FAITHFUL REPRESENTATIONS OF LIE GROUPS II

MORIKUNI GOTÔ

The present paper is a continuation of Part I.[1] In the introduction of Part I we have explained our main problems and sketched their results. In the present part we shall proceed to give complete proofs of them.

Notations and definitions in Part I shall be retained.

Semi-simple f.r. Lie groups

5. Let $\sum$ be a field and let a and b be matrices of degree n and m respectively with coefficients of $\sum$. We denote by $a \times b$ the Kronecker product of a and b, and by $a^{[k]}$ the k-times Kronecker product of a:

$$a^{[0]} = 1_1 \text{ (the unit matrix of degree 1)},$$

$$a^{[k]} = \overbrace{a \times a \times \dots \times a}^{k} \quad \text{for } k > 0.$$

Let now $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be the sequence of all eigen-values of a, repeated as many times as their multiplicities. Then $\varepsilon_1^{i(1)} \varepsilon_2^{i(2)} \dots \varepsilon_n^{i(n)}$ is an eigen-value of $a^{[k]}$ if $i(1), i(2), \dots$ are non-negative integers such that $\sum_{j=1}^{n} i(j) = k$. Suppose now that $a^{[k]}$ is the unit matrix (of degree n). Since every eigen-value of $a^{[k]}$ is 1, we have $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_n = \varepsilon$, where ε is a k-th root of unity. Let e_1 be an eigen-vector of a: $ae_1 = \varepsilon e_1$, and let e_2 be a vector, linearly independent of e_1. We operate $a^{[k]}$ to $e_2 \times e_1 \times \dots \times e_1$ and get $ae_2 = \varepsilon e_2$ easily. Hence *if $a^{[k]}$ is the unit matrix, then $a = \varepsilon 1_n$ where $\varepsilon^k = 1$.* Using this fact we shall prove the following algebraic

LEMMA 8. *Let $\sum$ be an algebraically closed field, and let $\mathfrak{G}$ be a group composed of non-singular matrices with coefficients of $\sum$. Let now $\mathfrak{D}$ be a finite central invariant subgroup of $\mathfrak{G}$. Then we can construct a faithful representation of $\mathfrak{G}/\mathfrak{D}$, which is an induced representation of $\mathfrak{G}$ on a certain tensor space, where we assume that the order of $\mathfrak{D}$ is indivisible by the characteristic of $\sum$.*

Proof. It is clearly sufficient to prove the lemma in case when the order q of $\mathfrak{D}$ is a prime number. Let d ($\neq e$) be an element of $\mathfrak{D}$. Since $\mathfrak{D}$ is comp-

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[1] Gotô [22].

letely reducible because q does not divide the characteristic of $\sum$, we may assume that d is of the form

$$d = \begin{pmatrix} \varepsilon^{i(1)}1_{j(1)} & & & 0 \\ & \varepsilon^{i(2)}1_{j(2)} & & \\ & & \ddots & \\ 0 & & & \varepsilon^{i(k)}1_{j(h)} \end{pmatrix},$$

where ε is a q-th root of unity and $i(s) \not\equiv i(t)$ mod. q if $s \neq t$. Since $gd = dg$ for any element g of $\mathfrak{G}$, every eigen-space of d is allowable by $\mathfrak{G}$:

$$\mathfrak{G} \ni g = \begin{pmatrix} g_1 & & & 0 \\ & g_2 & & \\ & & \ddots & \\ 0 & & & g_h \end{pmatrix}.$$

Clearly, we can determine non-negative integers a_{ls} ($l = 1, 2, \ldots, h-1$; $s = 1, 2, \ldots, h$) such that a solution $(x_1, \ldots, x_h)$ of the congruence equations

$$a_{l1}x_1 + a_{l2}x_2 + \ldots + a_{lh}x_h \equiv 0 \quad \text{mod. } q \qquad l = 1, 2, \ldots, h-1$$

satisfies the relations

$$x_s \equiv mi(s) \quad \text{mod. } q \qquad s = 1, 2, \ldots, h$$

for a suitably chosen integer m.

Now an induced representation of $\mathfrak{G}$ is given by

$$g \to \begin{pmatrix} g^* & 0 \\ 0 & g^{**} \end{pmatrix},$$

where

$$g^* = \begin{pmatrix} g_1^{[q]} & & & 0 \\ & g_2^{[q]} & & \\ & & \ddots & \\ 0 & & & g_h^{[q]} \end{pmatrix}.$$

and

$$g^{**} = \begin{pmatrix} g_1^{[a_{11}]} \times \ldots \times g_h^{[a_{1h}]} & & & 0 \\ & g_1^{[a_{21}]} \times \ldots \times g_h^{[a_{2h}]} & & \\ & & \ddots & \\ 0 & & & g_1^{[a_{h-11}]} \times \ldots \times g_h^{[a_{h-1h}]} \end{pmatrix}$$

Let $\mathfrak{Z}$ be the kernel of the representation. Since $z_s^{[q]}$ is the unit matrix for $z \in \mathfrak{Z}$, we have

$$z = \begin{pmatrix} \varepsilon^{f(1)} 1_{j(1)} & & & 0 \\ & \varepsilon^{f(2)} 1_{j(2)} & & \\ & & \ddots & \\ 0 & & & \varepsilon^{f(h)} 1_{j(h)} \end{pmatrix}.$$

Now since z^{**} is also the unit matrix we get that

$$a_{l1} f(1) + a_{l2} f(2) + \ldots + a_{lh} f(h) \equiv 0 \mod. q, \qquad l = 1, 2, \ldots, h-1$$

so that from the definition of a_{ls} there exists an integer m such that

$$f(s) \equiv m i(s) \mod. q \quad s = 1, 2, \ldots, h.$$

Hence we have $z \in \mathfrak{D}$, i.e. $\mathfrak{Z} \subseteqq \mathfrak{D}$. That $\mathfrak{D} \subseteqq \mathfrak{Z}$ is obvious, q.e.d.

LEMMA 9.[2] *Let $\mathfrak{G}$ be a semi-simple f.r.*[3] *Lie group and let $\mathfrak{G}^*$ be a Lie group homomorphic with $\mathfrak{G}$. Then $\mathfrak{G}^*$ is also f.r.*

Proof. We may suppose that $\mathfrak{G}/\mathfrak{N} \cong \mathfrak{G}^*$, where $\mathfrak{N}$ is a closed invariant subgroup of $\mathfrak{G}$. Let G be the Lie algebra of $\mathfrak{G}$, and let $\mathfrak{N}_0$ be the connected component of $\mathfrak{N}$ containing e. As the Lie algebra N of $\mathfrak{N}$ is an ideal of semi-simple G, there exists an ideal M of G such that

$$G = M + N, \quad M \cap N = 0.$$

We denote by $\mathfrak{M}$ the subgroup of $\mathfrak{G}$ generated by M. Then since M is semi-simple as an ideal of a semi-simple Lie algebra and $\mathfrak{G}$ is f.r., Corollary to Theorem 2 in Part I implies that $\mathfrak{M}$ is closed. Now from

$$\mathfrak{M}/\mathfrak{M} \cap \mathfrak{N}_0 \cong \mathfrak{G}/\mathfrak{N}_0 \quad \text{and} \quad \mathfrak{G}/\mathfrak{N}_0 \big/ \mathfrak{N}/\mathfrak{N}_0 \cong \mathfrak{G}/\mathfrak{N},$$

$\mathfrak{G}^*$ is homomorphic with $\mathfrak{M}$: $\mathfrak{M} \sim \mathfrak{G}^*$. On the other hand since the Lie algebra of $\mathfrak{M}$ is isomorphic to that of $\mathfrak{G}^*$, $\mathfrak{M}$ and $\mathfrak{G}^*$ are locally isomorphic, so that the kernel $\mathfrak{D}$ of the homomorphism $\mathfrak{M} \sim \mathfrak{G}^*$ is discrete. Then in virtue of Lemma 5 in Part I our assertion is a direct consequence of Lemma 8, q.e.d.

6. A Lie algebra G is called *complex* if G is isomorphic with a Lie algebra over the field C of complex numbers, and a Lie group is called *complex* when its Lie algebra is complex. When the structure of a Lie algebra G is given by

$$G = Px_1 + Px_2 + \ldots + Px_r, \qquad [x_i, x_j] = \sum_s \gamma_{ijs} x_s, \qquad \gamma_{ijs} \in P,$$

where $x_1, \ldots, x_r$ constitute a basis of G, the complex Lie algebra G_C difined by

$$G_C = Cx_1 + Cx_2 + \ldots + Cx_r, \quad [x_i, x_j] = \sum_s \gamma_{ijs} x_s$$

[2] Cf. Malcev [13].

[3] A connected Lie group is called faithfully representable (f.r.) if it admits a continuous faithful linear representation. See Part I.

is called the *complex form* of G, and conversely G is called a *real form* of G_C. A complex Lie algebra may have no real form at all, or have several real forms. Note that a Lie algebra is semi-simple if and only if its complex form is semi-simple.

Let G^* be a complex semi-simple Lie algebra. H. Weyl has, in his fundamental papers on semi-simple Lie groups,[4] proved the following important results: We can select a suitable basis $x_1, x_2, \ldots, x_r$ with respect to C so that

$$G^* = Cx_1 + Cx_2 + \ldots + Cx_r, \qquad [x_i, x_j] = \sum_s \gamma_{ijs} x_s, \qquad \gamma_{ijs} \in P$$

and moreover any Lie group which is generated by the real form

$$K = Px_1 + Px_2 + \ldots + Px_r$$

is always compact. Such a real form, called a *compact form* of G^*, is unique up to inner automorphisms.

Now consider a general real form of G^*.[5] Take an involutive automorphism τ, $\tau^2 = 1_r$, of a compact form K, and decompose K into eigen-spaces of τ:

$$K = K_1 + K_{-1},$$

where K_1, K_{-1} are eigen-spaces of τ belonging to eigen-values 1, -1 respectively. Now the subalgebra of G^* given by

$$G = K_1 + \sqrt{-1}\, K_{-1}$$

is a real form of G^*, and conversely all real forms of G^* can be obtained in such a way. Now isomorphic real forms of G^*, considered as subalgebras of G^*, are conjugated with respect to automorphisms. Therefore since any local automorphism of a simply connected Lie group can be extended to an automorphism in the large, *in the simply connected complex semi-simple Lie group generated by G^*, isomorphic real forms of G^* generate isomorphic subgroups.*

Next, we shall call a complex semi-simple Lie algebra *complex simple* if it has no proper complex ideal distinct from 0. It is easy to see that a complex simple Lie algebra, as well as a real form of a complex simple Lie algebra, is simple. Let G be a simple Lie algebra and G_C the complex form of G. Then only the following two cases are possible.[6]

Case 1. G_C is simple. Then G is a real form of G_C, which is complex simple.

Case 2. G_C is not simple. In this case G itself is complex simple.

From now on we shall call a real form of a complex simple Lie algebra *real*

4) Weyl [23].

5) Cartan [20]. See also Gantmacher [21].

6) E. g. Gantmacher [21].

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simple. A *real*, or *complex simple* Lie group is defined correspondingly.

Let now $\mathfrak{G}^*$ be a complex Lie group and G^* the Lie algebra of $\mathfrak{G}^*$. We shall call a subgroup $\mathfrak{G}$ of $\mathfrak{G}^*$ generated by a real form G of G^* a *real form* of $\mathfrak{G}^*$, and call $\mathfrak{G}^*$ a *complex form* of $\mathfrak{G}$. While a Lie algebra always has one and only one complex form, a Lie group may have no complex form at all, or have several complex forms.

LEMMA 10. *Let $\mathfrak{G}$ be a linear Lie group. If $\mathfrak{G}$ is compact or real simple, then there exists a complex-linear Lie group $\mathfrak{G}_C$ which contains $\mathfrak{G}$ as a real form.* Here we shall call a linear Lie group *complex-linear* if its linear Lie algebra contains any complex multiple of its matrix.

Proof. Let $G = Px_1 + \ldots + Px_r$ be the linear Lie algebra of $\mathfrak{G}$. It is clearly sufficient to show that x_i's are linearly independent with respect to C.

First suppose that $\mathfrak{G}$ is compact. Then since any compact linear group is equivalent to a subgroup of the unitary group, there exists a matrix a such that $a^{-1}Ga$ is composed of skew hermitian matrices. Suppose now that $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_r x_r = 0$ $\alpha_i \in C$. Then we have $\alpha_1 a^{-1}x_1 a + \alpha_2 a^{-1}x_2 a + \ldots + \alpha_r a^{-1}x_r a = 0$. Now since every $a^{-1}x_i a$ is skew hermitian, we have $\bar{\alpha}_1 a^{-1}x_1 a + \bar{\alpha}_2 a^{-1}x_2 a + \ldots + \bar{\alpha}_r a^{-1}x_r a = 0$, so that $\bar{\alpha}_1 x_1 + \bar{\alpha}_2 x_2 + \ldots + \bar{\alpha}_r x_r = 0$, where $\bar{\alpha}$ denotes the conjugate complex number of α. Then the linear independence of x_i's with respect to P implies that $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$.

Next let G be real simple. We know that its complex form G_C is simple. Hence the complex-linear Lie algebra spanned by x_i, $Cx_1 + Cx_2 + \ldots + Cx_r$, forms a faithful representation of G_C, q.e.d.

7. Let G be a Lie algebra and let $\hat{\mathfrak{G}}$ be a corresponding connected Lie group. If $\hat{\mathfrak{G}}$ is f.r. and any f.r. Lie group $\mathfrak{G}$, generated by G, is homomorphic with $\hat{\mathfrak{G}}$, we call $\hat{\mathfrak{G}}$ a *linear covering group* of G, or of an f.r. Lie group generated by G, or merely with respect to its structure.

THEOREM 3. *For any simple Lie algebra G there exists one and only one linear covering group $\hat{\mathfrak{G}}$. Furthermore any local isomorphism of $\hat{\mathfrak{G}}$ to any f.r. Lie group can be extended to a homomorphism in the large, in particular a local automorphism of $\hat{\mathfrak{G}}$ can be extended to an automorphism.*

The linear covering group of a complex simple Lie algebra is simply connected, and that of a real simple Lie algebra can be obtained as a real form of the simply connected group generated by its complex form.

Proof. A compact Lie group is f.r.[7] in virtue of the theory of almost pe-

[7] E. g. Chevalley [8], Chap. VI.

riodic functions. Let G^* be a complex (semi-) simple Lie algebra and K its compact form. Since K always generate a compact group by the above mentioned theorem of Weyl, there exists a simply connected linear Lie group $\mathfrak{K}$ generated by K. Now by Lemma 10 there exists a complex-linear Lie group $\mathfrak{K}_C$ which contains $\mathfrak{K}$ as a real form, whence the Lie algebra of $\mathfrak{K}_C$ is isomorphic with G^*. The space of $\mathfrak{K}_C$, the direct product of that of $\mathfrak{K}$ and the Euclidean space of a certain dimension, is also simply connected. Therefore a simply connected complex (semi-) simple Lie group is f.r.

Next let G be a real simple Lie algebra, and let $\tilde{\mathfrak{G}}$ be the simply connected Lie group corresponding to G and $\mathfrak{Z}$ the center of $\tilde{\mathfrak{G}}$. Denote by $\tilde{\mathfrak{G}}_C$ the simply connected Lie group corresponding to the complex form G_C of G, and by $\hat{\mathfrak{G}}$ a real form of $\tilde{\mathfrak{G}}_C$ whose Lie algebra is G. The structure of $\hat{\mathfrak{G}}$ is then determined uniquely. (See §6.) Let now $\hat{\mathfrak{Z}}$ be a discrete invariant subgroup of $\tilde{\mathfrak{G}}$ so that $\tilde{\mathfrak{G}}/\hat{\mathfrak{Z}} \cong \hat{\mathfrak{G}}$. Since $\hat{\mathfrak{G}}$ is f.r. and the center of $\hat{\mathfrak{G}}$ is isomorphic to $\mathfrak{Z}/\hat{\mathfrak{Z}}$, the index $[\mathfrak{Z} : \hat{\mathfrak{Z}}]$ must be finite by Lemma 5 in Part I.

Now let $\mathfrak{G}$ be an f.r. Lie group corresponding to G. The existence of a complex form $\mathfrak{G}_C$ of $\mathfrak{G}$ can be assured by Lemma 10. Since in the homomorphism $\tilde{\mathfrak{G}}_C \sim \mathfrak{G}_C$ there corresponds $\mathfrak{G}$ to $\hat{\mathfrak{G}}$, we have $\hat{\mathfrak{G}} \sim \mathfrak{G}$. Hence $\hat{\mathfrak{G}}$ is a linear covering group for G. We note also the fact that the order of the center of $\hat{\mathfrak{G}}$ is not less that that of $\mathfrak{G}$.

Suppose now that a linear representation f, $\tilde{\mathfrak{G}} \ni \tilde{g} \to f(\tilde{g})$, gives a faithful representation of $\hat{\mathfrak{G}}$, and let σ be an automorphism of $\tilde{\mathfrak{G}}$. The kernel of the representation of $\tilde{\mathfrak{G}}$ given by

$$\tilde{g} \to \begin{pmatrix} f(\tilde{g}) & 0 \\ 0 & f(\tilde{g}^{\sigma}) \end{pmatrix}$$

is clearly $\hat{\mathfrak{Z}} \cap \hat{\mathfrak{Z}}^{\sigma} = \mathfrak{D}$. Since $\tilde{\mathfrak{G}}/\mathfrak{D}$ is f.r., we have that $[\mathfrak{Z} : \mathfrak{D}] \leqq [\mathfrak{Z} : \hat{\mathfrak{Z}}] < \infty$. Since $\mathfrak{D}$ is contained in $\mathfrak{Z}$ we get $\hat{\mathfrak{Z}} = \hat{\mathfrak{Z}}^{\sigma} = \mathfrak{D}$, namely $\hat{\mathfrak{Z}}$ is characteristic. The uniqueness of the linear covering group of G may be proved by a similar argument.

Next let $\mathfrak{G}$ be an f.r. Lie group generated by G, and let φ_0 be a local isomorphism of $\hat{\mathfrak{G}}$ to $\mathfrak{G}$. Since $\hat{\mathfrak{Z}}$ is a characteristic subgroup of $\tilde{\mathfrak{G}}$, there exists a homomorphism φ so that $\varphi = \varphi_0$ locally, q.e.d.

In virtue of Lemma 9 and Theorem 3 *we can count up all simple f.r. Lie groups*, since all simply connected complex simple Lie groups are known.

For example in the four grand classes A_n, B_n, C_n ($n = 1, 2, \ldots$) and D_n ($n = 3, 4, \ldots$) of complex simple Lie algebras given by Lie-Killing-Cartan,[8]

[8] E. g. Cartan [3].

we are familiar with the simply connected groups for the classes A_n and C_n i.e. the unimodular groups and the symplectic groups,[9] and for B_n and D_n the classes of the proper orthogonal groups, the so-called spin representations[10] give us simply connected groups.

The necessity of the second example of non-f.r. Lie groups given by Cartan,[11] i.e. an arbitrary proper covering group of $\mathfrak{Sl}(2, P)$, is now clarified, because $\mathfrak{Sl}(2, P)$ is, as a real form of the complex unimodular group, which is simply connected, a linear covering group of itself.

THEOREM 4. *For any semi-simple Lie algebra G there exists one and only one linear covering group $\hat{\mathfrak{G}}$, and $\hat{\mathfrak{G}}$ has the property stated in the first half of Theorem* 3 *for a simple Lie algebra.*

Moreover $\hat{\mathfrak{G}}$ is decomposed into a direct product of simple Lie groups. If G is complex, $\hat{\mathfrak{G}}$ is simply conneted.

Proof. Decompose G into simple ideals: $G = G_1 + \ldots + G_s$, and let $\hat{\mathfrak{G}}_i$ be the linear covering group of G_i. Then the direct product

$$\hat{\mathfrak{G}} = \hat{\mathfrak{G}}_1 \times \ldots \times \hat{\mathfrak{G}}_s$$

is clearly the uniquely determined linear covering group of G, and other assertions are easy to prove by an analogous argument as in Theorem 3, q.e.d.

Now we can easily prove the following

COROLLARY 1. *A connected semi-simple Lie group is f.r. if and only if every* (*closed*) *simple invariant Lie sudgroup is f.r.*

Next let $\mathfrak{G}$ be a topological group. In the introduction of Part I we defined a notion of an (*l*)-*group*. We repeat the definition: If there exists a set of closed invariant subgroups $\{\mathfrak{N}_\alpha\}$ of $\mathfrak{G}$ such that

$$1)\quad \mathfrak{G}/\mathfrak{N}_\alpha \text{ is f.r.} \qquad \text{and} \qquad 2)\quad \bigcap \mathfrak{N}_\alpha = e,$$

then $\mathfrak{G}$ is called an (*l*)-*group.*

COROLLARY 2. *A connected semi-simple Lie group is f.r. if it is an* (*l*)-*group.*

Proof. Let $\mathfrak{G}$ be a connected semi-simple (*l*)-group and $\tilde{\mathfrak{G}}$ the universal covering group of $\mathfrak{G}$, and let $\hat{\mathfrak{G}}$ be the linear covering group of the Lie algebra of $\mathfrak{G}$. We may suppose that $\mathfrak{G} = \tilde{\mathfrak{G}}/\mathfrak{D}$ and $\hat{\mathfrak{G}} = \tilde{\mathfrak{G}}/\mathfrak{Z}$. Let z be an element of $\mathfrak{Z}$. Since $\mathfrak{G}/\mathfrak{N}_\alpha$ is f.r. there corresponds to z the unit element in the homomorphism $\tilde{\mathfrak{G}}(\sim \mathfrak{G}) \sim \mathfrak{G}/\mathfrak{N}_\alpha$, namely $z\mathfrak{D} \subseteqq \mathfrak{N}_\alpha$, whence $z\mathfrak{D} \subseteqq \mathfrak{D}$ because $\bigcap \mathfrak{N}_\alpha$ is the unit

9) E. g. Weyl [24].

10) Cartan [6].

11) Cartan [6]. Cf. §4 in Part I.

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element. Therefore we have $\mathfrak{Z} \subseteqq \mathfrak{D}$, namely $\mathfrak{G}$ is f.r., q.e.d

Solvable f.r. Lie groups

8. We first prove the following

LEMMA 11.[12] *A simply connected solvable Lie group is f.r.*

Proof. Let G be a solvable Lie algebra. Then there exists a linear Lie algebra G_1 isomorphic with G. Next since $G/D(G)$ is commutative, we may easily prove that there exists a linear Lie algebra G_2, composed of nilpotent matrices, which is isomorphic with $G/D(G)$. Then the repressntation of G given by

$$\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$$

is obviously faithful. Furthermore in this representation there corresponds a nilpotent matrix to any element of $D(G)$ by a theorem of Lie, and to any element which does not belong to $D(G)$ there corresponds a non-zero nilpotent matrix in the representation G_2. Hence the linear Lie algebra generates a simply connected linear Lie group by Lemma 6 in Part I, q.e.d.

LEMMA 12. *Let $\mathfrak{G}$ be a linear Lie group and let $\mathfrak{N}$ be a closed invariant subgroup, $\mathfrak{H}$ a closed vector subgroup of $\mathfrak{G}$. Suppose that $\mathfrak{G} = \mathfrak{H}\mathfrak{N}$, $\mathfrak{H} \cap \mathfrak{N} = e$. If a discrete subgroup $\mathfrak{D}$ of $\mathfrak{H}$ is contained in the center of $\mathfrak{G}$, then $\mathfrak{G}/\mathfrak{D}$ is f.r.*

Proof. Let m be the dimension of $\mathfrak{H}$. It is clearly sufficient to prove the lemma when $m = 1$. We may suppose that $\mathfrak{H}$ is a one-parameter group $\exp \lambda x$, where x is a matrix and λ a real parameter, furthermore that $\mathfrak{D}$ is generated by $\exp x = d$. We decompose the vector space M on which $\mathfrak{G}$ operates into eigen-spaces with respect to x:

$$M = M_\alpha + M_\beta + \ldots,$$

where $\alpha, \beta, \ldots$ denote eigen-values of x. Next we sum up eigen-spaces whose eigen-values are the same mod. $2\pi\sqrt{-1}$:

$$M = M_1 + M_2 + \ldots$$

$$\begin{cases} M_1 = M_{\alpha(1)} + M_{\beta(1)} + \ldots & \alpha(1) \equiv \beta(1) \equiv \ldots \quad \text{mod. } 2\pi\sqrt{-1}, \\ M_2 = M_{\alpha(2)} + M_{\beta(2)} + \ldots & \alpha(2) \equiv \beta(2) \equiv \ldots \quad \text{mod. } 2\pi\sqrt{-1}, \\ \ldots\ldots & \end{cases}$$

$M = M_1 + M_2 + \ldots$ is then clearly the eigen-space decomposition with respect to $d = \exp x$, whose eigen-values are $\exp \alpha(1) = \exp \beta(1) = \ldots, \exp \alpha(2), \ldots.$

[12] Cartan [5].

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Since d is contained in the center, the eigen-spaces $M_l (l = 1, 2, \ldots)$ are allowable by $\mathfrak{G}$, and hence $\mathfrak{G}$ is decomposed into the representations induced on M_l.

Now define matrices x_1 and x_3 by the equations

$$x_1 e_l = \alpha(l) e_l \quad \text{for} \quad e_l \in M_l \quad l = 1, 2, \ldots,$$
$$x_3 e_\rho = \rho e_\rho \quad \text{for} \quad e_\rho \in M_\rho \quad \rho = \alpha, \beta, \ldots,$$

and put

$$x_0 = x - x_3, \quad x_2 = x_3 - x_1,$$

then we have

$$x = x_0 + x_1 + x_2,$$

where x's are mutually commutative, x_0 is nilpotent, and x_1 and x_2 are semi-simple.[13] Since the eigen-values of x_2 are all integral multiple of $2\pi\sqrt{-1}$, we have $\exp x_2 = 1_n$ where n denotes the dimension of M, and hence

$$d = \exp x_0 \exp x_1.$$

Then since d is contained in the center, we can easily prove that the matrix x_0 is commutative with any matrix of $\mathfrak{G}$. Hence we have

$$[\exp \lambda(x_0 + x_1), \mathfrak{G}] = 0,$$

for any real number λ. Therefore the correspondence defined by

$$\mathfrak{G} \ni g = \exp \lambda x \cdot h \rightarrow \exp \lambda x_2 \cdot h \quad h \in \mathfrak{N}$$

gives a representation $\mathfrak{G}^*$ of $\mathfrak{G}$. Let $\mathfrak{Z}$ be the kernel of the homomorphism. The relations $\mathfrak{N} \cap \mathfrak{Z} = e$ and $\mathfrak{D} \subseteqq \mathfrak{Z}$ are now obvious.

Next since $\mathfrak{G}/\mathfrak{D}\mathfrak{N}$ is a one-dimensional toroidal group, there exists a linear representation $\mathfrak{G}^{**}$ of $\mathfrak{G}$ such that the kernel coincides with $\mathfrak{D}\mathfrak{N}$. Hence the representation of $\mathfrak{G}$ given by

$$\begin{pmatrix} \mathfrak{G}^* & 0 \\ 0 & \mathfrak{G}^{**} \end{pmatrix}$$

is clearly a faithful representation of $\mathfrak{G}/\mathfrak{D}$, q.e.d.

LEMMA 13. *Let $\mathfrak{G}$ be a connected Lie group and $\mathfrak{N}$ a nilpotent Lie invariant subgroup. Then any compact subgroup of $\mathfrak{N}$ is central in $\mathfrak{G}$.*

Proof. Let G be the Lie algebra of $\mathfrak{G}$, and N the subalgebra of G corresponding to $\mathfrak{N}$. Then an inner derivation of G induced by an element of N is always nilpotent. Hence the linear group $\mathfrak{N}^*$, corresponding to $\mathfrak{N}$ in the adjoint representation of $\mathfrak{G}$, is simply connected, by Lemma 6 in Part I, and contains no compact subgroup except e. Hence a compact subgroup $\mathfrak{K}$ is contained in the kernel of the adjoint representation, i.e. the center, q.e.d.

[13] A matrix is called semi-simple if it has simple elementary divisors.

THEOREM 5.[14] *Let $\mathfrak{G}$ be a connected solvable Lie group. The following conditions are all necessary and sufficient for $\mathfrak{G}$ to be f.r:* 1) $\mathfrak{G} = \mathfrak{A}\mathfrak{N}$ $\mathfrak{A} \cap \mathfrak{N} = e$, *where $\mathfrak{A}$ is a (maximal) compact subgroup and $\mathfrak{N}$ is a closed simply connected invariant subgroup.* 2) $C(\mathfrak{G})(=\overline{D(\mathfrak{G})})$ *is simply connected.* 3) *The center of $C(\mathfrak{G})$ is connected and simply connected.* 4) $\mathfrak{A} \cap D(\mathfrak{G}) = e$, *where $\mathfrak{A}$ is a maximal compact group of $\mathfrak{G}$.* We note here the fact that a compact connected solvable Lie group is commutative, and hence a toroidal group.

Proof. Let G be the Lie algebra of $\mathfrak{G}$ and A a subalgebra of G corresponding to a maximal compact subgroup $\mathfrak{A}$. First we shall proceed to prove the following implications: ($\mathfrak{G}$ is f.r.) $\to$ 4) $\to$ 1) $\to$ ($\mathfrak{G}$ is f.r.)

Let $\mathfrak{G}$ be f.r. Since there corresponds a semi-simple matrix for an element of A, and a nilpotent matrix for an element of $D(G)$ in any representation of $\mathfrak{G}$, we have $A \cap D(G) = 0$. On the other hand the commutator group $D(\mathfrak{G})$ is closed and simply connected by Lemma 7 in Part I. Hence we get $\mathfrak{A} \cap D(\mathfrak{G}) = e$, i.e. 4).

Next from 4) we have $A \cap D(G) = 0$, and hence there exists a subspace N of G such that $N \geqq D(G)$, $G = A + N$ $A \cap N = 0$. N is then obviously an ideal. Let $\tilde{\mathfrak{G}}$ be the universal covering group of $\mathfrak{G}$ and let $\tilde{\mathfrak{A}}, \tilde{\mathfrak{N}}$ be (closed) subgroups of $\tilde{\mathfrak{G}}$ generated by A, N respectively. Then the simple connectedness of the factor group $\tilde{\mathfrak{G}}/\tilde{\mathfrak{N}}$ by Lemma 3 in Part I implies that $\tilde{\mathfrak{A}} \cap \tilde{\mathfrak{N}} = e$. Let $\mathfrak{D}$ be the kernel of the homomorphism $\hat{\mathfrak{G}} \sim \mathfrak{G}$. Then $\mathfrak{D}$ is obviously contained in $\tilde{\mathfrak{A}}$ because $\mathfrak{A}$ is a maximal compact subgroup of $\mathfrak{G}$. Hence N generates a closed simply connected group $\mathfrak{N}$ in $\mathfrak{G}$. That $\mathfrak{A} \cap \mathfrak{N} = e$ follows immediately. Thus we get 1).

Now suppose that the condition 1) is satisfied. Then by Lemma 11 the universal covering group of $\mathfrak{G}$ is f.r., and using Lemma 12 we get a faithful representation of $\mathfrak{G}$. Thus we have the equivalence of the faithful representability, 1) and 4).

Now from the simple connectedness of the adjoint group of a connected nilpotent Lie group and by Lemma 3 in Part I we can easily prove that 2) is equivalent to 3). Next from 1) we get 2) because $C(\mathfrak{G})$ is contained in $\mathfrak{N}$. Suppose now 2) is satisfied for $\mathfrak{G}$. Then $D(\mathfrak{G})$ is, as a Lie subgroup of the simply connected $C(\mathfrak{G})$, closed, and hence coincides with $C(\mathfrak{G})$, q.e.d.

From Theorem 5 and Lemma 13 we get the following

COROLLARY 1.[15] *A connected nilpotent Lie group is f.r. if and only if it is*

[14] Cf. Malcev [11], [13].

[15] Cf. Birkhoff [19].

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a direct product of a compact group and a simply connected Lie gronp.

COROLLARY 2. *A connected solvable Lie* (l)*-group is f.r.*

Proof. Let $\mathfrak{G}$ be a connected solvable Lie group. Suppose that $\mathfrak{G}$ is not f.r. Then the center of $C(\mathfrak{G})$ contains a compact group $\mathfrak{K}(\neq e)$ by Theorem 5. Since $\mathfrak{K}$ is mapped into the unit matrix in any linear representation of $\mathfrak{G}$, $\mathfrak{G}$ is not an (l)-group. q.e.d.

Remark. A connected non-f.r. solvabie Lie group has no locally isomorphic (univalent) representation.

F.r. Lie groups

9. Let G be a Lie algebra and R the radical of G, and let R_1 be the intersection of R and $D(G)$. Then the factor algebra G/R_1 is decomposed into a direct sum of a semi-simple ideal S and the center Z:

$$G/R_1 = S + Z \qquad S \cap Z = 0 \qquad [S, Z] = 0$$

Let $\hat{\mathfrak{S}}$ be the linear covering group of S, and Z the m-dimensional vector group, where m denotes the dimension of Z. The direct product $\hat{\mathfrak{S}} \times \mathfrak{Z}$ is then clearly f.r., and furthermore there exists a faithful representation of $\hat{\mathfrak{S}} \times \mathfrak{Z}$ such that the center of the corresponding linear Lie algebra consists of nilpotent matrices. Let G_1 be the linear Lie algebra of such a representation. Let now G_2 be an arbitrary faithful representation of G. Then the representation of G defined by

$$G^* = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$$

is of course faithful.

Let now $\hat{\mathfrak{G}}$ be the linear Lie group generated by G^*. Then we may easily prove that the radical $\hat{\mathfrak{R}}$ of $\hat{\mathfrak{G}}$ is simply connected in such a way as in the proof of Lemma 11. Let $\mathfrak{S}$ be a maximal semi-simple Lie subgroup of $\hat{\mathfrak{G}}$. Since $\mathfrak{S}$ is closed and the center of $\mathfrak{S}$ is finite in virtue of § 4 in Part I, $\mathfrak{S} \cap \hat{\mathfrak{R}}$ is also a finite group, whence the simple connectedness of $\hat{\mathfrak{R}}$ implies that $\mathfrak{S} \cap \hat{\mathfrak{R}} = e$. On the other hand since $\hat{\mathfrak{G}}$ has a representation which is isomorphic to $\hat{\mathfrak{S}}$, we get that $\mathfrak{S} \cong \hat{\mathfrak{G}}/\hat{\mathfrak{R}} \sim \hat{\mathfrak{S}}$, whence we have $\mathfrak{S} \cong \hat{\mathfrak{S}}$ comparing the orders of the centers of $\mathfrak{S}$ and $\hat{\mathfrak{S}}$.

Now we shall prove that our $\hat{\mathfrak{G}}$ is a linear covering group of G. First let $\tilde{\mathfrak{G}}$ be the universal covering group of $\hat{\mathfrak{G}}$ and let $\mathfrak{G}$ be any f.r. Lie group corresponding to G. We may suppose that $\tilde{\mathfrak{G}}/\mathfrak{D} = \mathfrak{G}$ and $\tilde{\mathfrak{G}}/\mathfrak{Z} = \hat{\mathfrak{G}}$. Now let $\tilde{\mathfrak{S}}$ be a maximal semi-simple Lie subgroup of $\tilde{\mathfrak{G}}$. Since the space of $\hat{\mathfrak{G}}$ is a direct product of the space of $\hat{\mathfrak{S}}$ and the Euclidean space, $\mathfrak{Z}$ is contained in $\tilde{\mathfrak{S}}$. Since

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$\tilde{\mathfrak{S}}\mathfrak{D}/\mathfrak{D}(\cong \tilde{\mathfrak{S}}/\tilde{\mathfrak{S}}\cap\mathfrak{D})$, a maximal semi-simple Lie subgroup of $\mathfrak{G}$, is f.r., we get that $\tilde{\mathfrak{S}} \cap \mathfrak{D} \supseteqq \mathfrak{Z}$, whence $\mathfrak{Z}$ is contained in $\mathfrak{D}$: $\mathfrak{Z} \subseteqq \mathfrak{D}$. Hence $\mathfrak{G}$ is homomorphic with $\hat{\mathfrak{G}}$, namely $\hat{\mathfrak{G}}$ is a linear covering group. Here we note the fact that the discrete invariant subgroup $\mathfrak{Z}$ is characteristic, which will be seen easily. Now the uniqueness of the linear covering group for any Lie algebra may be proved as in the proof of Theorem 3.

Next let $\hat{\mathfrak{G}}$ be a linear covering group and $\mathfrak{N}$ a Lie invariant subgroup of $\hat{\mathfrak{G}}$. An analogous argument as in Lemma 3 in Part I shows that $\mathfrak{N}$ is closed and $\mathfrak{N}$ and $\hat{\mathfrak{G}}/\mathfrak{N}$ are both linear covering groups.

From the above considerations we may easily prove the following

THEOREM 6. *For any Lie algebra G there exists one and only one linear covering group $\hat{\mathfrak{G}}$.*

The radical $\hat{\mathfrak{R}}$ of $\hat{\mathfrak{G}}$ is simply connected, and a maximal semi-simple Lie subgroup $\hat{\mathfrak{G}}$ is a linear covering group such that

$$\hat{\mathfrak{G}} = \hat{\mathfrak{S}}\hat{\mathfrak{R}}, \quad \hat{\mathfrak{S}} \cap \hat{\mathfrak{R}} = e.$$

In particular, a linear covering group of a complex Lie algebra is simply connected.[16]

Any local homomorphism of a linear covering group onto any f.r. Lie group can be extended to a homomorphism in the large.

Remark. A connected Lie group is homomorphic with the linear covering group of its Lie algebra if its maximal semi-simple Lie subgroup is f.r.

THEOREM 7.[17] *Let $\mathfrak{G}$ be a connected Lie group and $\mathfrak{R}$ the radical of $\mathfrak{G}$. If a maximal semi-simple Lie subgroup $\mathfrak{S}$ and $\mathfrak{R}$ are f.r., then $\mathfrak{G}$ is also f.r.*, and vice, versa.

Proof. Let G be the Lie algebra of $\mathfrak{G}$ and R the radical of G. Since the radical $\mathfrak{R}$ is f.r. and solvable there exist closed subgroups $\mathfrak{A}$ and $\mathfrak{R}'$ of $\mathfrak{R}$ such that

$$\mathfrak{R} = \mathfrak{A}\mathfrak{R}', \quad \mathfrak{A} \cap \mathfrak{R}' = e,$$

where $\mathfrak{A}$ is a toroidal group and $\mathfrak{R}'$ is simply connected and invariant in $\mathfrak{R}$. Let $h(\lambda)$ be an everywhere dense one-parameter subgroup of $\mathfrak{A}$: $\overline{h(\lambda)} = \mathfrak{A}$, and let x be an element of G which generates $h(\lambda)$. Now we decompose G and R into eigenspaces of the inner derivation δ_x induced by x. Since δ_x is semi-simple because of the compactness of $\mathfrak{A}$, we may put

[16] Matsushima [15].

[17] Malcev [13].

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$$G = G_0 + G_1, \quad \delta_x G_0 = 0, \quad \delta_x G_1 = G_1,$$
$$R = R_0 + R_1, \quad \delta_x R_0 = 0, \quad \delta_x R_1 = R_1,$$

where G_0 and R_0 are subalgebras. Since it is clear that $G_0 \cap R = R_0$ and $G_0 + R = G$, we have

$$G_0/R_0 \cong G/R \quad \text{(semi-simple)}.$$

Hence R_0 is the radical of G_0. Let S be a maximal semi-simple subalgebra of G. Then S is evidently maximal semi-simple in G. Let A be the Lie algebra of $\mathfrak{A}$. Since $\delta_x G_0 = 0$, we have $[A, G_0] = 0$, and in particular $[A, S] = 0$. Now let us consider R as an S-module. Then A and $D(R)$ are both S-modules such that $A \cap D(R) = 0$. Hence from the complete reducibility of R there exists a submodule N containing $D(R)$ such that

$$R = A + N, \quad A \cap N = 0$$

N is, as an ideal of R and an S-module, an ideal of G. Let now $\mathfrak{N}$ be the subgroup of $\mathfrak{G}$ generated by N. $\mathfrak{N}$ is easily seen to be closed and simply connected. Thus we get the decomposition

$$\mathfrak{R} = \mathfrak{A}\mathfrak{N} \quad \mathfrak{A} \cap \mathfrak{N} = e,$$

where $\mathfrak{A}$ is a toroidal group and $\mathfrak{N}$ is a closed simply connected invariant subgroup of $\mathfrak{G}$.

Next let $\mathfrak{S}$ be the subgroup generated hy S. Since the center of $\mathfrak{S}$ is finite in virtue of Lemma 5 in Part I because $\mathfrak{S}$ is f.r., $\mathfrak{S}$ is closed by Theorem 2 in Part I. Then the compactness of $\mathfrak{A}$ implies that $\mathfrak{S}\mathfrak{A}$ is closed. Now since the center of $\mathfrak{S}\mathfrak{A}$ is compact and $\mathfrak{N}$ contains no compact subgroup except e, we have $\mathfrak{S}\mathfrak{A} \cap \mathfrak{N} = e$, whence $\mathfrak{G}$ *is topologically a direct product of* $\mathfrak{S}\mathfrak{A}$ *and* $\mathfrak{N}$: $\mathfrak{G} = (\mathfrak{S}\mathfrak{A})\mathfrak{N}$.[18]

Let now $\hat{\mathfrak{G}}$ be the linear covering group of G. Then by the remark to Theorem 6, $\hat{\mathfrak{G}}$ is a covering group of $\mathfrak{G}$. Let $\hat{\mathfrak{S}}$, $\mathfrak{A}$ and $\hat{\mathfrak{N}}$ be subgroups of $\hat{\mathfrak{G}}$ generated by S, A and N respectively. Then $\hat{\mathfrak{G}} = \hat{\mathfrak{S}}\hat{\mathfrak{A}}\hat{\mathfrak{N}}$ and the space of $\hat{\mathfrak{G}}$ is the direct product of the spaces of $\hat{\mathfrak{S}}$, $\hat{\mathfrak{A}}$ and $\hat{\mathfrak{N}}$. Since $\mathfrak{N}$ is simply connected, we have $\mathfrak{D} \subseteqq \hat{\mathfrak{S}}\hat{\mathfrak{A}}$, where $\mathfrak{D}$ denotes the kernel of the homomorphism $\hat{\mathfrak{G}} \sim \mathfrak{G}$.

18) Matsushima in [15] proved the following theorem: *Let* $\mathfrak{G}$ *be a connected Lie group and* $\mathfrak{H}$ *and* $\mathfrak{N}$ *closed subgroups. Let* H, N *be their Lie algebras. Then* $\mathfrak{G}$ *is f.r. if the following conditions are satisfied.* 1) $\mathfrak{N}$ *is a simply connected solvable invariant subgroup and* $\mathfrak{H}$ *is f.r.* 2) $\mathfrak{G} = \mathfrak{H}\mathfrak{N}$ $\mathfrak{H} \cap \mathfrak{A} = e$. 3) N *is completely reducible as an* H*-module.* Using this theorem we get *another proof* of Theorem 7, because $\mathfrak{S}\mathfrak{A}$, which is a direct product of f.r. $\mathfrak{S}$ and a toroidal group $\mathfrak{A}$ mod. a finite group, is f.r. by Lemma 8, and our N is clearly a completely reducible $(S + A)$-module. We note also the fact that our Theorem 7 and our decomposition $\mathfrak{G} = (\mathfrak{S}\mathfrak{A})\mathfrak{N}$ show *the validity of the converse of the above Theorem of Matsushima.*

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Now let $\mathfrak{B}$ be a maximal compact subgroup of $\mathfrak{S}$, and B its Lie algebra. Then $\mathfrak{K} = \mathfrak{B}\mathfrak{A}$ is obviously a maximal compact subgroup of $\mathfrak{S}\mathfrak{A}$, and the Lie algebra K of $\mathfrak{K}$ coincides with $A + B$. Let $\hat{\mathfrak{B}}$ be the subgroup of $\hat{\mathfrak{G}}$ generated by B. Since $\hat{\mathfrak{S}}$ is a covering group of $\mathfrak{S}$ of finite order, $\hat{\mathfrak{B}}$ is compact. Now since $\mathfrak{K}$ is a maximal compact subgroup of $\mathfrak{S}\mathfrak{A}$, we may easily conclude that $\hat{\mathfrak{K}} = \hat{\mathfrak{B}}\hat{\mathfrak{A}} \supseteq \mathfrak{D}$. Let A be of m dimensions. Then $\mathfrak{A}$ is the m-dimensional vector group. Hence $\mathfrak{D}$ is, as a discrete invariant subgroup of a direct product of a compact group and an m-dimensional vector group, of rank at most m. On the other hand, $\mathfrak{D}_1 = \mathfrak{D} \cap \hat{\mathfrak{A}}$ is obviously a free commutative group of rank m. Hence the factor group $\mathfrak{D}/\mathfrak{D}_1$ is finite. Now since $\hat{\mathfrak{S}}\hat{\mathfrak{R}}$ is a closed invariant subgroup and $\hat{\mathfrak{A}}$ is a vector group, Lemma 12 shows that $\hat{\mathfrak{G}}/\mathfrak{D}_1$ is f.r. Then, $\mathfrak{D}/\mathfrak{D}_1$ being a finite group, the relation

$$\mathfrak{G} \cong \hat{\mathfrak{G}}/\mathfrak{D} \cong \hat{\mathfrak{G}}/\mathfrak{D}_1 \Big/ \mathfrak{D}/\mathfrak{D}_1 ,$$

implies the faithful representability of $\mathfrak{G}$ by Lemma 8, q.e.d.

Corollary 1. *Let $\mathfrak{G}$ be an f.r. Lie group, and $\mathfrak{K}$ a maximal compact subgroup of $\mathfrak{G}$. Then there exists a closed simply connected solvable subgroup $\mathfrak{H}$ of $\mathfrak{G}$ such that*

$$\mathfrak{G} = \mathfrak{K}\mathfrak{H} = \mathfrak{H}\mathfrak{K}, \quad \mathfrak{H} \cap \mathfrak{K} = e .$$

Proof. The assertion was proved by K. Iwasawa[19] for adjoint groups of semi-simple Lie groups. By our Lemma 5 in Part I it is also valid for semi-simple f.r. Lie groups. Then the decomposition of an f.r. Lie group $\mathfrak{G}$:

$$\mathfrak{G} = (\mathfrak{S}\mathfrak{A})\mathfrak{N}, \quad \mathfrak{S}\mathfrak{A} \cap \mathfrak{N} = e ,$$

given in the proof of Theorem 7, easily implies our corollary, q.e.d.

Next let us call an f.r. group $\mathfrak{G}$ *completely reducible* if any linear representation of $\mathfrak{G}$ is completely reducible. Semi-simple, or compact f.r. groups are completely reducible, but a vector groop is not.

Let $\mathfrak{G}$ be a completely reducible f.r. group and G its Lie algebra. By a theorem of Cartan, an irreducible linear Lie algebra is a direct sum of a semi-simple ideal and its center which is composed of scalar matrices. Using this we can readily prove that G is decomposed into a direct sum of its center Z and a semi-simple ideal S: $G = S + Z$, $[S, Z] = 0$. Now let $\mathfrak{S}$ and $\mathfrak{Z}$ be subgroups of $\mathfrak{G}$ generated by S and Z respectively. Assume that $\mathfrak{Z}$ is not compact. Then the finiteness of the center of $\mathfrak{S}$ by Lemma 5 in Part I obviously implies that there exists a closed vector invariant subgroup of $\mathfrak{G}$ which is a direct factor. Hence $\mathfrak{G}$ is not completely reducible, contrary to the assumption.

[19] Iwasawa [10].

Thus $\mathfrak{Z}$ must be compact. Since the converse proposition is clear, we have the following

LEMMA 14. *An f.r. group $\mathfrak{G}$ is completely reducible if and only if the radical of $\mathfrak{G}$ is compact.*

Now the following corollary is an immediate consequence of Theorem 7. (Cf. also Note [13] of p. 103).

COROLLARY 2. *Let $\mathfrak{G}$ be a connected Lie group. It there exist closed subgroups $\mathfrak{T}$ and $\mathfrak{N}$ of $\mathfrak{G}$ such that $\mathfrak{G} = \mathfrak{T}\mathfrak{N}$ $\mathfrak{T} \cap \mathfrak{N} = e$, where $\mathfrak{T}$ is a completely reducible f.r. group and $\mathfrak{N}$ is a simply connected solvable invariant subgroup, then $\mathfrak{G}$ is f.r., and vice versa.*

10. In this § three theorems shall be proved concerning f.r. Lie groups.

THEOREM 8. *A connected Lie group is f.r. if it is an (l)-group.*

Proof. Let $\mathfrak{G}$ be a connected Lie group and let $\mathfrak{S}$ and $\mathfrak{R}$ be a maximal semi-simple Lie subgroup and the radical of $\mathfrak{G}$ rospectively. Suppose that $\mathfrak{G}$ is an (l)-group. Then $\mathfrak{S}$ and $\mathfrak{R}$ are also (l)-groups. Now Corollary 2 to Theorem 4 and Corollary 2 to Theorem 5 imply that $\mathfrak{S}$ and $\mathfrak{R}$ are both f.r., whence $\mathfrak{G}$ is f.r. by Theorem 7, q.e.d.

THEOREM 9. *Any f.r. Lie group is isomorphic with a closed subgroup of the general linear group of a certain degree.* In other words, for any f.r. Lie group there exists a topologically isomorphic linear representation with respect to the induced topology.

Proof. Let $\mathfrak{G}$ be an f.r. Lie group and $\mathfrak{R}$ the radical of $\mathfrak{G}$. The commutator subgroup $D(\mathfrak{G})$ is closed and the radical $\mathfrak{R}_1$ of $D(\mathfrak{G})$ is simply connected by Lemma 7 in Part I. Denote by $\mathfrak{A}$ a maximal compact subgroup of $\mathfrak{R}$. Then there exists a closed simply connected invariant subgroup $\mathfrak{N}$ such that $\mathfrak{R} = \mathfrak{A}\mathfrak{N}$, $\mathfrak{A} \cap \mathfrak{N} = e$, and $\mathfrak{N} \geqq \mathfrak{R}_1$. As $\mathfrak{A}$ is compact, $\mathfrak{A}D(\mathfrak{G})$ is a closed invariant subgroup. Let

$$\mathfrak{G} = (\mathfrak{S}\mathfrak{A})\mathfrak{N}, \quad \mathfrak{S}\mathfrak{A} \cap \mathfrak{N} = e$$

be a decomposition as in the proof of Theorem 7. Since $\mathfrak{S}\mathfrak{A}\mathfrak{R}_1$ coincides with $\mathfrak{A}D(\mathfrak{G})$ locally and they are both connected, we have

$$\mathfrak{S}\mathfrak{A}\mathfrak{R}_1 = \mathfrak{A}D(\mathfrak{G}),$$

whence $\mathfrak{A}D(\mathfrak{G}) \cap \mathfrak{N} = \mathfrak{R}_1$. Now from

$$\mathfrak{G}/\mathfrak{A}D(\mathfrak{G}) = \mathfrak{A}D(\mathfrak{G})\mathfrak{N}/\mathfrak{A}D(\mathfrak{G}) \cong \mathfrak{N}/\mathfrak{N} \cap \mathfrak{A}D(\mathfrak{G}) = \mathfrak{N}/\mathfrak{R}_1,$$

we see that $\mathfrak{G}/\mathfrak{A}D(\mathfrak{G})$ is a vector group. Hence there exists surely a representation of $\mathfrak{G}$ such that the kernel coincides with $\mathfrak{A}D(\mathfrak{G})$ and whose linear Lie

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algebra consists of nilpotent matrices. Let $\mathfrak{G}^*$ be such a representation.

Next let $\mathfrak{G}^{**}$ be a faithful representation of $\mathfrak{G}$. The representation defined by

$$\begin{pmatrix} \mathfrak{G}^* & 0 \\ 0 & \mathfrak{G}^{**} \end{pmatrix}$$

is of course faithful. On the other hand in this representation any non-zero element of the Lie algebra of $\mathfrak{N}$ is represented by a matrix with non-zero 0-eigenspace, whence $\mathfrak{N}$ is closed in the general linear group by Lemma 6 in Part I. Since $\mathfrak{A}$ is compact, the radical $\mathfrak{R} = \mathfrak{A}\mathfrak{N}$ is also represented by a closed group. Thus Lemma 5 and Theorem 2 in Part I imply our assertion, q.e.d.

Remark. The above two theorems give us apparently weaker but equivalent definitions of f.r. groups.

Examples of non-f.r. Lie groups have been given by Cartan and Birkhoff. (See § 4 in Part I and § 7). Here we shall prove a theorem which shows the necessity of the above examples, namely the following

THEOREM 10. *A connected Lie group $\mathfrak{G}$ is f.r. if and only if the following two conditions are satisfied.*

1) *All simple Lie subgroups are f.r.*
2) *The radical of $C(\mathfrak{G})$ is simply connected.*

Proof. If $\mathfrak{G}$ is f.r., the above two conditions are obvious. Conversely let us suppose that 1) and 2) are satisfied. Let $\mathfrak{S}$ be a maximal semi-simple Lie subgroup of $\mathfrak{G}$ and $\mathfrak{R}$ the radical of $\mathfrak{G}$. Since any simple invariant Lie subgroup of $\mathfrak{S}$ is f.r., $\mathfrak{S}$ is also f.r. by Corollary 1 to Theorem 4. Now since $C(\mathfrak{R})$ is contained in the radical of $C(\mathfrak{G})$, it is simply connected, whence $\mathfrak{R}$ is f.r. by Theorem 5. Therefore the theorem follows from Theorem 7, q.e.d.

Nagoya University

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