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ARTICLE

Making K_{r+1}-free graphs r-partite

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Abstract
The Erdős–Simonovits stability theorem states that for all $\varepsilon > 0$ there exists $\alpha > 0$ such that if G is a K_{r+1}-free graph on n vertices with $e(G) > \mathrm{ex}(n, K_{r+1}) - \alpha n^2$, then one can remove εn^2 edges from G to obtain an r-partite graph. Füredi gave a short proof that one can choose $\alpha = \varepsilon$. We give a bound for the relationship of α and ε which is asymptotically sharp as $\varepsilon \to 0$.

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1. Introduction

Erdős asked how many edges need to be removed in a triangle-free graph on n vertices in order to make it bipartite. He conjectured that the balanced blow-up of C_5 with class sizes $n/5$ is the worst case, and hence $n^2/25$ edges would always be sufficient. Together with Faudree, Pach and Spencer [6], he proved that one can remove at most $n^2/18$ edges to make a triangle-free graph bipartite.

Further, Erdős, Győri and Simonovits [7] proved that for graphs with at least $n^2/5$ edges, an unbalanced C_5 blow-up is the worst case. For $r \in \mathbb{N}$, let $D_r(G)$ denote the minimum number of edges which need to be removed to make G r-partite.

Theorem 1.1 (Erdős, Győri and Simonovits [7]). *Let G be a K_3-free graph on n vertices with at least $n^2/5$ edges. There exists an unbalanced C_5 blow-up of H with $e(H) \geqslant e(G)$ such that*

$$D_2(G) \leqslant D_2(H). \tag{1.1}$$

This proved the Erdős conjecture for graphs with at least $n^2/5$ edges. A simple probabilistic argument (*e.g.* [7]) settles the conjecture for graphs with at most $2/25n^2$ edges.

A related question was studied by Sudakov; he determined the maximum number of edges in a K_4-free graph which need to be removed in order to make it bipartite [16]. This problem for K_6-free graphs was solved by Hu, Lidický, Martins, Norin and Volec [11].

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We will study the question of how many edges in a K_{r+1}-free graph need, at most, to be removed to make it r-partite. For $n \in \mathbb{N}$ and a graph H, let $\mathrm{ex}(n, H)$ denote the Turán number, *i.e.* the maximum number of edges of an H-free graph. The Erdős–Simonovits theorem [8] for cliques states that for every $\varepsilon > 0$ there exists $\alpha > 0$ such that if G is a K_{r+1}-free graph on n vertices with $e(G) > \mathrm{ex}(n, K_{r+1}) - \alpha n^2$, then $D_r(G) \leqslant \varepsilon n^2$.

Füredi [9] gave a nice short proof of the statement that a K_{r+1}-free graph G on n vertices with at least $\mathrm{ex}(n, K_{r+1}) - t$ edges satisfies $D_r(G) \leqslant t$, and thus provided a quantitative version of the Erdős–Simonovits theorem.

In [11] Füredi's result was strengthened for some values of r. Roberts and Scott [15] showed that $D_r(G) = O(t^{3/2}/n)$ when $t \leqslant \delta n^2$, and that this result is sharp up to a constant factor. They also proved a more general result for H-free graphs where H is an edge-critical graph. For small t, we will determine asymptotically how many edges are needed. For very small t, it is already known [4] that G has to be r-partite, as the following theorem shows.

Theorem 1.2 (Brouwer [4]). *Let $r \geqslant 2$ and $n \geqslant 2r+1$ be integers. Let G be a K_{r+1}-free graph on n vertices with $e(G) \geqslant \mathrm{ex}(n, K_{r+1}) - \lfloor n/r \rfloor + 2$. Then*

$$D_r(G) = 0. \tag{1.2}$$

This phenomenon was also studied in [1], [10], [12] and [18]. We will be studying K_{r+1}-free graphs on fewer edges. For these, our main result is the following theorem.

Theorem 1.3. *Let $r \geqslant 2$ be an integer. Then, for all $n \geqslant 3r^2$ and for all $0 \leqslant \alpha \leqslant 10^{-7} r^{-12}$, the following holds. Let G be a K_{r+1}-free graph on n vertices with*

$$e(G) \geqslant \mathrm{ex}(n, K_{r+1}) - \alpha n^2. \tag{1.3}$$

Then

$$D_r(G) \leqslant \left(\frac{2r}{3\sqrt{3}} + o_\alpha(1)\right)\alpha^{3/2} n^2, \tag{1.4}$$

where $o_\alpha(1)$ is a term converging to 0 for α tending to 0.

Note that we did not try to optimize our bounds on n and α in the theorem.

The blow-up of a graph G is obtained by replacing every vertex $v \in V(G)$ with finitely many copies so that the copies of two vertices are adjacent if and only if the originals are.

For two graphs G and H, we define $G \otimes H$ to be the graph on the vertex set $V(G) \cup V(H)$ with $gg' \in E(G \otimes H)$ if and only if $gg' \in E(G)$, $hh' \in E(G \otimes H)$ if and only if $hh' \in E(H)$, and $gh \in E(G \otimes H)$ for all $g \in V(G)$, $h \in V(H)$.

We will prove that Theorem 1.3 is asymptotically sharp by describing an unbalanced blow-up of $K_{r-2} \otimes C_5$ that needs at least that many edges to be removed to make it r-partite. Our extremal example appeared first (with different class sizes) in a paper by Andrásfai, Erdős and Sós [2].

Theorem 1.4. *Let $r, n \in \mathbb{N}$ and $0 < \alpha < 1/(4r^4)$. Then there exists a K_{r+1}-free graph G on n vertices with*

$$e(G) \geqslant \mathrm{ex}(n, K_{r+1}) - \alpha n^2 + \frac{4r}{3\sqrt{3}}\alpha^{3/2} n^2 - \frac{2r(r-3)}{9}\alpha^2 n^2$$

and

$$D_r(G) \geqslant \frac{2r}{3\sqrt{3}}\alpha^{3/2} n^2.$$

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In Kang and Pikhurko's proof [12] of Theorem 1.2, the case $e(G) = \mathrm{ex}(n, K_{r+1}) - \lfloor n/r \rfloor + 1$ is studied. In this case they constructed a family of K_{r+1}-free non-r-partite graphs, which includes our extremal graph, for that number of edges.

We conjecture that our extremal example needs the most edges removed to make it r-partite among all K_{r+1}-free graphs with many edges.

Conjecture 1.5. *Let $r \geqslant 2$ be an integer and let n be sufficiently large. Then there exists $\alpha_0 > 0$ such that for all $0 \leqslant \alpha \leqslant \alpha_0$ the following holds. For every K_{r+1}-free graph G on n vertices there exists an unbalanced $K_{r-2} \otimes C_5$ blow-up H on n vertices with $e(H) \geqslant e(G)$ such that*

$$D_r(G) \leqslant D_r(H). \tag{1.5}$$

This conjecture can be seen as a generalization of Theorem 1.1. Note that Conjecture 1.5 was recently proved by Korándi, Roberts and Scott [13]. We recommend the interested reader to read the excellent survey [14] by Nikiforov. He gives a good overview on further related stability results, for example on guaranteeing large induced r-partite subgraphs of K_{r+1}-free graphs.

We organize the paper as follows. In Section 2 we prove Theorem 1.3 and in Section 3 we give the sharpness example, *i.e.* we prove Theorem 1.4.

2. Proof of Theorem 1.3

In this section we prove the following version of Theorem 1.3, which gives better control over the error term.

Theorem 2.1. *Let $r \geqslant 2$ be an integer. Then, for all $n \geqslant 3r^2$ and for all $0 \leqslant \alpha \leqslant 10^{-7}r^{-12}$, the following holds. Let G be a K_{r+1}-free graph on n vertices with*

$$e(G) \geqslant \mathrm{ex}(n, K_{r+1}) - \alpha n^2. \tag{2.1}$$

Then

$$D_r(G) \leqslant \left(\frac{2r}{3\sqrt{3}} + 30r^3\alpha^{1/6}\right)\alpha^{3/2}n^2. \tag{2.2}$$

Let G be an n-vertex K_{r+1}-free graph with $e(G) \geqslant \mathrm{ex}(n, K_{r+1}) - t$, where $t = \alpha n^2$. We will assume that n is sufficiently large. Furthermore, by Theorem 1.2 we can assume that

$$\alpha \geqslant \frac{\lfloor n/r \rfloor - 2}{n^2} \geqslant \frac{1}{2rn}. \tag{2.3}$$

This also implies that $t \geqslant r$ because $n \geqslant 3r^2$. During our proof we will make use of Turán's theorem and a version of Turán's theorem for r-partite graphs on multiple occasions. Turán's theorem [17] determines the maximum number of edges in a K_{r+1}-free graph.

Theorem 2.2 (Turán [17]). *Let $r \geqslant 2$ and $n \in \mathbb{N}$. Then*

$$\frac{n^2}{2}\left(1 - \frac{1}{r}\right) - \frac{r}{2} \leqslant \mathrm{ex}(n, K_{r+1}) \leqslant \frac{n^2}{2}\left(1 - \frac{1}{r}\right).$$

Let $K(n_1, \ldots, n_r)$ denote the complete r-partite graph whose r colour classes have sizes $n_1, \ldots, n_r$, respectively. Turán's theorem for r-partite graphs states the following.

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Theorem 2.3 (folklore). *Let $r \geqslant 2$ and $n_1, \ldots, n_r \in \mathbb{N}$ satisfying $n_1 \leqslant \cdots \leqslant n_r$. For a K_r-free subgraph H of $K(n_1, \ldots, n_r)$, we have*

$$e(H) \leqslant e(K(n_1, \ldots, n_r)) - n_1 n_2.$$

For a proof of this folklore result see [3, Lemma 3.3], for example.

We denote the maximum degree of G by $\Delta(G)$. For two disjoint subsets U, W of $V(G)$, write $e(U, W)$ for the number of edges in G with one endpoint in U and the other endpoint in W. We write $e^c(U, W)$ for the number of non-edges between U and W, *i.e.* $e^c(U, W) = |U||W| - e(U, W)$.

Füredi [9] used Erdős's degree majorization algorithm [5] to find a vertex partition with some useful properties. We include the proof for completeness.

Lemma 2.1 (Füredi [9]). *Let $t, r, n \in \mathbb{N}$ and G be an n-vertex K_{r+1}-free graph with $e(G) \geqslant \mathrm{ex}(n, K_{r+1}) - t$. Then there exists a vertex partition $V(G) = V_1 \cup \cdots \cup V_r$ such that*

$$\sum_{i=1}^{r} e(G[V_i]) \leqslant t, \quad \Delta(G) = \sum_{i=2}^{r} |V_i| \quad \textit{and} \quad \sum_{1 \leqslant i < j \leqslant r} e^c(V_i, V_j) \leqslant 2t. \tag{2.4}$$

Proof. Let $x_1 \in V(G)$ be a vertex of maximum degree. Define $V_1 := V(G) \setminus N(x_1)$ and $V_1^+ = N(x_1)$. Iteratively, let x_i be a vertex of maximum degree in $G[V_{i-1}^+]$. Let $V_i := V_{i-1}^+ \setminus N(x_i)$ and $V_i^+ = V_{i-1}^+ \cap N(x_i)$. Since G is K_{r+1}-free this process stops at $i \leqslant r$ and thus gives a vertex partition $V(G) = V_1 \cup \cdots \cup V_r$. Summing up the degrees of vertices in V_1, we have

$$2e(G[V_1]) + e(V_1, V_1^+) = \sum_{x \in V_1} \deg(x) \leqslant |V_1||V_1^+|,$$

and similarly for the other classes

$$2e(G[V_i]) + e(V_i, V_i^+) = \sum_{x \in V_1} \deg_{G[V_{i-1}^+]}(x) \leqslant |V_i||V_i^+|.$$

Adding up these inequalities we get

$$\mathrm{ex}(n, K_{r+1}) - t + \sum_{i=1}^{r} e(G[V_i]) = e(G) + \sum_{i=1}^{r} e(G[V_i]) \leqslant \sum_{i=1}^{r-1} |V_i||V_i^+| \leqslant \mathrm{ex}(n, K_{r+1}),$$

implying

$$\sum_{i=1}^{r} e(G[V_i]) \leqslant t.$$

By construction,

$$\sum_{i=2}^{r} |V_i| = |V_1^+| = |N(x_1)| = \Delta(G).$$

Let H be the complete r-partite graph with vertex set $V(G)$ and all edges between V_i and V_j for $1 \leqslant i < j \leqslant r$. The graph H is r-partite and thus has at most $\mathrm{ex}(n, K_{r+1})$ edges. Finally, since G has at most t edges not in H and at least $\mathrm{ex}(n, K_{r+1}) - t$ edges in total, at most $2t$ edges of H can be missing from G, giving us

$$\sum_{1 \leqslant i < j \leqslant r} e^c(V_i, V_j) \leqslant 2t$$

and proving the last inequality. □

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For this vertex partition we can get bounds on the class sizes.

Lemma 2.2. *For all $i \in [r]$,*

$$|V_i| \in \left\{ \frac{n}{r} - \frac{5}{2}\sqrt{\alpha}n, \frac{n}{r} + \frac{5}{2}\sqrt{\alpha}n \right\},$$

and thus also

$$\Delta(G) \leqslant \frac{r-1}{r}n + \frac{5}{2}\sqrt{\alpha}n.$$

Proof. We know that

$$\sum_{1 \leqslant i < j \leqslant r} |V_i||V_j| \geqslant e(G) - \sum_{i=1}^{r} e(G[V_i]) \geqslant \left(1 - \frac{1}{r}\right)\frac{n^2}{2} - \frac{r}{2} - 2t.$$

Also

$$\sum_{1 \leqslant i < j \leqslant r} |V_i||V_j| = \frac{1}{2}\sum_{i=1}^{r} |V_i|(n - |V_i|) = \frac{n^2}{2} - \frac{1}{2}\sum_{i=1}^{r} |V_i|^2.$$

Thus we can conclude that

$$\sum_{i=1}^{r} |V_i|^2 \leqslant \frac{n^2}{r} + r + 4t. \tag{2.5}$$

Now let $x = |V_1| - n/r$. Then

$$\begin{aligned} \sum_{i=1}^{r} |V_i|^2 &= \left(\frac{n}{r} + x\right)^2 + \sum_{i=2}^{r} |V_i|^2 \\ &\geqslant \left(\frac{n}{r} + x\right)^2 + \frac{(\sum_{i=2}^{r} |V_i|)^2}{r-1} \\ &\geqslant \left(\frac{n}{r} + x\right)^2 + \frac{(n(1-1/r) - x)^2}{r-1} \\ &\geqslant \frac{n^2}{r} + x^2. \end{aligned} \tag{2.6}$$

Combining this with (2.5), we get

$$|x| \leqslant \sqrt{r + 4t} \leqslant \frac{5}{2}\sqrt{t} = \frac{5}{2}\sqrt{\alpha}n,$$

and thus

$$\frac{n}{r} - \frac{5}{2}\sqrt{\alpha}n \leqslant |V_1| \leqslant \frac{n}{r} + \frac{5}{2}\sqrt{\alpha}n.$$

In a similar way we get the bounds on the sizes of the other classes. □

Lemma 2.3. *The graph G contains r vertices $x_1 \in V_1, \ldots, x_r \in V_r$ which form a K_r, and for every i*

$$\deg(x_i) \geqslant n - |V_i| - 5r\alpha n.$$

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Proof. Let $V_i^c := V(G) \setminus V_i$. We call a vertex $v_i \in V_i$ *small* if $|N(v_i) \cap V_i^c| < |V_i^c| - 5r\alpha n$ and *big* otherwise. For $1 \leqslant i \leqslant r$, let B_i denote the set of big vertices inside class V_i. There are at most

$$\frac{4t}{5r\alpha n} = \frac{4}{5r}n$$

small vertices in total as otherwise (2.4) is violated. Thus in each class there are at least $n/10r$ big vertices, *i.e.* $|B_i| \geqslant n/10r$. The number of missing edges between the sets $B_1, \ldots, B_r$ is at most $2t < \frac{1}{100r^2}n^2$. Thus, using Theorem 2.3, we can find a K_r with one vertex from each B_i. □

Lemma 2.4. *There exists a vertex partition $V(G) = X_1 \cup \cdots \cup X_r \cup X$ such that the X_i are independent sets, $|X| \leqslant 5r^2\alpha n$ and*

$$\frac{n}{r} - 3\sqrt{\alpha}n \leqslant |X_i| \leqslant \frac{n}{r} + 3r\sqrt{\alpha}n$$

for all $1 \leqslant i \leqslant r$.

Proof. By Lemma 2.3 we can find vertices $x_1, \ldots, x_r$ forming a K_r and having $\deg(x_i) \geqslant n - |V_i| - 5r\alpha n$. Define X_i to be the common neighbourhood of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r$ and $X = V(G) \setminus (X_1 \cup \cdots \cup X_r)$. Since G is K_{r+1}-free, the X_i are independent sets. Now we bound the size of X_i using the bounds on the sets V_i. Since every x_j has at most $|V_j| + 5r\alpha n$ non-neighbours, we get

$$|X_i| \geqslant n - \sum_{\substack{1 \leqslant j \leqslant r \\ j \neq i}} (|V_j| + 5r\alpha n) \geqslant |V_i| - 5r^2\alpha n \geqslant \frac{n}{r} - 3\sqrt{\alpha}n \tag{2.7}$$

and

$$\sum_{i=1}^{r} \deg(x_i) \geqslant n(r-1) - 5r^2\alpha n. \tag{2.8}$$

A vertex $v \in V(G)$ cannot be incident to all of the vertices $x_1, \ldots, x_r$, because G is K_{r+1}-free. Further, every vertex from X is not incident to at least two of the vertices $x_1, \ldots, x_r$. Thus

$$\sum_{i=1}^{r} \deg(x_i) \leqslant n(r-1) - |X|. \tag{2.9}$$

Combining (2.8) with (2.9), we conclude that

$$|X| \leqslant 5r^2\alpha n.$$

For the upper bound on the sizes of the sets X_i we get

$$|X_i| \leqslant n - \sum_{\substack{1 \leqslant j \leqslant r \\ j \neq i}} |X_j| \leqslant n - \frac{r-1}{r}n + 3r\sqrt{\alpha}n = \frac{n}{r} + 3r\sqrt{\alpha}n. \tag{2.10}$$

□

We now bound the number of non-edges between $X_1, \ldots, X_r$.

Lemma 2.5. *We have*

$$\sum_{1 \leqslant i < j \leqslant r} e^c(X_i, X_j) \leqslant t + e(X, X^c) + |X|^2 - \left(1 - \frac{1}{r}\right)n|X| + r.$$

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Proof.

$$\begin{aligned}\frac{n^2}{2}\left(1-\frac{1}{r}\right)-\frac{r}{2}-t\leqslant e(G)&\\ &= e(X,X^c)+e(X)+\sum_{1\leqslant i<j\leqslant r} e(X_i,X_j)\\ &\leqslant e(X,X^c)+\frac{|X|^2}{2}+\left(1-\frac{1}{r}\right)\left(\frac{(n-|X|)^2}{2}\right)-\sum_{1\leqslant i<j\leqslant r} e^c(X_i,X_j). \quad (2.11)\end{aligned}$$

This gives the statement of the lemma. □

Let

$$\bar{X}=\left\{v\in X \mid \deg_{X_1\cup\cdots\cup X_r}(v)\geqslant \frac{r-2}{r}n+3\alpha^{1/3}n\right\}\quad\text{and}\quad \hat{X}:=X\setminus\bar{X}.$$

Let $d\in[0,1]$ such that $|\bar{X}|=d|X|$. Further, let $k\in[0,5r^2]$ such that $|X|=k\alpha n$. Now we shall further develop the upper bound from Lemma 2.5.

Lemma 2.6. *We have*

$$\sum_{1\leqslant i<j\leqslant r} e^c(X_i,X_j)\leqslant 20r^2\alpha^{4/3}n^2+\left(1-(1-d)\frac{1}{r}k\right)\alpha n^2.$$

Proof. By Lemma 2.5,

$$\begin{aligned}\sum_{1\leqslant i<j\leqslant r} e^c(X_i,X_j)&\leqslant t+e(X,X^c)+|X|^2-\left(1-\frac{1}{r}\right)n|X|+r\\
&\leqslant t+d|X|\Delta(G)+(1-d)|X|\left(\frac{r-2}{r}n+3\alpha^{1/3}n\right)+|X|^2-\left(1-\frac{1}{r}\right)n|X|+r\\
&\leqslant t+d|X|\left(n\frac{r-1}{r}+\frac{5}{2}\sqrt{\alpha}n\right)+(1-d)|X|\left(\frac{r-2}{r}n+3\alpha^{1/3}n\right)\\
&\qquad+|X|^2-\left(1-\frac{1}{r}\right)n|X|+r\\
&\leqslant \frac{5}{2}d|X|\sqrt{\alpha}n+3(1-d)|X|\alpha^{1/3}n+|X|^2+t+n|X|\frac{d-1}{r}+r\\
&\leqslant \frac{5}{2}k\alpha^{3/2}n^2+3k\alpha^{4/3}n^2+|X|^2+\left(1-(1-d)\frac{1}{r}k\right)\alpha n^2+r\\
&\leqslant \frac{25}{2}r^2\alpha^{3/2}n^2+15r^2\alpha^{4/3}n^2+25r^4\alpha^2n^2+\left(1-(1-d)\frac{1}{r}k\right)\alpha n^2+r\\
&\leqslant 20r^2\alpha^{4/3}n^2+\left(1-(1-d)\frac{1}{r}k\right)\alpha n^2. \quad (2.12)\end{aligned}$$

□

Let

$$C:=20r^2\alpha^{4/3}+\left(1-(1-d)\frac{1}{r}k\right)\alpha. \quad (2.13)$$

For every vertex $u \in X$ there is no K_r in $N_{X_1}(u) \cup \cdots \cup N_{X_r}(u)$. Thus, by applying Theorem 2.3 and Lemma 2.6, we get

$$\min_{i \neq j} |N_{X_i}(u)||N_{X_j}(u)| \leqslant \sum_{1 \leqslant i<j \leqslant r} e^c(X_i, X_j) \leqslant Cn^2. \tag{2.14}$$

Bound (2.14) implies in particular that every vertex $u \in X$ has degree at most $\sqrt{C}n$ to one of the sets $X_1, \ldots, X_r$, that is,

$$\min_i |N_{X_i}(u)| \leqslant \sqrt{C}n. \tag{2.15}$$

Therefore we can partition $\hat{X} = A_1 \cup \cdots \cup A_r$ such that every vertex $u \in A_i$ has at most $\sqrt{C}n$ neighbours in X_i.

By the following calculation, for every vertex $u \in \bar{X}$ the second smallest neighbourhood to the sets X_i has size at least $\alpha^{1/3}n$:

$$\min_{i \neq j} |N_{X_i}(u)| + |N_{X_j}(u)| \geqslant \frac{r-2}{r}n + 3\alpha^{1/3}n - (r-2)\left(\frac{n}{r} + 3r\sqrt{\alpha}n\right) \geqslant 2\alpha^{1/3}n, \tag{2.16}$$

where we used the definition of $\bar{X}$ and Lemma 2.4. Combining the lower bound on the second smallest neighbourhood with (2.14), we can conclude that for every $u \in \bar{X}$

$$\min_i |N_{X_i}(u)| \leqslant \frac{C}{\alpha^{1/3}}n. \tag{2.17}$$

Hence we can partition $\bar{X} = B_1 \cup \cdots \cup B_r$ such that every vertex $u \in B_i$ has at most $C\alpha^{-1/3}n$ neighbours in X_i. Consider the partition $A_1 \cup B_1 \cup X_1, A_2 \cup B_2 \cup X_2, \ldots, A_r \cup B_r \cup X_r$. By removing all edges inside the classes, we end up with an r-partite graph. We have to remove at most

$$\begin{aligned}
&e(X) + d|X|\frac{C}{\alpha^{1/3}}n + (1-d)|X|\sqrt{C}n \\
&\quad \leqslant 6r^2\alpha^{5/3}n^2 + (1-d)k\sqrt{C}\alpha n^2 \\
&\quad \leqslant 6r^2\alpha^{5/3}n^2 + (1-d)k\left(\sqrt{20r^2\alpha^{4/3}} + \sqrt{\left(1-(1-d)\frac{1}{r}k\right)\alpha}\right)\alpha n^2 \\
&\quad \leqslant 6r^2\alpha^{5/3}n^2 + 5r^2\sqrt{20r^2\alpha^{4/3}}\alpha n^2 + (1-d)k\sqrt{\left(1-(1-d)\frac{1}{r}k\right)\alpha}\alpha n^2 \\
&\quad \leqslant 6r^2\alpha^{5/3}n^2 + 5\sqrt{20}r^3\alpha^{5/3}n^2 + \frac{2r}{3\sqrt{3}}\alpha^{3/2}n^2 \\
&\quad \leqslant \left(\frac{2r}{3\sqrt{3}} + 30r^3\alpha^{1/6}\right)\alpha^{3/2}n^2
\end{aligned} \tag{2.18}$$

edges. We have used (2.15), (2.17) and the fact that

$$(1-d)k\sqrt{1-(1-d)\frac{k}{r}} \leqslant \frac{2r}{3\sqrt{3}},$$

which can be seen by setting $z = (1-d)k$ and finding the maximum of $f(z) := z\sqrt{1-z/r}$, which is obtained at $z = 2r/3$.

3. Sharpness example

In this section we will prove Theorem 1.4, *i.e.* that the leading term from Theorem 1.3 is best possible.

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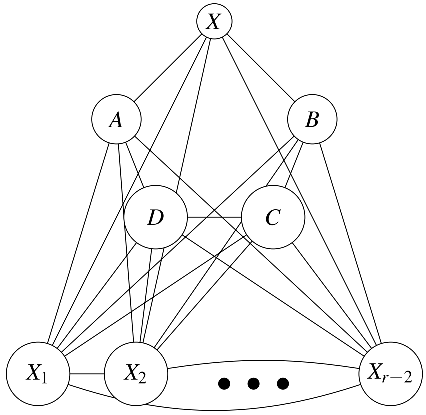

Figure 1. The graph G.

Proof of Theorem 1.4. Let G be the graph with vertex set $V(G) = A \cup X \cup B \cup C \cup D \cup X_1 \cdots \cup X_{r-2}$, where all classes $A, X, B, C, D, X_1, \ldots, X_{r-2}$ form independent sets; A, X, B, C, D form a complete blow-up of a C_5, where the classes are named in cyclic order; and for each $1 \leqslant i \leqslant r-2$, every vertex from X_i is incident to all vertices from $V(G) \setminus X_i$. See Figure 1 for an illustration of G.

The sizes of the classes are

$$|X| = \frac{2r}{3}\alpha n, \quad |A| = |B| = \sqrt{\frac{\alpha}{3}}n, \quad |C| = |D| = \frac{1-(2r/3)\alpha}{r}n - \sqrt{\frac{\alpha}{3}}n, \quad |X_i| = \frac{1-(2r/3)\alpha}{r}n.$$

The smallest class is X and the second smallest are A and B. By deleting all edges between X and A ($|X||A| = (2r/(3\sqrt{3}))\alpha^{3/2}n^2$), we get an r-partite graph. Since the classes A and X are the two smallest class sizes, the smallest canonical cut is of size $(2r/(3\sqrt{3}))\alpha^{3/2}n^2$. A result by Erdős, Győri and Simonovits [7, Theorem 7] states that there is a canonical 'edge deletion' achieving the minimum of $D_r(G)$. Hence

$$D_r(G) \geqslant \frac{2r}{3\sqrt{3}}\alpha^{3/2}n^2.$$

Let us now count the number of edges of G. The number of edges incident to X is

$$\begin{aligned} e(X, X^c) &= \left(\frac{2r}{3}\alpha\right)\left(2\sqrt{\frac{\alpha}{3}}\right)n^2 + \left(\frac{2r}{3}\alpha\right)\left(\frac{1-(2r/3)\alpha}{r}(r-2)\right)n^2 \\ &= \left(\frac{2}{3}(r-2)\alpha + \frac{4r}{3\sqrt{3}}\alpha^{3/2} - \frac{4r(r-2)}{9}\alpha^2\right)n^2. \end{aligned} \tag{3.1}$$

Using that $|A| + |C| = |B| + |D| = |X_1|$, we have that the number of edges inside $A \cup B \cup C \cup D \cup X_1 \cup \cdots \cup X_{r-2}$ is

$$\begin{aligned} e(X^c) &= |X_1|^2\binom{r}{2} - |A||B| \\ &= \left(\frac{1-(2r/3)\alpha}{r}n\right)^2\binom{r}{2} - \frac{1}{3}\alpha n^2 \\ &= \frac{1}{r^2}\binom{r}{2}n^2 - \frac{4r}{3}\frac{1}{r^2}\alpha\binom{r}{2}n^2 + \frac{4}{9}\alpha^2\binom{r}{2}n^2 - \frac{1}{3}\alpha n^2 \\ &= \left(1-\frac{1}{r}\right)\frac{n^2}{2} - \frac{2}{3}(r-1)\alpha n^2 - \frac{1}{3}\alpha n^2 + \frac{4}{9}\alpha^2\binom{r}{2}n^2. \end{aligned} \tag{3.2}$$

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Thus the number of edges of G is

$$\begin{aligned} e(G) &= e(X^c) + e(X, X^c) \\ &= \left(1 - \frac{1}{r}\right)\frac{n^2}{2} - \alpha n^2 + \frac{4r}{3\sqrt{3}}\alpha^{3/2}n^2 - \frac{2r(r-3)}{9}\alpha^2 n^2 \\ &\geqslant \mathrm{ex}(n, K_{r+1}) - \alpha n^2 + \frac{4r}{3\sqrt{3}}\alpha^{3/2}n^2 - \frac{2r(r-3)}{9}\alpha^2 n^2, \end{aligned} \tag{3.3}$$

where we applied Turán's theorem in the last step. □

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References

[1] Amin, K., Faudree, J., Gould, R. J. and Sidorowicz, E. (2013) On the non-$(p-1)$-partite K_p-free graphs. *Discuss. Math. Graph Theory* **33** 9–23.
[2] Andrásfai, B., Erdős, P. and Sós, V. T. (1974) On the connection between chromatic number, maximal clique and minimal degree of a graph. *Discrete Math.* **8** 205–218.
[3] Balogh, J., Morris, R., Samotij, W. and Warnke, L. (2016) The typical structure of sparse K_{r+1}-free graphs. *Trans. Amer. Math. Soc.* **368** 6439–6485.
[4] Brouwer, A. E. (1981) *Some Lotto Numbers From an Extension of Turán's Theorem*, Vol. 152 of Afdeling Zuivere Wiskunde [Department of Pure Mathematics]. Mathematisch Centrum, Amsterdam.
[5] Erdős, P. (1970) On the graph theorem of Turán. *Mat. Lapok* **21** 249–251.
[6] Erdős, P., Faudree, R., Pach, J. and Spencer, J. (1988) How to make a graph bipartite. *J. Combin. Theory Ser. B* **45** 86–98.
[7] Erdős, P., Győri, E. and Simonovits, M. (1992) How many edges should be deleted to make a triangle-free graph bipartite? In *Sets, Graphs and Numbers (Budapest, 1991)*, Vol. 60 of Colloquia Mathematica Societatis János Bolyai, pp. 239–263. North-Holland.
[8] Erdős, P. and Simonovits, M. (1966) A limit theorem in graph theory. *Studia Sci. Math. Hungar* **1** 51–57.
[9] Füredi, Z. (2015) A proof of the stability of extremal graphs, Simonovits' stability from Szemerédi's regularity. *J. Combin. Theory Ser. B* **115** 66–71.
[10] Hanson, D. and Toft, B. (1991) k-saturated graphs of chromatic number at least k. *Ars Combin.* **31** 159–164.
[11] Hu, P., Lidický, B., Martins, T., Norin, S. and Volec, J. Large multipartite subgraphs in H-free graphs. Manuscript.
[12] Kang, M. and Pikhurko, O. (2005) Maximum K_{r+1}-free graphs which are not r-partite. *Mat. Stud.* **24** 12–20.
[13] Korándi, D., Roberts, A. and Scott, A. (2020) Exact stability for Turán's theorem. arXiv:2004.10685
[14] Nikiforov, V. (2011) Some new results in extremal graph theory. In *Surveys in Combinatorics 2011*, Vol. 392 of London Mathematical Society Lecture Note Series, pp. 141–181. Cambridge University Press.
[15] Roberts, A. and Scott, A. (2018) Stability results for graphs with a critical edge. arXiv:1610.08389. https://doi.org/10.1016/j.ejc.2018.07.004
[16] Sudakov, B. (2007) Making a K_4-free graph bipartite. *Combinatorica* **27** 509–518.
[17] Turán, P. (1941) Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok* **48** 436–452.
[18] Tyomkyn, M. and Uzzell, A. J. (2015) Strong Turán stability. *Electron. J. Combin.* **22** P3-9.

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