, \qquad i=1,\cdots,n\,.$$

But

$$(P_i p)(x+y) = E_x(P_i p)(y)\,,$$

and

$$\sum_\beta p_\beta(x) P^{\beta+e_i} p(y) = P_i\Big(\sum_\beta p_\beta(x) P^\beta p(y)\Big)$$
$$= P_i(p(x+y)) = P_i E_x p(y)\,.$$

Therefore each P_i is a translation invariant operator. Using Proposition 1.1.2 and $P_i p_0(x) = 0$, we obtain (i) of the delta set. Hence, each differential expression $p_i(\partial)$ of P_i is expressed by

$$p_i(\partial) = \sum_{k=1}^n c_{ik}\partial_k + \sum_{|\beta|>1} c_{i,\beta}\partial^\beta, \qquad i=1,\cdots,n\,.$$

By (ii) of binomial type, the determinant $|c_{ij}|_{i,j=1,\cdots,n}$ does not vanish. Hence, (ii) of the delta set holds. Q.E.D.

§ 2. The expansion theorem

We study next the expansion of a translation invariant operator in terms of a delta set and its multiple powers. The difficulties caused by convergence questions are minimal, and we refuse to discuss them in this paper.

The following theorem generalizes the Taylor expansion formula to delta sets and their basic polynomials.

THEOREM 1.2.1. (First Expansion Formula). *Let T be a translation invariant operator, and $\{P_1,\cdots,P_n\}$ be a delta operator with a basic set $p_\alpha(x)$. Then*

$$T = \sum_\alpha a_\alpha P^\alpha$$

with

$$a_\alpha = [T p_\alpha(x)]_{x=0}\,.$$

Proof. Since $p_\alpha(x)$ are of binomial type, we may write the binomial identity as

$$p_\alpha(x+y) = \sum_\beta p_\beta(x) P^\beta p_\alpha(y)\,,$$

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where, for $\alpha - \beta \notin Z_+^n$, we regard $P^\beta p_\alpha(x)$ as vanishing. Apply T to the both sides regarding x as the variable and y as a parameter and get

$$Tp_\alpha(x+y) = \sum_\beta Tp_\beta(x) P^\beta p_\alpha(y) .$$

By Proposition 1.1.3, this expansion can be extended to all polynomials. Setting x equal to zero, we obtain the expansion formula. Q.E.D.

This theorem gives an explicit interpretation to the generating function of a sequence of basic polynomials.

COROLLARY 1.2.2. *Let* $\{P_1, \cdots, P_n\}$ *be a delta set with basic polynomials* $p_\alpha(x)$, *and set* $P_i = p_i(\partial)$, $i = 1, \cdots, n$. *Let* $\{q_1(\xi), \cdots, q_n(\xi)\}$ *be the inverse formal power series. Then*

$$\sum_\alpha p_\alpha(x)\xi^\alpha = \exp \sum_{i=1}^n x_i q_i(\xi) .$$

Proof. Expand the translation operator E_a in terms of $\{P_1, \cdots, P_n\}$ by the first expansion formula. Then, we have

$$\sum_\alpha p_\alpha(a) P^\alpha = E_a ,$$

a formula which can be considered as a generalization of Taylor's formula. By Proposition 1.1.2, we get

$$\sum_\alpha p_\alpha(a) p^\alpha(\zeta) = \exp \sum_{i=1}^n a_i \zeta_i ,$$

whence the conclusion, upon setting $\xi_i = p_i(\zeta)$, $i = 1, \cdots, n$, and $a = x$. Q.E.D.

Next, we obtain a useful characterization of basic polynomials as the following:

COROLLARY 1.2.3. *Given any sequence of constants* $c_i(\alpha)$, $i = 1, \cdots, n$, $(\alpha \in Z_+^n - 0)$ *with the determinant* $|c_i(e_j)|_{i,j=1,\cdots,n} \neq 0$, *there exists a unique sequence of basic polynomials* $p_\alpha(x)$ *such that*

$$[\partial_i p_\alpha(x)]_{x=0} = c_i(\alpha) .$$

Proof. Set

$$q_i(\xi) = \sum_\alpha c_i(\alpha)\xi^\alpha , \qquad i = 1, \cdots, n .$$

The above corollary gives a unique sequence of basic polynomials $p_\alpha(x)$

to satisfy

$$\exp \sum_{i=1}^{n} x_i \sum_{\alpha} c_i(\alpha)\xi^{\alpha} = \sum_{\alpha} p_{\alpha}(x)\xi^{\alpha} .$$

Operating ∂_i to the both sides and setting x equal to zero, we get the relation of this corollary. Q.E.D.

§ 3. Closed forms

For the first time we introduce operators that are not translation invariant. Let $p(x)$ be a polynomial. Multiplying each term of $p(x)$ by the variables x_i, $i = 1, \cdots, n$, we obtain a new polynomial $x_i p(x)$. Call this *multiplication operator* and we denote it by $\boldsymbol{x}_i$, $i = 1, \cdots, n$.

LEMMA 1.3.1. *Let the symbol of $f(\partial)$ be $f(\xi)$. Then its commutator*

$$[f(\partial), \boldsymbol{x}_i] = f(\partial)\boldsymbol{x}_i - \boldsymbol{x}_i f_i(\partial) , \qquad i = 1, \cdots, n ,$$

corresponds to a translation invariant operator with the symbol $(\partial_i f)(\xi)$.

The proof is a straightforward verification.

In this section, let $(\partial_i f)(\partial)$ be the differential operator corresponding to the symbol $(\partial_i f)(\xi)$.

As well known, the delta set $\{\partial_1, \cdots, \partial_n\}$ and the multiplication operators $\boldsymbol{x}_i$, $i = 1, \cdots, n$, satisfy the *Heisenberg Weyl relation*, that is,

$$[\partial_i, \partial_j] = [\boldsymbol{x}_i, \boldsymbol{x}_j] = 0 ,$$

and

$$[\partial_i, \boldsymbol{x}_j] = \delta_{ij}, \ i, j = 1, \cdots, n .$$

Now we construct another operators for generic delta set to satisfy the Heisenberg Weyl relation.

THEOREM 1.3.2. *Let $\{p_1(\xi), \cdots, p_n(\xi)\}$ be a symbol of the differential expression of the delta set $\{P_1, \cdots, P_n\}$. Setting the inverse Jacobi matrix:*

$$\begin{pmatrix} \partial_1 p_1(\xi), & \cdots, & \partial_n p_1(\xi) \\ \vdots & & \vdots \\ \partial_1 p_n(\xi), & \cdots, & \partial_n p_n(\xi) \end{pmatrix}^{-1} = (b_{ij}(\xi))_{i,j=1,\cdots,n} ,$$

then we have the following relations for i, $j = 1, \cdots, n$:

(i) $[p_i(\partial), p_j(\partial)] = 0$;

(ii) $[\sum_{k=1}^{n} \boldsymbol{x}_k b_{ki}(\partial), \sum_{k=1}^{n} \boldsymbol{x}_k b_{kj}(\partial)] = 0$;

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(iii) $[p_i(\partial), \sum_{k=1}^n x_k b_{kj}(\partial)] = \delta_{ij}$.

Proof. By Proposition 1.1.2, the proof of (i) is trivial. We can show (iii) immediately. Indeed, by Lemma 1.3.1,

$$\begin{aligned}&\left[p_i(\partial), \sum_{k=1}^n x_k b_{kj}(\partial)\right]\\ &= p_i(\partial)\sum_{k=1}^n x_k b_{kj}(\partial) - \sum_{k=1}^n x_k b_{kj}(\partial)p_i(\partial)\\ &= \sum_{k=1}^n (p_i(\partial)x_k - x_k p_i(\partial))b_{kj}(\partial)\\ &= \sum_{k=1}^n (\partial_k p_i)(\partial)b_{kj}(\partial)\\ &= \delta_{ij} .\end{aligned}$$

Lastly we prove (ii). The left side of (ii) is the following:

$$\begin{aligned}&\left[\sum_{k=1}^n x_k b_{ki}(\partial), \sum_{k=1}^n x_k b_{kj}(\partial)\right]\\ &= \sum_{k,l=1}^n x_k b_{ki}(\partial)x_l b_{lj}(\partial) - x_k b_{kj}(\partial)x_l b_{li}(\partial)\\ &= \sum_{k=1}^n x_k\left\{\sum_{l=1}^n [b_{ki}(\partial), x_l]b_{lj}(\partial) - [b_{kj}(\partial), x_l]b_{li}(\partial)\right\} .\end{aligned}$$

Therefore we have only to prove the following:

$$\sum_{l=1}^n (\partial_l b_{ki})(\xi)b_{lj}(\xi) - (\partial_l b_{kj})(\xi)b_{li}(\xi) = 0 . \tag{1.3.1}$$

Setting the inverse functions of $\{p_1(\xi), \cdots, p_n(\xi)\}$ by $\{q_1(\zeta), \cdots, q_n(\zeta)\}$, we have

$$\partial p_i(q_1(\zeta), \cdots, q_n(\zeta))/\partial\zeta_j = \delta_{ij} .$$

Hence, setting

$$(\xi_1, \cdots, \xi_n) = (q_1(\zeta), \cdots, q_n(\zeta)) ,$$

we have

$$\sum_{k=1}^n (\partial_k p_i)(\xi)(\partial_j q_k)(\zeta) = \delta_{ij} , \tag{1.3.2}$$

and so, for $i, j = 1, \cdots, n$,

$$b_{ij}(\xi) = (\partial_j q_i)(\zeta) .$$

Using (1.3.2) and

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$$\partial/\partial\xi_i = \sum_{k=1}^{n} (\partial_i p_k)(\xi)\partial/\partial\zeta_k ,$$

we can replace the left side of (1.3.1) as the following:

$$\begin{aligned}
&\sum_{l=1}^{n} (\partial_l b_{ki})(\xi) b_{lj}(\xi) - (\partial_l b_{kj})(\xi) b_{li}(\xi) \\
&\quad = \sum_{l=1}^{n} \partial(\partial_i q_k(\zeta))/\partial\xi_l \, (\partial_j q_l)(\zeta) - \partial(\partial_j q_k(\zeta))/\partial\xi_l \, (\partial_i q_l)(\zeta) \\
&\quad = \sum_{l,m=1}^{n} (\partial_l p_m)(\xi)(\partial_m\partial_i q_k)(\zeta)(\partial_j q_l)(\zeta) - (\partial_l p_m)(\xi)(\partial_m\partial_j q_k)(\zeta)(\partial_i q_l)(\zeta) \\
&\quad = \sum_{m=1}^{n} \delta_{jm}(\partial_m\partial_i q_k)(\zeta) - \delta_{im}(\partial_m\partial_j q_k)(\zeta) \\
&\quad = 0 .
\end{aligned}$$

Hence, we obtain (1.3.1). Q.E.D.

Now, we obtain a creation operator for a sequence of basic polynomials. In the theory of classical polynomials [11], it is well known as Rodrigues' formula.

COROLLARY 1.3.3 (Rodrigues' Formula). *Using the notation of Theorem* 1.3.2, *we have for the basic polynomial* $p_\alpha(x)$ *of* $\{P_1, \cdots, P_n\}$

$$p_{\alpha+e_i}(x) = (\alpha_i + 1)^{-1}\Big(\sum_{k=1}^{n} x_k b_{ki}(\partial)\Big) p_\alpha(x) , \qquad i = 1, \cdots, n .$$

PROPOSITION 1.3.4. *Let* $\{P_1, \cdots, P_n\}$ *be a normal delta set and* $p_\alpha(x)$ *be the normal basic polynomials for* $\{P_1, \cdots, P_n\}$. *Then the differential expression* $\{p_1(\partial), \cdots, p_n(\partial)\}$ *of* $\{P_1, \cdots, P_n\}$ *satisfies the following conditions*:

(1.3.3) $$p_i(\partial) = \partial_i q_i(\partial) ,$$

and

(1.3.4) $$q_i(0) = 1 , \qquad i = 1, \cdots, n$$

if and only if the constants $[\partial_i p_\alpha(x)]_{x=0} = a_i(\alpha)$, $i = 1, \cdots, n$, *in Corollary* 1.2.3 *satisfy the following*

(1.3.5) $$a_i(\alpha) = 0 \qquad \text{whenever } \alpha - e_i \notin Z_+^n .$$

Proof. Set

$$p_i(\partial) = \sum_\alpha c_i(\alpha)\partial^\alpha , \qquad i = 1, \cdots, n$$

and

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$$p_\alpha(x) = \sum_\beta a(\alpha;\, \beta) x^\beta/\beta!\ .$$

The property (iii) of the basic polynomial gives

$$\sum_\beta c_i(\beta) a(\alpha;\, \beta + \gamma) = a(\alpha - e_i;\, \gamma)\,, \qquad i = 1, \cdots, n. \tag{1.3.6}$$

Setting $\gamma = 0$ in (1.3.6) and using Proposition 1.1.3.

$$c_i(\alpha) + \sum_{|\beta| < |\alpha|} a(\alpha;\, \beta) c_i(\beta) = \delta_{\alpha, e_i}\ . \tag{1.3.7}$$

Iterating (1.3.7), we obtain

$$c_i(\alpha) = \sum_{k=0} (-1)^k \sum_{|\alpha| > |\beta_1| > \cdots > |\beta_k|} a(\alpha;\, \beta_1) \cdots a(\beta_{k-1};\, \beta_k) \delta_{\beta_k, e_i}\,, \quad i = 1, \cdots, n\ . \tag{1.3.8}$$

Also the binomial identity (iii) gives, for $\beta = \beta_1 + \cdots + \beta_l$,

$$a(\alpha;\, \beta) = \sum_{\alpha = \alpha_1 + \cdots + \alpha_l} a(\alpha_1;\, \beta_1) \cdots a(\alpha_l;\, \beta_l)\ . \tag{1.3.9}$$

Suppose that (1.3.5) holds. Note that $a(\alpha;\, e_i) = a_i(\alpha)$. By (1.3.9), if $\beta - e_i$ is contained in Z^n_+, $a(\alpha;\, \beta)$ vanishes except the case of $\alpha - e_i \in Z^n_+$. Hence, by (1.3.8), we see that if $\alpha - e_i$ is not contained in Z^n_+, $c_i(\alpha)$ is equal to zero. So, (1.3.3) and (1.3.4) hold.

Conversely, suppose that (1.3.3) and (1.3.4) hold. For $|\alpha| = 1$, it is trivial to prove (1.3.5). Inducing for the length of α, that is, $|\alpha|$, we assume that for $|\alpha| < \ell - 1$, (1.3.5) holds. By using (1.3.8) and (1.3.9), we easily prove (1.3.5) similar to the proof of the sufficient condition. Q.E.D.

To prove the transfer formula, we need a property of a determinant with noncommutative entries.

Let A_{ij}, $i, j = 1, \cdots, n$, be linear operators. We define the *determinant with the noncommutative entries* A_{ij} such that

$$\begin{vmatrix} A_{11}, & \cdots, & A_{1n} \\ A_{n1}, & \cdots, & A_{nn} \end{vmatrix} \equiv \sum_\sigma \varepsilon(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)}\,,$$

where σ runs in the permutation group of order n, and $\varepsilon(\sigma)$ is the signature of σ.

Let $\mathfrak{S}$ be an algebra generated by operators A_i, B_j, $i, j = 1, \cdots, n$ and the identity operator I. Then we define the *operators* $\hat{A}_i$, $\hat{B}_i$ *and* $\hat{B}^*_j$, $i, j = 1, \cdots, n$, *on* $\mathfrak{S}$ such that for $X \in \mathfrak{S}$,

$$\hat{A}_i X = A_i X\,, \qquad \hat{B}_i X = B_i X \tag{1.3.10}$$

and

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$$\hat{B}_j^* X = X B_j . \tag{1.3.11}$$

LEMMA 1.3.5. *Suppose that B_i commutes B_j for $i, j = 1, \cdots, n$. Then we have following*

$$\begin{vmatrix} \hat{A}_1\hat{B}_1^* - [\hat{A}_1, \hat{B}_1], & -[\hat{A}_1, \hat{B}_2], \cdots, & -[\hat{A}_1, \hat{B}_n] \\ -[\hat{A}_2, \hat{B}_1], & \hat{A}_2\hat{B}_2^* - [\hat{A}_2, \hat{B}_2], \cdots, & -[\hat{A}_2, \hat{B}_n] \\ \vdots & \ddots & \vdots \\ -[\hat{A}_n, \hat{B}_1], & \cdots\cdots\cdots\cdots, & \hat{A}_n\hat{B}_n^* - [\hat{A}_n \hat{B}_n] \end{vmatrix} I = \begin{vmatrix} \hat{B}_1\hat{A}_1, \hat{B}_2\hat{A}_1, & \cdots\cdots\cdots\cdots\cdots\cdots, & \hat{B}_n\hat{A}_1 \\ -[\hat{A}_2, \hat{B}_1], & \hat{A}_2\hat{B}_2^* - [\hat{A}_2, \hat{B}_2], \cdots, & -[\hat{A}_2, \hat{B}_n] \\ \vdots & \ddots & \vdots \\ -[\hat{A}_n, \hat{B}_1], & \cdots\cdots\cdots\cdots, \hat{A}_n\hat{B}_n^* & -[\hat{A}_n, \hat{B}_n] \end{vmatrix} I, \tag{1.3.12}$$

where $[\hat{A}_i, \hat{B}_j]$ is the commutator:

$$\hat{A}_i\hat{B}_j - \hat{B}_j\hat{A}_i$$

Proof. From the definition of the determinant, we have only to prove the identity:

$$\sum_\sigma \varepsilon(\sigma)\hat{A}_1(\delta_{1\sigma(1)}\hat{B}_1^* - \hat{B}_{\sigma(1)})(\delta_{2\sigma(2)}\hat{A}_2\hat{B}_2^* - [\hat{A}_2, \hat{B}_{\sigma(2)}]) \cdots (\delta_{n\sigma(n)}\hat{A}_n\hat{B}_n^* - [\hat{A}_n, \hat{B}_{\sigma(n)}])I = 0 . \tag{1.3.13}$$

Using

$$[\hat{B}_i, \hat{B}_j] = 0, \qquad i, j = 1, \cdots, n,$$

we have

$$\sum_\sigma \varepsilon(\sigma)\hat{A}_1(\delta_{1\sigma(1)}\hat{B}_1^* - \hat{B}_{\sigma(1)})(\delta_{1\sigma(2)}\hat{B}_1^* - \hat{B}_{\sigma(2)})\hat{A}_2(\delta_{3\sigma(3)}\hat{A}_3\hat{B}_3^* - [\hat{A}_3, \hat{B}_{\sigma(3)}]) \cdots (\delta_{n\sigma(n)}\hat{A}_n\hat{B}_n^* - [\hat{A}_n, \hat{B}_{\sigma(n)}])I = 0 . \tag{1.3.14}$$

Hence, adding the left side of (1.3.13) to (1.3.14), we obtain

$$\sum_\sigma \varepsilon(\sigma)\hat{A}_1(\delta_{1\sigma(1)}\hat{B}_1^* - \hat{B}_{\sigma(1)})\hat{A}_2((\delta_{1\sigma(2)}\hat{B}_1^* + \delta_{2\sigma(2)}\hat{B}_2^*) - \hat{B}_{\sigma(2)}) \times (\delta_{3\sigma(3)}\hat{A}_3\hat{B}_3^* - [\hat{A}_3, \hat{B}_{\sigma(3)}]) \cdots (\delta_{n\sigma(n)}\hat{A}_n\hat{B}_n^* - [\hat{A}_n, \hat{B}_{\sigma(n)}])I . \tag{1.3.15}$$

Going on the similar method, we obtain

$$\begin{aligned} &\sum_\sigma \varepsilon(\sigma)\hat{A}_1(\delta_{1\sigma(1)}\hat{B}_1^* - \hat{B}_{\sigma(1)})(\delta_{2\sigma(2)}\hat{A}_2\hat{B}_2^* - [\hat{A}_2, \hat{B}_{\sigma(2)}]) \\ &\qquad \cdots (\delta_{n\sigma(n)}\hat{A}_n\hat{B}_n^* - [\hat{A}_n, \hat{B}_{\sigma(n)}])I \\ &= \sum_\sigma \varepsilon(\sigma)\hat{A}_1(\delta_{1\sigma(1)}\hat{B}_1^* - \hat{B}_{\sigma(1)})\hat{A}_2((\delta_{1\sigma(2)}\hat{B}_1^* + \delta_{2\sigma(2)}\hat{B}_2^*) - \hat{B}_{\sigma(2)}) \\ &\qquad \cdots \hat{A}_n\Big(\sum_{k=1}^n \delta_{k\sigma(n)}\hat{B}_k^* - \hat{B}_{\sigma(n)}\Big)I . \end{aligned} \tag{1.3.16}$$

Using

$$\Big(\sum_{k=1}^{n} \delta_{k\sigma(n)} \hat{B}_k^* - \hat{B}_{\sigma(n)}\Big) I = 0\,,$$

we obtain (1.3.13). Q.E.D.

Now, we obtain a useful formula, that is, "Transfer Formula".

THEOREM 1.3.6. (Transfer Formula). *Let $p_\alpha(x)$ be a normal basic polynomial for the normal delta set $\{P_1, \cdots, P_n\}$. Then for some differential operators $P(\alpha; \partial)$ with constant coefficients depending on a parameter α, $p_\alpha(x)$ is represented by*

$$p_\alpha(x) = P(\alpha; \partial) x^\alpha/\alpha!\,, \tag{1.3.17}$$

if and only if the i-th component of the differential expression $\{p_1(\partial), \cdots, p_n(\partial)\}$ is divided by ∂_i: for some differential operators $\{q_1(\partial), \cdots, q_n(\partial)\}$,

$$p_i(\partial) = \partial_i q_i(\partial)\,, \qquad i = 1, \cdots, n. \tag{1.3.18}$$

So, we have the followings:

(i)
$$p_\alpha(x) = \begin{vmatrix} (\partial_1 p_1)(\partial), & \cdots, & (\partial_n p_1)(\partial) \\ \vdots & & \vdots \\ (\partial_1 p_n)(\partial), & \cdots, & (\partial_n p_n)(\partial) \end{vmatrix} q_1(\partial)^{-\alpha_1 - 1} \cdots q_n(\partial)^{-\alpha_n - 1} x^\alpha/\alpha!\,,$$

(ii) *in the algebra $\mathfrak{S}$ generated by the operators $q_i(\partial)^{-\alpha_i}$ and x_j, $i, j = 1, \cdots, n$, setting $A_i = q_i(\partial)^{-\alpha_i}$ and $B_j = x_j$, and using the notation in Lemma* 1.3.5, *we have*

$$p_\alpha(x) = \begin{vmatrix} \widehat{q_1(\partial)^{-\alpha_1}} \hat{x}_1^* - (\partial_1 \widehat{q_1^{-\alpha_1}})(\partial), & \cdots, & -(\partial_n \widehat{q_1^{-\alpha_1}})(\partial) \\ \vdots & \cdots & \vdots \\ -(\partial_1 \widehat{q_n^{-\alpha_n}})(\partial), & \cdots, & \widehat{q_n(\partial)^{-\alpha_n}} \hat{x}_n^* - (\partial_n \widehat{q_n^{-\alpha_n}})(\partial) \end{vmatrix} I \cdot x^{\alpha - e_1 - \cdots - e_n}/\alpha!\,;$$

(iii) *by using notations of* (ii),

$$p_\alpha(x) = \begin{vmatrix} \hat{x}_1 \widehat{q_1(\partial)^{-\alpha_1}}, & \cdots\cdots\cdots, & \hat{x}_n \widehat{q_1(\partial)^{-\alpha_1}} \\ -(\partial_1 \widehat{q_2^{-\alpha_2}})(\partial), & \widehat{q_2(\partial)^{-\alpha_2}} \hat{x}_2^* - (\partial_2 \widehat{q_2^{-\alpha_2}})(\partial), \cdots, & -(\partial_n \widehat{q_2^{-\alpha_2}})(\partial) \\ -(\partial_1 \widehat{q_n^{-\alpha_n}})(\partial), & \cdots\cdots\cdots, & \widehat{q_n(\partial)^{-\alpha_n}} \hat{x}_n^* - (\partial_n \widehat{q_n^{-\alpha_n}})(\partial) \end{vmatrix}$$
$$\times\, I \cdot x^{\alpha - e_1 - \cdots - e_n}/\alpha!\,.$$

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Proof. First, we shall prove the necessary condition. Set the followings

$$P(\alpha; \partial) = \sum_{\beta} b(\alpha; \beta)\partial^{\beta},$$

and

$$p_{\alpha}(x) = \sum_{\beta} a(\alpha; \beta)x^{\beta}/\beta!\,.$$

By (1.3.17), we have

$$a(\alpha; \beta) = b(\alpha; \alpha - \beta) \qquad \text{if } \alpha - \beta \in \boldsymbol{Z}_{+}^{n}\,. \tag{1.3.19}$$

$$a(\alpha\colon \beta) = 0 \qquad \text{if } \alpha - \beta \notin \boldsymbol{Z}_{+}^{n}\,. \tag{1.3.20}$$

Therefore setting $\beta = e_k$ in (1.3.20), Proposition 1.3.4 gives (1.3.18).

Next, we show that the right sides of (i) and (ii) define the same sequence. Indeed,

$$\begin{aligned}
&\begin{vmatrix} (\partial_1 p_1)(\partial), & \cdots, & (\partial_n p_1)(\partial) \\ \vdots & & \vdots \\ (\partial_1 p_n)(\partial), & \cdots, & (\partial_n p_n)(\partial) \end{vmatrix} q_1(\partial)^{-\alpha_1 - 1} \cdots q_n(\partial)^{-\alpha_n - 1} x^{\alpha}/\alpha! \\
&\quad = \sum_{\sigma} \varepsilon(\sigma)(\partial_{\sigma(1)} p_1)(\partial) \cdots (\partial_{\sigma(n)} p_n)(\partial) q_1(\partial)^{-\alpha_1 - 1} \cdots q_n(\partial)^{-\alpha_n - 1} x^{\alpha}/\alpha! \\
&\quad = \sum_{\sigma} \varepsilon(\sigma)(\delta_{1\sigma(1)} q_1(\partial) + \partial_1(\partial_{\sigma(1)} q_1)(\partial)) \\
&\qquad\qquad \cdots (\delta_{n\sigma(n)} q_n(\partial) + \partial_n(\partial_{\sigma(n)} q_n)(\partial)) q_1(\partial)^{-\alpha_1 - 1} \cdots q_n(\partial)^{-\alpha_n - 1} x^{\alpha}/\alpha! \\
&\quad = \sum_{\sigma} \varepsilon(\sigma)(\delta_{1\sigma(1)} q_1(\partial)^{-\alpha_1} - \alpha_1^{-1} \partial_1(\partial_{\sigma(1)} q_1^{-\alpha_1})(\partial)) \\
&\qquad\qquad\qquad \cdots (\delta_{n\sigma(n)} q_n(\partial)^{-\alpha_n} - \alpha_n^{-1} \partial_n(\partial_{\sigma(n)} q_n^{-\alpha_n})(\partial)) x^{\alpha}/\alpha! \\
&\quad = \sum_{k=0}^{n} (-1)^k \sum_{j_1 < \cdots < j_k, \sigma} \varepsilon(\sigma) q_1(\partial)^{-\alpha_1} \cdots (\partial_{j_{\sigma(1)}} q_{j_1}^{-\alpha_{j_1}})(\partial) \\
&\qquad\qquad\qquad \cdots (\partial_{j_{\sigma(k)}} q_{j_k}^{-\alpha_{j_k}})(\partial) \cdots q_n(\partial)^{-\alpha_n} x^{\alpha - e_{j_1} - \cdots - e_{j_k}}/\alpha!\,.
\end{aligned}$$

Here the permutation σ of the last equation is an element in the permutation group of order k. By using the notations of (ii), the last equation is equal to (ii). The proof of (ii) $\Rightarrow$ (iii) is straightforward verification in virtue of Lemma 1.3.5. Lastly we prove the sufficient condition. Replace the right side of (i) by $\tilde{p}_{\alpha}(x)$. Then it is easy to see

$$p_i(\partial)\tilde{p}_{\alpha}(x) = \tilde{p}_{\alpha - e_i}(x), \qquad i = 1, \cdots, n.$$

By (iii), we have

$$\tilde{p}_{\alpha}(0) = 0 \qquad \text{for } \alpha \in \boldsymbol{Z}_{+}^{n} - 0\,.$$

For $\alpha = 0$, (i) and the conditions of the normal delta set give

$$\tilde{p}_0(x) = 1\,.$$

Therefore we see that $\tilde{p}_\alpha(x)$ is a basic polynomial for $\{P_1 \cdots P_n\}$, and by Proposition 1.1.4, $\tilde{p}_\alpha(x)$ is equal to $p_\alpha(x)$. Q.E.D.

§ 4. Examples

In this section, we discuss a generalization of typical nontrivial examples to appear in the theory of combinatorics and stochastic processes, that is, the Abel polynomials and the Gould polynomials (cf. [3], [4], [5], [6], [8], [10]). In the case of one variable, it is known in Rota, etc. [9] that the Abel polynomial $x(x-na)^{n-1}/n!$ is the basic polynomial for the delta operator $d/dx \exp(ad/dx)$, and the Gould polynomial $x(x+bn)^{-1}(x+bn)_n/n!$ is the basic polynomial for the delta operator $(\exp(d/dx)-1)\exp(-b\, d/dx)$. Here the symbol $(x)_n$ is equal to the lower factorial $x(x-1)\cdots(x-n+1)$. We generalize the delta operator and, by using the transfer formula, give the basic polynomial for the delta set. In this section we use, for convenience, $\boldsymbol{e}(x)$ in stead of $\exp x$.

(i) A generalization of the Abel polynomials.

We consider the following normal delta set:

$$p_i(\partial) = \partial_i \boldsymbol{e}(\langle a_i, \partial\rangle) \equiv \partial_i q_i(\partial)\,, \qquad i = 1, \cdots, n,$$

where $\langle a_i, \partial\rangle$ is equal to $\sum_{k=1}^n a_{ik}\partial_k$. Then by the transfer formula (iii), the basic polynomial $p_\alpha(x)$ is calculated as follows:

$$p_\alpha(x) = \begin{vmatrix} \hat{\boldsymbol{x}}_1 \widehat{q_1(\partial)^{-\alpha_1}}, & \cdots\cdots\cdots & , \hat{\boldsymbol{x}}_n \widehat{q_1(\partial)^{-\alpha_1}} \\ -[\widehat{q_2(\partial)^{-\alpha_2}}, \boldsymbol{x}_1], & \widehat{q_2(\partial)^{-\alpha_2}}\hat{\boldsymbol{x}}_2^* - [\widehat{q_2(\partial)^{-\alpha_2}}, \boldsymbol{x}_2], \cdots, & -[\widehat{q_2(\partial)^{-\alpha_2}}, \boldsymbol{x}_n] \\ \vdots & \ddots & \vdots \\ -[\widehat{q_n(\partial)^{-\alpha_n}}, \boldsymbol{x}_1], & \cdots\cdots\cdots & , \widehat{q_n(\partial)^{-\alpha_n}}\hat{\boldsymbol{x}}_n^* - [\widehat{q_n(\partial)^{-\alpha_n}}, \boldsymbol{x}_n] \end{vmatrix} \times I \cdot x^{\alpha - e_1 - \cdots - e_n}/\alpha!$$

$$= \begin{vmatrix} \hat{\boldsymbol{x}}_1 \boldsymbol{e}(\widehat{-\alpha_1\langle a_1, \partial\rangle}), & \cdots\cdots\cdots & , \hat{\boldsymbol{x}}_n \boldsymbol{e}(\widehat{-\alpha_1\langle a_1, \partial\rangle}) \\ \alpha_2 a_{21} \boldsymbol{e}(\widehat{-\alpha_2\langle a_2, \partial\rangle}), & (\hat{\boldsymbol{x}}_2^* + \alpha_2 a_{22})\boldsymbol{e}(\widehat{-\alpha_2\langle a_2, \partial\rangle}), \cdots, & \alpha_2 a_{2n} \boldsymbol{e}(\widehat{-\alpha_2\langle a_2, \partial\rangle}) \\ \vdots & \ddots & \\ \alpha_n a_{n1} \boldsymbol{e}(\widehat{-\alpha_n\langle a_n, \partial\rangle}), & \cdots\cdots\cdots & , (\hat{\boldsymbol{x}}_n^* + \alpha_n a_{nn})\boldsymbol{e}(\widehat{-\alpha_n\langle a_n, \partial\rangle}) \end{vmatrix} \times I \cdot x^{\alpha - e_1 - \cdots - e_n}/\alpha!$$

$$= \begin{vmatrix} \hat{x}_1, \cdots\cdots\cdots\cdots\cdots, \hat{x}_n \\ \alpha_2 a_{21}, \hat{x}_2^* + \alpha_1 a_{22}, \cdots, \alpha_2 a_{2n} \\ \ddots \\ \alpha_n a_{n1}, \cdots\cdots, \hat{x}_n^* + \alpha_n a_{nn} \end{vmatrix}$$

$$\prod_{i=1}^{n} (e(\widehat{-\alpha_i \langle a_i, \partial \rangle}))I \cdot x^{\alpha - e_1 - \cdots - e_n}/\alpha! .$$

Replacing the above equation by the determinant of commutative entries, we have

$$p_\alpha(x) = \sum_{k=0}^{n-1} \sum_{1=j_0<\cdots<j_k} \begin{vmatrix} x_{j_0}, \cdots\cdots, x_{j_k} \\ \alpha_{j_1} a_{j_1 j_0}, \cdots, \alpha_{j_1} a_{j_1 j_k} \\ \vdots \qquad \vdots \\ \alpha_{j_k} a_{j_k j_0}, \cdots, \alpha_{j_k} a_{j_k j_k} \end{vmatrix}$$

$$\prod_{i=1}^{n} (e(-\alpha_i \langle a_i, \partial \rangle)) x^{\alpha - e_{j_1} - \cdots - e_{j_k}}/\alpha!$$

$$= \sum_{k=0}^{n-1} \sum_{1=j_0<\cdots<j_k} \begin{vmatrix} x_{j_0}, \cdots\cdots, x_{j_k} \\ \alpha_{j_1} a_{j_1 j_0}, \cdots, \alpha_{j_1} a_{j_1 j_k} \\ \vdots \qquad \vdots \\ \alpha_{j_k} a_{j_k j_0}, \cdots, \alpha_{j_k} a_{j_k j_k} \end{vmatrix}$$

$$\prod_{l=1}^{n} \left(x_l - \sum_{i=1}^{n} \alpha_i a_{il} \right)^{\alpha_l - \delta_{l j_1} - \cdots - \delta_{l j_k}} \Big/ \alpha! .$$

Hence, we obtain the Abel polynomials with n variables:

$$p_\alpha(x) = \begin{vmatrix} x_1, \cdots\cdots\cdots\cdots\cdots, x_n \\ \alpha_2 a_{21}, x_2 - \sum_{k \neq 2} \alpha_k a_{k2}, \cdots, \alpha_2 a_{2n} \\ \vdots \qquad \ddots \qquad \vdots \\ \alpha_n a_{n1}, \cdots\cdots, x_n - \sum_{k \neq n} \alpha_k a_{kn} \end{vmatrix}$$

$$\prod_{j=1}^{n} \left(x_j - \sum_{i=1}^{n} \alpha_i a_{ij} \right)^{\alpha_j - 1} \Big/ \alpha! .$$

Let us give some properties of the Abel polynomials with n variables.

(a) A generalization of the Abel's identity (cf. [1]).
In virtue of Theorem 1.2.1, the Abel's identity is stated as

$$(x + y)^\alpha/\alpha! = \sum_{\alpha = \beta + \gamma} p_\beta(x) \prod_{i=1}^{n} \left(y_i + \sum_{j=1}^{n} \beta_j a_{ij} \right)^{\gamma_i} \Big/ \gamma_i! .$$

(b) A generalization of the first Abel inverse relation ([8], p. 93).
Using Corollary 1.2.2, we obtain

$$a_\alpha = \sum_{\alpha=\beta+\gamma} p_\beta(x) b_\gamma \,,$$
$$b_\alpha = \sum_{\alpha=\beta+\gamma} p_\beta(-x) a_\gamma \,.$$

(c) The orthogonal relation is the following:

$$\delta_{\alpha,\alpha_0} = \sum_{\alpha-\alpha_0=\beta+\gamma} p_\beta(x) p_\gamma(-x) \,.$$

The other properties of the Abel polynomials with n variables will be taken in the following papers.

(ii) A generalization of the Gould polynomials.

We consider the following normal delta set:

$$p_i(\partial) = (e(\partial_i) - 1)e(-\langle a_i, \partial\rangle) \,, \qquad i = 1, \cdots, n \,,$$

where $\langle a_i, \partial\rangle$ is equal to $\sum_{j=1}^{n} a_{ij}\partial_j$.

As well known, the lower factorial $(x)_n/n!$ is the basic polynomial for the delta operator $e(d/dx) - 1$. Hence, by the transfer formula (i) in the case of one variable, we see

$$(1.4.1) \qquad (d/dx(e(d/dx) - 1)^{-1})^{n+1} e(d/dx) x^n = (x)_n \,.$$

In virtue of the transfer formula (i) and (1.4.1), the basic polynomial $p_\alpha(x)$ is calculated as follows:

$$p_\alpha(x) = \begin{vmatrix} -a_{11}(e(\partial_1) - 1) + e(\partial_1), & -a_{12}(e(\partial_1) - 1), & \cdots, & -a_{1n}(e(\partial_1) - 1) \\ -a_{21}(e(\partial_2) - 1), & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -a_{n1}(e(\partial_n) - 1), & \cdots\cdots\cdots, & & -a_{nn}(e(\partial_n) - 1) + e(\partial_n) \end{vmatrix}$$
$$\prod_{i=1}^{n} e(\alpha_i\langle a_i, \partial\rangle) \prod_{i=1}^{n} (\partial_i(e(\partial_i) - 1)^{-1})^{\alpha_i+1} x^\alpha/\alpha!$$
$$= \begin{vmatrix} -a_{11}(e(\partial_1) - 1) + e(\partial_1), & -a_{12}(e(\partial_1) - 1), & \cdots, & -a_{1n}(e(\partial_1) - 1) \\ -a_{21}(e(\partial_2) - 1), & -a_{22}(e(\partial_2) - 1) + e(\partial_2), & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ -a_{n1}(e(\partial_n) - 1), & \cdots\cdots\cdots\cdots\cdots\cdots, & & -a_{nn}(e(\partial_n) - 1) + e(\partial_n) \end{vmatrix}$$
$$\prod_{j=1}^{n} \Big(x_j - 1 + \sum_{i=1}^{n} \alpha_i a_{ij}\Big)_{\alpha_j} \Big/ \alpha! \,.$$

Here, we note that

$$(-a_{ll}(e(\partial_l) - 1) + e(\partial_l))\Big(x_l - 1 + \sum_{j=1}^{n} \alpha_j a_{jl}\Big)_{\alpha_l}$$
$$= \Big(x_l + \sum_{j\neq l} \alpha_j a_{jl}\Big)\Big(x_l + \sum_{j=1}^{n} \alpha_j a_{jl}\Big)^{-1}\Big(x_l + \sum_{j=1}^{n} \alpha_j a_{jl}\Big)_{\alpha_l} \,.$$

Hence, we obtain the Gould polynomials with n variables:

$$(1.4.2)\qquad p_\alpha(x)=\begin{vmatrix} x_1+\sum_{k\neq 1}\alpha_k a_{k1}, & -\alpha_1 a_{12}, & \cdots, & -\alpha_1 a_{1n} \\ -\alpha_2 a_{21}, & x_2+\sum_{k\neq 2}\alpha_k a_{k2}, & \cdots, & -\alpha_2 a_{2n} \\ \vdots & \vdots & \ddots & \\ -\alpha_n a_{n1}, & \cdots\cdots & \cdots, & x_n+\sum_{k\neq n}\alpha_k a_{kn}\end{vmatrix}$$
$$\prod_{l=1}^{n}\Big(x_l+\sum_{j=1}^{n}\alpha_j a_{jl}\Big)^{-1}\Big(x_l+\sum_{j=1}^{n}\alpha_j a_{jl}\Big)_{\alpha_l}\Big/\alpha!\,.$$

Let us give some properties of the Gould polynomials with n variables.

(a) *A* generalization of Gould's inversion formula (cf. [5] (3.1) and (3.2)).

Since

$$p^\alpha(\partial)=\sum_{\alpha=\beta+\gamma}(-1)^{|\gamma|}\binom{\alpha}{\beta}e\Big(\langle\beta,\partial\rangle-\sum_{i=1}^{n}\alpha_i\langle a_i,\partial\rangle\Big),$$

by Theorem 1.2.1, we find that

$$F(\alpha)=\sum_{\alpha=\beta+\gamma}(-1)^{|\gamma|}\binom{\alpha}{\beta}f(\beta-\langle\alpha,a^*\rangle)$$

is the inverse of

$$f(x)=\sum_{\alpha}p_\alpha(x)F(\alpha)\,,$$

where the notation $\binom{\alpha}{\beta}$ is equal to $\binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n}$, and the vector $\langle\alpha,a^*\rangle$ is $(\sum_{k=1}^{n}\alpha_k a_{kl},\cdots,\sum_{k=1}^{n}\alpha_k a_{kn})$.

(b) Setting $p_\alpha(x)\equiv p_\alpha(x;a)$ in (1.4.2), we give the connection constants of $p_\alpha(x;a)$ in terms of $p(x;a-b)$ (cf. [5], (5.5)).

Now,

$$e(\langle b_i-a_i,\partial\rangle)(e(\partial_i)-1)p_\alpha(x;a)$$
$$=p_{\alpha-e_i}(x+b_i;a)\,,$$

and, therefore

$$e\Big(\sum_{i=1}^{n}\beta_i\langle b_i-a_i,\partial\rangle\Big)\prod_{j=1}^{n}(e(\partial_j)-1)^{\beta_j}p_\alpha(x;a)$$
$$=p_{\alpha-\beta}\Big(x+\sum_{i=1}^{n}\beta_i b_i;a\Big)\,,$$

whence, by Theorem 1.2.1,

$$p_\alpha(x+y;a) = \sum_{\alpha=\beta+\gamma} p_\beta(x;a-b)p_\gamma\Big(y+\sum_{i=1}^{n}\beta_i b_i;a\Big). \tag{1.4.3}$$

By a change of parameters, we also obtain

$$p_\alpha(x+y;a-b) = \sum_{\alpha=\beta+\gamma} p_\beta(x;a)p_\gamma\Big(y-\sum_{i=1}^{n}\beta_i b_i;a-b\Big). \tag{1.4.4}$$

(c) A generalization of Gould's main theorem (cf. [5] (5.3) and (5.4)). Using (1.4.3) and (1.4.4), we obtain the inversion formula:

$$f_\alpha(x+y) = \sum_{\alpha=\beta+\gamma} F_\beta(x)p_\gamma\Big(y-\sum_{i=1}^{n}\beta_i b_i;a-b\Big),$$

$$F_\alpha(x+y) = \sum_{\alpha=\beta+\gamma} f_\beta(x)p_\gamma\Big(y+\sum_{i=1}^{n}\beta_i b_i;a\Big).$$

The other properties of the Gould polynomials with n variables will be taken in the following papers.

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