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The Geometry of Weakly Self-dual Kähler Surfaces

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Abstract. We study Kähler surfaces with harmonic anti-self-dual Weyl tensor. We provide an explicit local description, which we use to obtain the complete classification in the compact case. We give new examples of extremal Kähler metrics, including Kähler–Einstein metrics and conformally Einstein Kähler metrics. We also extend some of our results to almost Kähler 4-manifolds, providing new examples of Ricci-flat almost Kähler metrics which are not Kähler.

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Introduction

Self-dual Kähler surfaces have been considered in several recent works, in particular in a paper by R. Bryant [13], where self-dual Kähler surfaces appear as the four-dimensional case of a comprehensive study of Bochner-flat Kähler manifolds in all dimensions, and in a paper by two of the authors [8], where a generic equivalence has been established between self-dual Kähler surfaces and self-dual Hermitian Einstein metrics and where an explicit local description of the latter is provided.

Whereas self-dual surfaces are easily proved to be extremal, i.e., admitting a Hamiltonian Killing vector field whose momentum map is the scalar curvature, it was an a priori unexpected fact, independently discovered in the above works, that they actually admit a second Hamiltonian Killing vector field; moreover, a crucial observation of R. Bryant [13], is that the momentum map of the latter is the Pfaffian of the normalized Ricci form. Since these Killing vector fields commute, this also provides a link with the work of H. Pedersen and the second author [17], where an explicit local classification of self-dual Einstein metrics with two commuting Killing vector fields is obtained, without the hypothesis that they are Hermitian.

In this paper, we show that we can relax the assumption of self-duality, and establish the same bi-Hamiltonian structure for *weakly self-dual* Kähler surfaces, i.e., Kähler surfaces whose anti-self-dual Weyl tensor W^- is harmonic; this fact has its origins in the basic *Matsumoto–Tanno identity* for such surfaces, recently rediscovered by W. Jelonek [32], and leads to a surprisingly simple explicit expression, generalizing an expression found by Bryant in the self-dual case. On the other hand, we also observe that in Calabi's family of extremal Kähler metrics on the first Hirzebruch surface F_1, there is a unique (and completely explicit) weakly self-dual metric up to homothety. This is in contrast to the self-dual case, where the compact (smooth) examples are all locally symmetric [20].

Our results concerning weakly self-dual Kähler surfaces may be summarized as follows (definitions and more precise statements are given in the body of the paper).

THEOREM. *Let* (M, g, J, ω) *be a weakly self-dual Kähler surface. Then* (g, J) *is a bi-extremal Kähler metric in the sense that the scalar curvature and the Pfaffian of the normalized Ricci form of* (g, J) *are Poisson-commuting momentum maps for Hamiltonian Killing vector fields* K_1 *and* K_2 *respectively. Furthermore, on each connected component of* M *one of the following holds.*

(i) K_1 *and* K_2 *are linearly independent on a dense open set. Then* (g, J, ω) *has the explicit local form* (41–45)*, depending on an arbitrary polynomial of degree* 4 *and an arbitrary constant which is zero if and onlyif g is self-dual, cf. Theorem* 2.
(ii) K_1 *is nonvanishing on a dense open set, but* $K_1 \wedge K_2$ *is identically zero. Then* (g, J, ω) *is locally of cohomogeneity one and is given explicitly by the Calabi construction, cf. Theorem* 4.
(iii) K_1 *and* K_2 *vanish identically. Then g has parallel Ricci curvature* (*hence is either Kähler–Einstein or locally a Kähler product of two Riemann surfaces of constant curvatures*).

If (M, g, J, ω) *is compact and connected, then it necessarily belongs to case* (ii) *or* (iii) *above, and in case* (ii) (M, g, J, ω) *is isomorphic to the weakly self-dual Calabi extremal metric on* F_1 (*see Theorem* 5).

The path to proving this result touches upon various important themes in Kähler geometry, and along the way we introduce further ideas, results and examples. A key aspect of our approach is to study weakly self-dual Kähler surfaces within a more general setting. The Matsumoto–Tanno identity for weakly self-dual Kähler surfaces is equivalent to the fact that the primitive part ρ_0 of the Ricci form ρ of (g, J) satisfies an overdetermined linear differential equation. On the open set where ρ_0 is nonvanishing, the equation means that ρ_0 defines a conformally Kähler Hermitian structure I inducing the opposite orientation to J. Many of the properties of weakly self-dual Kähler surfaces are simple consequences of the fact that the ρ is a closed J-invariant 2-form, whose primitive part satisfies this equation. In particular,

we prove in Proposition 2 that two of the algebraic invariants (essentially the trace and the Pfaffian) of *any* such 2-form φ are Poisson-commuting momentum maps for Hamiltonian Killing vector fields. Therefore, throughout the work, we develop the theory of Kähler surfaces with such 'Hamiltonian' 2-forms φ, which include other interesting examples in addition to weakly self-dual Kähler surfaces.

First of all we study the generic case that the Hamiltonian Killing vector fields are linearly independent. This means that the Kähler structure (g, J, ω) is toric, and in Theorem 1 we characterize the class of toric Kähler structures arising in this way from Hamiltonian 2-forms. Whereas toric Kähler surfaces in general depend essentially on an arbitrary function of two variables [1, 29], our toric surfaces, which we call 'ortho-toric', have an explicit form—given in Proposition 8—depending only on two arbitrary functions of one variable. This has the practical advantage that curvature conditions lead to ordinary differential equations for these functions. In particular, we are able to obtain explicitly all of the extremal toric Kähler structures in our class, including some new examples of Kähler metrics which are conformally Einstein, but neither self-dual nor anti-self-dual, and also some explicit Kähler–Einstein metrics. The weakly self-dual metrics in this family are classified in Theorem 2.

The case that the Hamiltonian Killing vector fields associated to a Hamiltonian 2-form φ are linearly dependent, but not both zero, is closely related to the Calabi construction of Kähler metrics on line bundles over a Riemann surface [14]. We provide, in Theorem 3, a geometric local characterization of these Kähler metrics: they are the Kähler metrics (g, J), with a Killing vector field K such that the almost Hermitian pair (g, I), where I is equal to J on span of $\{K, JK\}$ but $-J$ on the orthogonal distribution, is conformally Kähler. Over a fixed Riemann surface Σ, the general form of these Kähler metrics 'of Calabi type' again depends essentially only on functions of one variable from which it is easy to recover the Calabi extremal metrics. We present these in Proposition 14: the Riemann surface Σ has constant curvature, and the metrics have local cohomogeneity one under $U(2)$, $U(1,1)$ or a central extension of the Heisenberg group Nil. The weakly self-dual Calabi extremal metrics are classified in Theorem 4: there is a four parameter family, one of which is globally defined on the first Hirzebruch surface. The existence of such a metric has been independently observed by Jelonek [33].

The proof of the above theorem is completed by classifying the compact weakly self-dual Kähler surfaces. A partial classification for real analytic Kähler surfaces has been recently obtained by Jelonek [32], but we improve it in two respects: first, as speculated by Jelonek, we are able to remove the assumption of real-analyticity; second we prove that the only (non-product non-Kähler–Einstein) weakly self-dual Kähler metric on a ruled surface, is the Calabi extremal example on the first Hirzebruch surface F_1.

The paper is organized as follows. In the first section, we establish some basic facts concerning weakly self-dual Kähler surfaces; in particular, using general properties of Hamiltonian 2-forms, we show that weakly self-dual Kähler surfaces are bi-extremal

and that their anti-self-dual Weyl tensor is degenerate (some facts proved in this section also appear in Jelonek's paper [32]). We also present the rough classification, Proposition 6, that allows us to deduce the above theorem from Theorems 2, 4 and 5. The generic, toric case, and Theorem 2, are described in the second section, whereas the third section treats the Calabi examples and Theorem 4. The classification of compact weakly self-dual Kähler surfaces is given in Section 4.

In the final section we show that some of our results generalize to the class of almost Kähler 4-manifolds (M, g, J, ω) whose Ricci tensor is J-invariant. We first observe that the Calabi construction gives rise to new (local) examples of self-dual, Ricci-flat almost Kähler 4-manifolds (see Example 2), which provide further local counterexamples (cf. [9, 44]) to the still open Goldberg conjecture which states that a *compact* Einstein almost Kähler manifold must be Kähler–Einstein. Next we consider compact almost Kähler 4-manifolds with J-invariant Ricci tensor which are weakly self-dual, i.e., have harmonic anti-self-dual Weyl tensor. We show in Theorem 6 that weak-self-duality has strong consequences for the integrability of the corresponding almost complex structure, providing another interesting link with the Goldberg conjecture. As an application of this global result, we prove that a compact almost Kähler 4-manifold has constant sectional curvatures on the Lagrangian 2-planes if and only if it is a self-dual Kähler surface (see Corollary 1).

1. Weakly Self-dual Kähler Surfaces

1.1. THE MATSUMOTO–TANNO IDENTITY

A *Kähler surface* (M, g, J) is an oriented Riemannian four-dimensional manifold equipped with a self-dual complex structure J, such that $\nabla J = 0$, where ∇ denotes the Levi-Civita connection of g. The Kähler form is the J-invariant self-dual 2-form $\omega(\cdot, \cdot) = \langle J\cdot, \cdot\rangle$; ω is closed and (M, ω) is a symplectic manifold.

The vector bundle $\Lambda^+ M$ of self-dual 2-forms is the orthogonal direct sum of the trivial bundle generated by ω and of the bundle of J-anti-invariant 2-forms, whereas the bundle $\Lambda^- M$ coincides with the bundle of *primitive* – or *trace-free* – J-invariant 2-forms.

We denote by R, Ric, Ric_0, Scal, $W = W^+ + W^-$, the curvature, the Ricci tensor, the trace-free part of Ric, the scalar curvature (i.e., the trace of Ric), and the Weyl tensor, expressed as the sum of its $\pm$-self-dual components $W^{\pm}$.

The *Ricci form*, ρ, of a Kähler surface is the J-invariant 2-form defined by $\rho(\cdot, \cdot) = \mathrm{Ric}(J\cdot, \cdot)$; ρ is closed and, up to a factor 2π, is a representative of the first Chern class of (M, J) in de Rham cohomology; the trace-free part of ρ is denoted by ρ_0.

DEFINITION 1. A Kähler surface (M, g, J) is weakly self-dual if its anti-self-dual Weyl tensor W^- is harmonic, i.e., satisfies $\delta^g W^- = 0$, where the codifferential δ^g acts on W^- as on a 2-form with values in $\Lambda^- M$.

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Because of the Bianchi identity, the weak self-duality condition can also be defined in terms of the Ricci tensor; from this point of view, it can also be considered as a *weak Einstein condition*, in the sense that every Einstein metric is weakly self-dual. The link is provided by the *Cotton–York* tensor $C_{X,Y}(Z)$ of the Riemannian metric g. Recall that the Cotton–York tensor is defined by

$$C_{X,Y}(Z) = -(\nabla_X h)(Y, Z) + (\nabla_Y h)(X, Z),$$

where $h = \frac{1}{2}\mathrm{Ric}_0 + \frac{1}{24}\mathrm{Scal}\, g$ denotes the *normalized Ricci tensor* of g.

The normalized Ricci tensor is the form in which the Ricci tensor appears in the well-known decomposition of the Riemannian curvature $R = h \wedge \mathrm{Id} + W$ (cf., e.g., [12]); therefore, via the (differential) Bianchi identity, the $\pm$-self-dual components, $C^{\pm}$, of the Cotton–York tensor are linked to the $\pm$-self-dual components of the Weyl tensor by $\delta W^{+} = C^{+}$ and $\delta W^{-} = C^{-}$.

Definition 1 can thus be rephrased as follows:

DEFINITION 2. A Kähler surface is weakly self-dual if its Cotton–York tensor is self-dual.

The normalized Ricci tensor plays a natural role throughout this work. For this reason, to simplify formulae, we write $s = \frac{1}{6}\mathrm{Scal}$ for the *normalized scalar curvature*, which is the trace of h.

LEMMA 1 ([43, 32]). *For any Kähler surface* (M, g, J, ω) *we have*

$$\nabla_X \rho_0 = -2C^{-}(JX) - \tfrac{1}{2}\mathrm{d}s(X)\omega + \tfrac{1}{2}(\mathrm{d}s \wedge JX^{\flat} - J\,\mathrm{d}s \wedge X^{\flat}). \tag{1}$$

In particular, the Kähler surface (M, g, J, ω) *is weakly self-dual if and only if the following Matsumoto–Tanno identity*

$$\nabla_X \rho_0 = -\tfrac{1}{2}\mathrm{d}s(X)\omega + \tfrac{1}{2}(\mathrm{d}s \wedge JX^{\flat} - J\,\mathrm{d}s \wedge X^{\flat}) \tag{2}$$

is satisfied (*for any vector field* X).

Proof. Since $2h = \mathrm{Ric} - sg$, the Cotton–York tensor of any Kähler surface can be written as follows:

$$\begin{aligned} 2C_{X,Y}(Z) &= -2(\nabla_X h)(Y, Z) + 2(\nabla_Y h)(X, Z) \\ &= -(\nabla_X \mathrm{Ric})(Y, Z) + (\nabla_Y \mathrm{Ric})(X, Z) + \mathrm{d}s(X)\langle Y, Z\rangle - \mathrm{d}s(Y)\langle X, Z\rangle \\ &= -(\nabla_X \rho)(Y, JZ) + (\nabla_Y \rho)(X, JZ) + ds(X)\langle Y, Z\rangle - \mathrm{d}s(Y)\langle X, Z\rangle \\ &= (\nabla_{JZ}\rho)(X, Y) + \mathrm{d}s(X)\langle Y, Z\rangle - \mathrm{d}s(Y)\langle X, Z\rangle. \end{aligned}$$

(In order to obtain the last line from the preceding one, we use the fact that $\mathrm{d}\rho = 0$.) We then have

$$(\nabla_Z \rho)(X, Y) = -2C_{X,Y}(JZ) - \mathrm{d}s(X)\langle JY, Z\rangle + \mathrm{d}s(Y)\langle JX, Z\rangle,$$

or, equivalently,

$$\begin{aligned} (\nabla_Z \rho_0)(X, Y) = -2C_{X,Y}(JZ) &- \tfrac{3}{2}\mathrm{d}s(Z)\langle JX, Y\rangle - \\ &- \mathrm{d}s(X)\langle JY, Z\rangle + \mathrm{d}s(Y)\langle JX, Z\rangle. \end{aligned} \tag{3}$$

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The anti-self-dual component of (3) gives identity (1); the last statement follows immediately. □

1.2. TWISTOR 2-FORMS AND KÄHLER METRICS

On any Riemannian manifold (M, g), if Φ is an anti-self-dual 2-form, i.e., a section of the vector bundle $\Lambda^- M$, then $\nabla\Phi$ is a section of the vector bundle $T^*M \otimes \Lambda^- M$. This bundle has an orthogonal direct sum decomposition

$$T^*M \otimes \Lambda^- M = V^{(0)}M \oplus V^{(1)}M, \tag{4}$$

in accordance with the algebraic decomposition $\mathbb{R}^4 \otimes \Lambda^-\mathbb{R}^4 = \mathbb{R}^4 \oplus (\Sigma_+ \otimes \Sigma^3_-)$ into irreducible sub-representations under the action of the orthogonal group. In (4), $V^{(0)}M$ corresponds to the factor $\mathbb{R}^4$, hence is isomorphic to T^*M, whereas $V^{(1)}M$ corresponds to the factor $\Sigma_+ \otimes \Sigma^3_-$, so it is the kernel of the natural contraction $T^*M \otimes \Lambda^- M \to T^*M$. The projection of the connection to $V^{(0)}M$ may be identified with the exterior derivative or divergence on anti-self-dual 2-forms (which are related by the Hodge $*$ operator), while the projection to $V^{(1)}M$ is often called the *twistor* or *Penrose operator* on anti-self-dual 2-forms.

DEFINITION 3. An anti-self-dual 2-form Φ is called a twistor 2-form if $\nabla\Phi$ is a section of the sub-bundle $V^{(0)}M$ of $T^*M \otimes \Lambda^- M$.

Any nonvanishing section Φ of $\Lambda^- M$ can be written uniquely as $\Phi = \lambda\,\omega_I$, where $\lambda = |\Phi|/\sqrt{2}$ is a positive function and ω_I is the Kähler form of an anti-self-dual almost-complex structure I on (M, g).

LEMMA 2 ([45]). *If $\Phi = \lambda\,\omega_I$ is a nonvanishing section of $\Lambda^- M$, then Φ is a twistor 2-form if and only if the almost-Hermitian pair $(\bar{g} = \lambda^{-2}g, I)$ is Kähler, with Kähler form $\bar{\omega} = \lambda^{-2}\omega_I = \lambda^{-3}\Phi$.*

Proof. Φ is a twistor 2-form if and only if there exists a 1-form γ such that, at each point of M, $\nabla\Phi = \sum_{i=1}^3 I_i\gamma \otimes \omega_i$, where the triple $(I_1, I_2, I_3 = I_1I_2)$ is any positively oriented, orthonormal frame of (anti-self-dual) almost-complex structures at that point, and ω_i is the Kähler form of I_i.

If $\Phi = \lambda\,\omega_I$ is a nonvanishing twistor 2-form on M, then, by choosing $I_1 = I$, we have

$$\lambda\nabla I = (I\gamma - \mathrm{d}\lambda) \otimes I + I_2\gamma \otimes I_2 + I_3\gamma \otimes I_3.$$

Since the norm of I is constant, this equality implies that $I\gamma = \mathrm{d}\lambda$. Now observe that the equation

$$\lambda\nabla I = I_3\mathrm{d}\lambda \otimes I_2 - I_2\mathrm{d}\lambda \otimes I_3 \tag{5}$$

is equivalent to I being parallel with respect to the Levi-Civita connection of $\bar{g} = \lambda^{-2}g$. Hence, if Φ is a twistor 2-form $(\bar{g}, I)$ is Kähler. Conversely, if $(\bar{g}, I)$ is Kähler then (5) holds, from which it follows that $\nabla\Phi = \nabla(\lambda\,\omega_I)$ is the section of $V^{(0)}M$ corresponding to the 1-form $\gamma = -I\mathrm{d}\lambda$. □

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If (M, g, J, ω) is a Kähler surface then a 2-form is anti-self-dual if and only if it is trace-free and J-invariant, and there is the following reformulation of Definition 3.

LEMMA 3. *Let (M, g, J, ω) be a Kähler surface and let φ_0 be an anti-self-dual 2-form. Then φ_0 is a twistor 2-form if and only if there is a 1-form β such that*

$$\nabla_X \varphi_0 = -\beta(X)\omega + \beta \wedge JX^\flat - J\beta \wedge X^\flat \tag{6}$$

for any vector field X.

Proof. It is easily checked that the right-hand side of (6) is (the contraction with X of) the general form of a section of $V^{(0)}M \cong T^*M$. □

By the contracted Bianchi identity, C^- is a section of $V^{(1)}M$, and so the last statement of Lemma 1 can be rephrased as follows.

LEMMA 4. *A Kähler surface is weakly self-dual if and only if the trace-free part ρ_0 of the Ricci form ρ is a twistor 2-form.*

Together with Lemma 2, this implies:

PROPOSITION 1. *On the open set M_0 where $\rho_0 = \lambda\,\omega_I$ does not vanish, a Kähler surface (M, g, J, ω) is weakly self-dual if and only if the pair $(\bar{g} = \lambda^{-2}g, I)$ is Kähler.*

In particular it follows that on M_0, the self-dual Weyl tensor of $\bar{g}$, with the orientation induced by I, is degenerate, and equal to $\frac{3}{4}\bar{s}\,\bar{\omega} \otimes_0 \bar{\omega}$, where $\bar{s}$ is the (normalized) scalar curvature of $\bar{g}$, $\bar{\omega} = \lambda^{-2}\omega_I$, and $\bar{\omega} \otimes_0 \bar{\omega}$ stands for the traceless part of $\bar{\omega} \otimes \bar{\omega}$ viewed as an endomorphism of $\Lambda^+ M$. By the conformal covariance of the Weyl tensor, it follows that on M_0, the *anti-self-dual* Weyl tensor of g (using the orientation induced by J) is given by

$$W^- = \frac{3}{4}\kappa\,\omega_I \otimes_0 \omega_I, \tag{7}$$

where

$$\kappa = \bar{s}\lambda^{-2} \tag{8}$$

is the conformal scalar curvature of the Hermitian pair (g, I), which is related to the Riemannian scalar curvature of g by

$$\kappa - s = \delta\theta - |\theta|^2; \tag{9}$$

here θ denotes the *Lee form* of the pair (g, I), defined by $\mathrm{d}\omega_I = -2\theta \wedge \omega_I$, see [21, 7]. Note that we have normalized the conformal scalar curvature by a factor $\frac{1}{6}$ to be consistent with our normalization of the scalar curvature s.

Notice that this only uses the fact that M_0 admits a nonvanishing twistor 2-form, namely ρ_0. On the other hand, ρ_0 is not anarbitrary twistor 2-form: by Lemma 1, the 1-form β defined by $\nabla\rho_0$ using (6), is equal to $\frac{1}{2}\mathrm{d}s$, and so is exact. This fact, which is equivalent to the fact that the Ricci form $\rho = \rho_0 + \frac{3}{2}s\omega$ is closed, will be exploited in the next subsection.

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1.3. HAMILTONIAN 2-FORMS

DEFINITION 4. A Hamiltonian 2-form on a Kähler surface (M, g, J, ω) is a closed J-invariant 2-form φ whose trace-free (i.e., anti-self-dual) part φ_0 is a twistor 2-form.

On a weakly self-dual Kähler surface the Ricci form ρ is Hamiltonian. In general, Hamiltonian 2-forms are characterized by an analogue of the Matsumoto–Tanno identity (2).

LEMMA 5. *Let (M, g, J, ω) be a Kähler surface. Then a J-invariant 2-form $\varphi = \varphi_0 + \frac{3}{2}\sigma\,\omega$ is Hamiltonian if and only if*

$$\nabla_X \varphi_0 = -\tfrac{1}{2}\mathrm{d}\sigma(X)\omega + \tfrac{1}{2}(\mathrm{d}\sigma \wedge JX^\flat - J\mathrm{d}\sigma \wedge X^\flat) \tag{10}$$

for any vector field X.

Proof. This is immediate from Lemma 3: in (6), $\mathrm{d}\varphi_0 = -\frac{3}{2}d\sigma \wedge \omega$ if and only if $\beta = \frac{1}{2}\mathrm{d}\sigma$. □

In order to explain the use of the term 'Hamiltonian', we recall the following definition.

DEFINITION 5. A real function f on a Kähler manifold (M, g, J, ω) is a (real) holomorphy potential if the gradient $\mathrm{grad}_g f$ is a holomorphic vector field, i.e., preserves J; equivalently, f is a holomorphy potential if $J\mathrm{grad}_g f$ is a Killing vector field with respect to g.

A holomorphy potential f is therefore a momentum map for a Hamiltonian Killing vector field with respect to the symplectic form ω.

To any J-invariant 2-form $\varphi = \varphi_0 + \frac{3}{2}\sigma\,\omega$, we may associate a *normalized* 2-form $\tilde{\varphi} = \frac{1}{2}\varphi_0 + \frac{1}{4}\sigma\,\omega$. For example, if φ is the Ricci form ρ, then $\tilde{\varphi}$ is the 2-form $\tilde{\rho}$ associated to the normalized Ricci tensor: $\tilde{\rho}(\cdot,\cdot) = h(J\cdot,\cdot)$.

We are going to show that if φ is Hamiltonian, then the trace and Pfaffian of $\tilde{\varphi}$ are holomorphy potentials.

Recall that, in general, the *Pfaffian* $\mathrm{pf}(\psi)$ of a 2-form ψ is defined by $\frac{1}{2}\mathrm{pf}(\psi) = *(\psi \wedge \psi)$, where $*$ is the Hodge operator; alternatively, $\psi \wedge \psi = \frac{1}{4}\mathrm{pf}(\psi)\,\omega \wedge \omega$.

Since $\tilde{\varphi} = \frac{1}{2}\varphi_0 + \frac{1}{4}\sigma\,\omega$, its Pfaffian π is given by

$$\pi = \tfrac{1}{4}\sigma^2 - \tfrac{1}{2}|\varphi_0|^2 = \Big(\frac{\sigma}{2} + \lambda\Big)\Big(\frac{\sigma}{2} - \lambda\Big), \tag{11}$$

where $\lambda = |\varphi_0|/\sqrt{2}$. We write this product as $\pi = \xi\eta$ so that $\sigma = \xi + \eta$ and $\lambda = \frac{1}{2}(\xi - \eta)$.

PROPOSITION 2. *Let (M, g, J, ω) be a Kähler surface and let $\varphi = \varphi_0 + \frac{3}{2}\sigma\,\omega$ be a Hamiltonian 2-form. Then the trace σ and the Pfaffian π of $\tilde{\varphi} = \frac{1}{2}\varphi_0 + \frac{1}{4}\sigma\,\omega$ are Poisson-commuting holomorphy potentials.*

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Proof. (i) Identity (10) can be written in terms of $\tilde{\varphi}$ as

$$\nabla_X \tilde{\varphi} = \tfrac{1}{4}(\mathrm{d}\sigma \wedge JX^\flat - J\mathrm{d}\sigma \wedge X^\flat). \tag{12}$$

Differentiating again and skew-symmetrizing, we get

$$R_{X,Y} \cdot \tilde{\varphi} = [R_{X,Y}, \tilde{\varphi}] = \tfrac{1}{4}(\nabla_Y \mathrm{d}\sigma \wedge JX^\flat - \nabla_X \mathrm{d}\sigma \wedge JY^\flat - J\nabla_Y \mathrm{d}\sigma \wedge X^\flat + J\nabla_X \mathrm{d}\sigma \wedge Y^\flat).$$

Since $[R_{X,Y}, \tilde{\varphi}]$ is J-invariant in X, Y, it follows that

$$\begin{aligned} &\nabla_Y \mathrm{d}\sigma \wedge JX - J\nabla_Y \mathrm{d}\sigma \wedge X - \nabla_X \mathrm{d}\sigma \wedge JY + J\nabla_X \mathrm{d}\sigma \wedge Y \\ &\quad = -\nabla_{JY} \mathrm{d}\sigma \wedge X - J\nabla_{JY} \mathrm{d}\sigma \wedge JX + \nabla_{JX} \mathrm{d}\sigma \wedge Y + J\nabla_{JX} \mathrm{d}\sigma \wedge JY, \end{aligned}$$

hence

$$S(Y) \wedge JX - JS(Y) \wedge X - S(X) \wedge JY + JS(X) \wedge Y = 0, \tag{13}$$

where $S(X) = \nabla_X \mathrm{d}\sigma + J\nabla_{JX} \mathrm{d}\sigma$.

As an algebraic object, S is a symmetric, J-anti-commuting, endomorphism of TM; hence by contracting (13) with a vector field Z and taking the trace over Y and Z, we see that $S = 0$, and therefore, σ is a holomorphy potential.

(ii) From (12) we derive $(\nabla_X \tilde{\varphi}) \wedge \tilde{\varphi} = \frac{1}{2}(X \wedge J\mathrm{d}\sigma \wedge \tilde{\varphi})$, hence

$$\mathrm{d}\pi = -2 * (J\mathrm{d}\sigma \wedge \tilde{\varphi}). \tag{14}$$

From this, we infer (again using (12)):

$$\begin{aligned} \nabla_X \mathrm{d}\pi &= -2 * (J\nabla_X \mathrm{d}\sigma \wedge \tilde{\varphi} + J\mathrm{d}\sigma \wedge \nabla_X \tilde{\varphi}) \\ &= -2 * (J\nabla_X \mathrm{d}\sigma \wedge \tilde{\varphi} + \tfrac{1}{4} J\mathrm{d}\sigma \wedge \mathrm{d}\sigma \wedge JX). \end{aligned} \tag{15}$$

The second term of the right-hand side of (15) is clearly J-invariant; the first term is J-invariant as well since σ is a holomorphy potential and $\tilde{\varphi}$ is J-invariant. Hence π is also a holomorphy potential.

(iii) By contracting Equation (14) with $J\mathrm{d}\sigma$, we see that σ and π Poisson-commute. □

Since σ and π Poisson-commute, the Killing vector fields $K_1 := J\mathrm{grad}\,\sigma$ and $K_2 := J\mathrm{grad}\,\pi$ commute. Also $\omega(K_1, K_2) = 0$.

Remark 1. The Killing vector fields K_1 and K_2 need not be nonzero or independent in general. In particular K_1 and K_2 both vanish if σ is constant, since φ is then parallel by (10).

Notice, however, that $K_1^{1,0}$, $K_2^{1,0}$, and hence $K_1^{1,0} \wedge K_2^{1,0}$, are holomorphic, so that on each connected component of M there are three possibilities: □

(i) $K_1 \wedge K_2$ is nonvanishing on a dense open set.
(ii) K_1 is nonvanishing on a dense open set, but $K_1 \wedge K_2$ vanishes identically;
(iii) K_1 and K_2 vanish identically;

One consequence of this remark is the following lemma.

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LEMMA 6. *If φ is a Hamiltonian 2-form on a Kähler surface M, then the open set M_0, where φ_0 is non-zero, is empty or dense in each connected component of M.*

Proof. On each component of M where $K_1 = J\mathrm{grad}_g\sigma$ is nonzero, the set U where $\mathrm{d}\sigma$ is nonvanishing is dense, hence $\nabla\varphi_0$ is nonvanishing on U and so the complement of the zero set of φ_0 is dense in this connected component. On the other hand, if K_1 is identically zero on a component, then φ_0 is parallel on that component, hence identically zero or everywhere nonzero. □

When K_1 and K_2 vanish identically, φ does not contain much information about the geometry of M (it could be just a constant multiple of ω, even zero). In the other two cases, however, we shall obtain an explicit classification of Kähler surfaces with a Hamiltonian 2-form. The keys to these classifications are Proposition 2 and the following observation.

PROPOSITION 3. *Let φ be a Hamiltonian 2-form on a Kähler surface M and write $\sigma = \xi + \eta$ and $\pi = \xi\eta$ for the trace and Pfaffian of $\tilde{\varphi}$.*

Then on each connected component of M where φ_0 is not identically zero, $\mathrm{d}\xi$ and $\mathrm{d}\eta$ are orthogonal.

Proof. The contraction of (10) with φ_0 yields

$$\langle\nabla_X\varphi_0, \varphi_0\rangle = \tfrac{1}{2}(\varphi_0(\mathrm{d}\sigma, JX) - \varphi_0(J\mathrm{d}\sigma, X)) = -\varphi_0(J\mathrm{d}\sigma, X)$$

and, hence,

$$2\lambda\,\mathrm{d}\lambda = \mathrm{d}(\lambda^2) = \tfrac{1}{2}\mathrm{d}(|\varphi_0|^2) = -\varphi_0(J\mathrm{d}\sigma). \tag{16}$$

Since $(\varphi_0 \circ J)^2 = \lambda^2 I\mathrm{d}$, we deduce that $\varphi_0(J\mathrm{d}\lambda) = -\frac{1}{2}\lambda\,\mathrm{d}\sigma$ and therefore

$$\varphi_0 \circ J\left(\frac{\mathrm{d}\sigma}{2} + \mathrm{d}\lambda\right) = -\lambda\left(\frac{\mathrm{d}\sigma}{2} + \mathrm{d}\lambda\right), \qquad \varphi_0 \circ J\left(\frac{\mathrm{d}\sigma}{2} - \mathrm{d}\lambda\right) = \lambda\left(\frac{\mathrm{d}\sigma}{2} - \mathrm{d}\lambda\right).$$

This means that the 1-forms $\mathrm{d}\xi$ and $\mathrm{d}\eta$, wherever they are nonzero, are eigenforms for the symmetric endomorphism $-\varphi_0 \circ J$, corresponding to the eigenvalues λ and $-\lambda$, respectively; in particular, they are orthogonal on the open set M_0 where λ (i.e., φ_0) is nonzero. However, by Lemma 6 this open set is empty or dense in each connected component of M, and the result follows. □

Remark 2. On the open set M_0 where λ is nonzero, so that $\varphi_0 = \lambda\,\omega_I$, observe that Equation (16) may be rewritten

$$I\mathrm{d}\sigma = 2J\mathrm{d}\lambda. \tag{17}$$

Indeed, supposing only that φ_0 is a twistor 2-form, $\bar{\omega} = \lambda^{-3}\varphi_0$ is closed, and so, if $\varphi = \varphi_0 + \frac{3}{2}\sigma\,\omega$,

$$\mathrm{d}\varphi = \mathrm{d}\big(\varphi_0 + \tfrac{3}{2}\sigma\,\omega\big) = 3\lambda^2\mathrm{d}\lambda \wedge \bar{\omega} + \tfrac{3}{2}\mathrm{d}\sigma \wedge \omega = 3\big(\mathrm{d}\lambda \wedge \omega_I + \tfrac{1}{2}\mathrm{d}\sigma \wedge \omega\big).$$

Hence, Equation (17) holds if and only if φ is closed. □

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1.4. BI-EXTREMAL KÄHLER SURFACES

In the case that (M, g, J, ω) is weakly self-dual and $\varphi_0 = \rho_0$, we can take $\varphi = \rho$, so that σ is the (normalized) scalar curvature s, and π, the Pfaffian p of the normalized Ricci form $\tilde{\rho}(\cdot, \cdot) = h(J\cdot, \cdot)$; evidently, $\tilde{\rho} = \frac{1}{2}\rho_0 + \frac{1}{4}s\,\omega$.

Recall that a Kähler metric is said to be *extremal* if the scalar curvature is a holomorphy potential.

DEFINITION 6. A Kähler metric is called bi-extremal if both the (normalized) scalar curvature $s = \mathrm{tr}_\omega \tilde{\rho}$ and the Pfaffian $p = \mathrm{pf}(\tilde{\rho})$ of the normalized Ricci form $\tilde{\rho}$ are holomorphy potentials.

Note that the potential function p appearing in the above definition is *not* the Pfaffian of the *usual* Ricci form $\rho = \rho_0 + \frac{3}{2}s\omega$; thus, our definition for bi-extremality differs from the one given in [42, 31] (compare Theorem 5 below and [31, Th. 1.1 & Prop. 3.8]).

Proposition 2 immediately implies:

PROPOSITION 4. *A weakly self-dual Kähler metric is bi-extremal.*

On a bi-extremal Kähler surface, the holomorphy potentials s and p automatically Poisson-commute, since K_2 preserves g, hence s, so that $\mathrm{d}s(K_2) = 0$.

For a weakly self-dual Kähler surface, Proposition 3 generically implies that $\mathrm{d}\xi$ and $\mathrm{d}\eta$ are orthogonal, where $s = \xi + \eta$ and $p = \xi\eta$. We shall see in Sections 2 and 3 that a bi-extremal Kähler surface satisfying this orthogonality condition is weakly self-dual.

1.5. THE BACH TENSOR

The *Bach tensor*, B, of an n-dimensional Riemannian manifold (M, g) is defined by

$$B_{X,Y} = \sum_{i=1}^{n} (-(\nabla_{e_i} C)_{e_i,X}(Y) + (W_{e_i,X} h(e_i), Y)), \tag{18}$$

where, we recall, C is the Cotton–York tensor and h is the normalized Ricci tensor (here, $\{e_i\}$ is an arbitrary g-orthonormal frame). When $n = 4$, the Bach tensor is conformal covariant of weight -2, i.e., $B^{\phi^{-2}g} = \phi^2 B^g$, and B can be indifferently expressed in terms of W^+ or of W^-. Specifically

$$\begin{aligned} B_{X,Y} &= 2\sum_{i=1}^{n} (-(\nabla_{e_i} C^+)_{e_i,X}(Y) + (W^+_{e_i,X} h(e_i), Y)) \\ &= 2\sum_{i=1}^{n} (-(\nabla_{e_i} C^-)_{e_i,X}(Y) + (W^-_{e_i,X} h(e_i), Y)); \end{aligned} \tag{19}$$

in particular, the Bach tensor vanishes whenever W^+, W^- or Ric_0 vanishes.

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If (M, g, J) is a Kähler surface, the Bach tensor is easily computed by using the above identity and the fact that $W^+ = \frac{3}{4} s\omega \otimes_0 \omega$. Indeed, if B^+ and B^-, denote the J-invariant and J-anti-invariant parts of B, we get

$$B^+ = s\,\mathrm{Ric}_0 + 2(\nabla \mathrm{d}s)_0^+, \qquad B^- = -(\nabla \mathrm{d}s)^-, \tag{20}$$

where $(\nabla \mathrm{d}s)_0^+$ is the J-invariant trace-free part of the Hessian and $(\nabla \mathrm{d}s)^-$ is the J-anti-invariant part. (This formula is due to Derdziński [21], where it was obtained by a different argument.)

It follows that B is J-invariant if and only if (M, g, J) is an extremal Kähler surface; moreover, if this holds, we get $B = s\,\mathrm{Ric}_0 + 2(\nabla \mathrm{d}s)_0$. On the open set U where s has no zero, the vanishing of B then means that the conformally related metric $\tilde{g} = s^{-2} g$ is Einstein. Therefore, on U, the following two statements are thus equivalent (see also [21]):

(i) The Bach tensor of the Kähler surface (M, g, J) vanishes;
(ii) (M, g, J) is extremal and the conformally related metric $\tilde{g} = s^{-2} g$ is Einstein.

Note also that when B is J-invariant, it is determined by the associated anti-self-dual 2-form $\tilde{B}(\cdot, \cdot) = B(J\cdot, \cdot)$, which is also given by

$$\tilde{B} = (\mathrm{d}J\mathrm{d}s)_0 + s\rho_0. \tag{21}$$

On a weakly self-dual Kähler surface, $C^- = 0$, while $W^- = \frac{3}{4}\kappa\omega_I \otimes_0 \omega_I$. It follows that $\tilde{B}$ is a multiple of $\kappa\rho_0$, which vanishes if and only if $W^- = 0$ or $\rho_0 = 0$.

PROPOSITION 5. *A weakly self-dual Kähler surface is Bach-flat* (*i.e., has vanishing Bach tensor*) *if and only if it is self-dual or Kähler–Einstein.*

1.6. ROUGH CLASSIFICATION OF WEAKLY SELF-DUAL KÄHLER SURFACES

We have seen in Lemma 6 that the open-set M_0, on which the trace-free part φ_0 of a Hamiltonian 2-form is nonzero, is empty or dense in each connected component of M. For a weakly self-dual Kähler surface (M, g, J, ω), the Ricci form ρ is Hamiltonian, and M_0 is the set of points at which g is not Kähler–Einstein. Because ρ is closely linked to the anti-self-dual Weyl tensor of M, we can obtain more information about M_0 except when g is self-dual.

We first recall the general fact, first observed by A. Derdziński in [21], that for any Kähler surface (M, g, J) with nonvanishing scalar curvature s, the conformally related metric $\tilde{g} = s^{-2} g$ satisfies $\delta^{\tilde{g}} W^+ = 0$; moreover, up to rescaling, $\tilde{g}$ is the unique metric in the conformal class $[g]$ that satisfies this property. This follows from the fact that the self-dual Cotton–York tensor C^+ of any Kähler surface can be written as

$$C^+(X) = -W^+\left(\frac{\mathrm{d}s}{s} \wedge X\right); \tag{22}$$

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on the other hand, the self-dual Cotton–York tensors of two conformally related metrics g and $f^{-2}g$ are related by

$$C^{+,f^{-2}g}(X) = C^{+,g}(X) + W^{+}\left(\frac{\mathrm{d}f}{f} \wedge X\right);$$

it follows from (22) that the self-dual Cotton–York tensor of the metric $s^{-2}g$ vanishes identically; moreover, as W^{+} has no kernel (for s nonvanishing), the latter property characterizes $s^{-2}g$ up to a constant multiple.

LEMMA 7. *On each connected component of the open set M_0 where ρ_0 does not vanish, the scalar curvature $\bar{s}$ of $\bar{g}$ is a constant multiple of λ^{-1}, i.e.,*

$$\bar{s} = c\lambda^{-1}, \tag{23}$$

where λ is the positive eigenvalue of Ric_0 *and c is a constant.*

Proof. We apply the preceding argument to the Kähler pair $(\bar{g}, I)$ on M_0 and to $g = \lambda^2\bar{g}$; by hypothesis, g satisfies $\delta^g W^- = C^- = 0$, where $W^-$is actually the *self-dual* Weyl tensor of g for the orientation induced by I; from the above mentioned uniqueness property, it follows that, wherever $\bar{s}$ is non-zero, g coincides with $\bar{s}^{-2}\bar{g}$ up to rescaling, i.e., that $\bar{s}$ is a locally constant multiple of λ^{-1}. However, the same holds on the interior of the zero set of $\bar{s}$. Hence by the continuity of $\bar{s}$ on M_0, $\bar{s} = c\lambda^{-1}$ for some constant c on each connected component of M_0. □

A more global statement may be obtained using the conformal scalar curvature $\kappa = \bar{s}\lambda^{-2}$. Since the anti-self-dual Weyl tensor of g is given by $W^- = \frac{3}{4}\kappa\omega_I \otimes_0 \omega_I$, it follows that κ^2 is equal to $|W^-|^2$ on M_0, up to a numerical factor, and hence we may extend κ continuously to the closure of M_0. Also λ is globally defined and continuous.

Therefore, using the fact that the closure of M_0 is a union of connected components by Lemma 6, we can rewrite Lemma 7.

LEMMA 8. *Let (M, g, J, ω) be a weakly self-dual Kähler surface. Then, on each component of M where ρ_0 is not identically zero, the conformal scalar curvature κ of (g, I) is linked to λ by*

$$\kappa\lambda^3 = c, \tag{24}$$

where c is the constant of Lemma 7. Moreover, $c = 0$ if and only if $W^- = 0$ on that component.

This lemma yields the following rough classification of weakly self-dual Kähler surfaces (see also [32]).

PROPOSITION 6. *Let (M, g, J, ω) be a weakly self-dual, connected, Kähler surface. Then either:*

(i) *ρ_0 is identically zero so (g, J) is Kähler–Einstein; or*
(ii) *the scalar curvature s of g is constant, but ρ_0 is not identically zero; then, (g, J) is locally the Kähler product of two Riemann surfaces of constant curvatures; or*

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(iii) *s is not constant and g is self-dual; or*
(iv) *W^- and ρ_0 have no zero: then, the Kähler metric $(\bar{g} = \lambda^{-2}g, I)$ of Proposition 1 is extremal and globally defined on M; in particular, W^- is degenerate everywhere.*

Proof. If s is constant, then by (2) the Ricci form is parallel. Hence either g is locally irreducible, and is Einstein, or (g, J) is locally the Kähler product of two Riemann surfaces of constant curvatures.

If s is not constant, then by Lemma 6 the open set M_0 where ρ_0 does not vanish is an open dense subset of M. However, by Lemma 8, $\kappa\lambda^3$ is constant. If this constant is zero, then κ must vanish identically and M is self-dual; otherwise κ and λ have no zero on M, so $M_0 = M$, the Kähler pair $(\bar{g} = \lambda^{-2}g, I)$ is defined on M and W^- is degenerate, but nonzero everywhere. As observed in Section 1.5, for a weakly self-dual Kähler surface the Bach form $\tilde{B}$ is a multiple of ω_I. Since B is a conformally covariant tensor, it follows that the Bach tensor of $\bar{g}$ is I-invariant, showing that $(\bar{g}, I)$ is an extremal Kähler metric. □

Kähler–Einstein metrics and Kähler products of Riemann surfaces clearly *are* weakly self-dual. Since these are well studied, we henceforth assume that s is not constant (on any component), i.e., K_1 is nonvanishing on a dense open set. In Section 2, we analyse the generic case that K_1 and K_2 are independent, while Section 3 is devoted to the case that $K_1 \wedge K_2$ vanishes identically (but K_1 is non-zero). In both sections, we obtain an explicit local classification within the more general framework of Kähler surfaces with a Hamiltonian 2-form.

2. Ortho-toric Kähler Surfaces

2.1. TORIC KÄHLER SURFACES

We have seen in Section 1 that on a Kähler surface with a Hamiltonian 2-form φ—in particular on a weakly self-dual Kähler surface—the trace σ and the Pfaffian π of the associated normalized 2-form $\tilde{\varphi}$ are holomorphy potentials for Hamiltonian Killing vector fields $K_1 = J\operatorname{grad}\sigma$ and $K_2 = J\operatorname{grad}\pi$. Furthermore, σ and π Poisson-commute, i.e., $\omega(K_1, K_2) = 0$.

A (usually compact) Kähler surface (M, g, J, ω), with holomorphic Killing vector fields K_1 and K_2 which are independent on a dense open set and satisfy $\omega(K_1, K_2) = 0$, is said to be *toric*. We begin this section by recalling the local theory of such surfaces, and we therefore assume that K_1 and K_2 are everywhere independent and that x_1 and x_2 are globally defined momentum maps for K_1 and K_2.

The condition $\omega(K_1, K_2) = 0$ is equivalent to the fact that x_1 and x_2 commute for the Poisson bracket determined by ω. Hence, also $[K_1, K_2] = 0$, and since K_1, K_2, JK_1 and JK_2 are all holomorphic, they all commute. In particular the rank 2 distributions Π, generated by K_1 and K_2, and $J\Pi$, generated by JK_1 and JK_2, are integrable.

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These distributions $\Pi, J\Pi$ are also orthogonal, since $\langle JK_1, K_2\rangle = 0$. It follows that K_1, K_2, JK_1 and JK_2 form a frame. Since they commute, the 1-forms in the dual coframe are closed, and may be written $\mathrm{d}t_1, \mathrm{d}t_2, J\mathrm{d}t_1, J\mathrm{d}t_2$, where t_1 and t_2 are only given locally and up to an additive constant. Now observe that

$$J\mathrm{d}t_1 = \frac{|K_2|^2\mathrm{d}x_1 - \langle K_1, K_2\rangle \mathrm{d}x_2}{|K_1 \wedge K_2|^2},$$
$$J\mathrm{d}t_2 = \frac{|K_1|^2\mathrm{d}x_2 - \langle K_1, K_2\rangle \mathrm{d}x_1}{|K_1 \wedge K_2|^2},$$

and so

$$J\mathrm{d}t_i = \sum_{j=1,2} G_{ij}\mathrm{d}x_j \qquad (i = 1, 2),$$

where G_{ij} is a positive definite symmetric matrix of functions of x_1 and x_2 (note that $K_i = \partial/\partial t_i$). These 1-forms are closed if and only if G_{ij} is the Hessian of a function of x_1 and x_2. The following well-known explicit classification is then readily obtained (see [1, 29]).

PROPOSITION 7. *Let G_{ij} be a positive definite 2×2 symmetric matrix of functions of 2-variables x_1, x_2 with inverse G^{ij}. Then the metric*

$$\sum_{i,j}(G_{ij}\mathrm{d}x_i\mathrm{d}x_j + G^{ij}\mathrm{d}t_i\mathrm{d}t_j)$$

is almost-Kähler with Kähler form

$$\omega = \mathrm{d}x_1 \wedge \mathrm{d}t_1 + \mathrm{d}x_2 \wedge \mathrm{d}t_2$$

and has independent Hamiltonian Killing vector fields $\partial/\partial t_1, \partial/\partial t_2$ with Poisson-commuting momentum maps x_1 and x_2. Any almost Kähler structure with such a pair of Killing vector fields is of this form (where the t_i are locally defined up to an additive constant), and is Kähler if and only if G_{ij} is the Hessian of a function of x_1 and x_2.

2.2. THE ORTHO-TORIC CASE

Propositions 2 and 3 motivate the following definition.

DEFINITION 7. A Kähler surface (M, g, J, ω) is ortho-toric if it admits two independent Hamiltonian Killing vector fields with Poisson-commuting momentum maps $\xi\eta$ and $\xi + \eta$ such that $\mathrm{d}\xi$ and $\mathrm{d}\eta$ are orthogonal.

An explicit classification of ortho-toric Kähler metrics follows from Proposition 7 by changing variables and imposing the orthogonality of $\mathrm{d}\xi$ and $\mathrm{d}\eta$. However, since the coordinate change is awkward, and we have not spelt out the proof of Proposition 7, we give a self-contained proof of this classification.

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PROPOSITION 8. *The almost-Hermitian structure* (g, J, ω) *defined by*

$$g = (\xi - \eta)\left(\frac{\mathrm{d}\xi^2}{F(\xi)} - \frac{\mathrm{d}\eta^2}{G(\eta)}\right) + \frac{1}{\xi - \eta}(F(\xi)(\mathrm{d}t + \eta\,\mathrm{d}z)^2 - G(\eta)(\mathrm{d}t + \xi\mathrm{d}z)^2), \tag{25}$$

$$\begin{aligned} J\mathrm{d}\xi &= \frac{F(\xi)}{\xi - \eta}(\mathrm{d}t + \eta\mathrm{d}z), \qquad & J\mathrm{d}t &= -\frac{\xi\mathrm{d}\xi}{F(\xi)} - \frac{\eta\mathrm{d}\eta}{G(\eta)}, \\ J\mathrm{d}\eta &= \frac{G(\eta)}{\eta - \xi}(\mathrm{d}t + \xi\mathrm{d}z), \qquad & J\mathrm{d}z &= \frac{\mathrm{d}\xi}{F(\xi)} + \frac{\mathrm{d}\eta}{G(\eta)}, \end{aligned} \tag{26}$$

$$\omega = \mathrm{d}\xi \wedge (\mathrm{d}t + \eta\mathrm{d}z) + \mathrm{d}\eta \wedge (\mathrm{d}t + \xi\mathrm{d}z) \tag{27}$$

is an ortho-toric Kähler structure for any functions F, G *of one variable. Every ortho-toric Kähler surface is of this form, where* t, z *are locally defined up to an additive constant.*

Proof. (i) The Kähler form may be written

$$\omega = \mathrm{d}(\xi + \eta) \wedge \mathrm{d}t + \mathrm{d}(\xi\eta) \wedge \mathrm{d}z$$

which is certainly closed. If is also immediate that $\partial/\partial t$ and $\partial/\partial z$ are Hamiltonian Killing vector fields with Poisson-commuting momentum maps $\xi + \eta$ and $\xi\eta$. Since $\mathrm{d}t + iJ\mathrm{d}t$ and $\mathrm{d}z + iJ\mathrm{d}z$ are closed, J is integrable, and the Kähler surface is clearly ortho-toric.

(ii) Conversely, suppose that (g, J, ω) is an ortho-toric Kähler surface with Killing vector fields K_1, K_2. Since the dual frame to K_1, K_2, JK_1, JK_2 consists of closed 1-forms, we may write it as $\mathrm{d}t, \mathrm{d}z, J\mathrm{d}t, J\mathrm{d}z$, where t and z are locally defined up to an additive constant. Note also that $\mathrm{d}\xi, \mathrm{d}\eta, \mathrm{d}t, \mathrm{d}z$ are linearly independent 1-forms—where $\xi + \eta$ and $\xi\eta$ are the momentum maps of K_1 and K_2—so we may use (ξ, η, t, z) as a coordinate system.

Since $(J\mathrm{d}z)(K_1) = 0$ and $(J\mathrm{d}z)(K_2) = 0$ we may write

$$J\mathrm{d}z = \frac{\mathrm{d}\xi}{F} + \frac{\mathrm{d}\eta}{G}$$

for some functions F and G (of ξ and η). The equations

$$0 = (J\mathrm{d}z)(JK_1) = -\langle J\mathrm{d}z, \mathrm{d}\xi + \mathrm{d}\eta\rangle$$

and

$$1 = (J\mathrm{d}z)(JK_2) = -\langle J\mathrm{d}z, \eta\mathrm{d}\xi + \xi\mathrm{d}\eta\rangle$$

give $F = |\mathrm{d}\xi|^2(\xi - \eta)$ and $G = |\mathrm{d}\eta|^2(\eta - \xi)$, using the fact that $\mathrm{d}\xi$ and $\mathrm{d}\eta$ are orthogonal. A similar argument tells us that

$$J\mathrm{d}t = -\frac{\xi\mathrm{d}\xi}{F} - \frac{\eta\mathrm{d}\eta}{G}.$$

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for the same functions F and G. Now since $J\mathrm{d}t$ and $J\mathrm{d}z$ are closed, we obtain $(\xi-\eta)F_\eta = 0$ and $(\eta-\xi)G_\xi = 0$, so that $F = F(\xi)$ and $G = G(\eta)$. The Kähler form ω is evidently given by (27), and since we know $J\mathrm{d}t$ and $J\mathrm{d}z$, we readily obtain (26), and hence the metric (25). □

Any ortho-toric Kähler surface (M, g, J, ω) comes equipped with an anti-self-dual almost-complex structure, I, whose Kähler form, ω_I is defined by

$$\begin{aligned}\omega_I &= \frac{\mathrm{d}\xi\wedge J\mathrm{d}\xi}{|\mathrm{d}\xi|^2} - \frac{\mathrm{d}\eta\wedge J\mathrm{d}\eta}{|\mathrm{d}\eta|^2}\\ &= \mathrm{d}\xi\wedge(\mathrm{d}t+\eta\mathrm{d}z) - \mathrm{d}\eta\wedge(\mathrm{d}t+\xi\mathrm{d}z);\end{aligned} \tag{28}$$

equivalently

$$\begin{aligned} I\mathrm{d}\xi &= J\mathrm{d}\xi = \frac{F(\xi)}{\xi-\eta}(\mathrm{d}t+\eta\mathrm{d}z), \qquad I\mathrm{d}t = -\frac{\xi\mathrm{d}\xi}{F(\xi)} + \frac{\eta\mathrm{d}\eta}{G(\eta)},\\ I\mathrm{d}\eta &= -J\mathrm{d}\eta = -\frac{G(\eta)}{\eta-\xi}(\mathrm{d}t+\xi\mathrm{d}z), \qquad I\mathrm{d}z = \frac{\mathrm{d}\xi}{F(\xi)} - \frac{\mathrm{d}\eta}{G(\eta)}.\end{aligned} \tag{29}$$

PROPOSITION 9. *For any ortho-toric Kähler surface, the almost-Hermitian pair* $(\bar g = (\xi-\eta)^{-2}g, I)$ *is Kähler.*

Proof. Clearly $I\mathrm{d}t$ and $I\mathrm{d}z$ are closed, so I is integrable. From (28), we easily infer that the Lee form θ of the Hermitian pair (g, I), defined by $\mathrm{d}\omega_I = -2\theta\wedge\omega_I$, is

$$\theta = -\mathrm{d}\log|\xi-\eta|. \tag{30}$$

It follows that $\bar\omega := (\xi-\eta)^{-2}\omega_I$ is closed, i.e., the pair $(\bar g = (\xi-\eta)^{-2}g, I)$ is Kähler. □

In particular, on any ortho-toric Kähler surface, the anti-self-dual Weyl tensor—which is the *self-dual* Weyl tensor of g for the orientation induced by I—is degenerate: $W^- = \kappa\omega_I\otimes_0\omega_I$, where κ is the conformal scalar curvature of the Hermitian pair (g, I).

Remark 3. The vector fields K_1 and K_2 are still Killing with respect to $\bar g$ and Hamiltonian with respect to $\bar\omega$, with momentum maps $-(1/\xi-\eta)$ and $-(\xi+\eta)/(2(\xi-\eta))$ respectively. However, the Kähler metric $(\bar g, I)$ is *not* ortho-toric in general, as it can be checked using Lemma 9 below.

Combining Propositions 2, 3 with 8 and 9, we obtain the following theorem.

THEOREM 1. *A Kähler surface is ortho-toric if and only if it admits a Hamiltonian* 2*-form whose associated Killing vector fields are independent. The Kähler structure is then given explicitly in terms of two arbitrary functions* F, G *of one variable by* (25)–(27).

Indeed, using (28)–(29), notice that, by definition, $J\mathrm{d}(\xi+\eta) = I\mathrm{d}(\xi-\eta)$, cf. Remark 2, so the Hamiltonian 2-form φ is $\frac{1}{2}(\xi-\eta)\omega_I + (\xi+\eta)\omega$.

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2.3. ORTHO-TORIC WEAKLY SELF-DUAL KÄHLER SURFACES

The curvature of an ortho-toric Kähler surface is entirely determined by the scalar curvature s of g, the conformal scalar curvature κ of the Hermitian pair (g, I), and the trace-free part ρ_0 of the Ricci form of (g, J).

LEMMA 9. *For any ortho-toric Kähler surface (M, g, J, ω), ρ_0 is a multiple μ of the Kähler form ω_I of the Hermitian pair (g, I), and μ, s, κ are given by*

$$\mu = \frac{F'(\xi) - G'(\eta)}{2(\xi - \eta)^2} - \frac{F''(\xi) + G''(\eta)}{4(\xi - \eta)}, \tag{31}$$

$$s = -\frac{F''(\xi) - G''(\eta)}{6(\xi - \eta)}, \tag{32}$$

$$\kappa = -\frac{F''(\xi) - G''(\eta)}{6(\xi - \eta)} + \frac{F'(\xi) + G'(\eta)}{(\xi - \eta)^2} - \frac{2(F(\xi) - G(\eta))}{(\xi - \eta)^3}. \tag{33}$$

In particular, on the open subset of M where μ has no zero, the anti-self-dual almost-complex structure determined by ρ_0 is equal to I.

Proof. From (27), we infer that the volume-form $v_g = \frac{1}{2}\omega \wedge \omega$ of g is given by

$$v_g = -(\xi - \eta)\mathrm{d}\xi \wedge \mathrm{d}\eta \wedge \mathrm{d}t \wedge \mathrm{d}z,$$

since t and z are the real parts of J-holomorphic coordinates. By putting $v_0 = \mathrm{d}t \wedge J\mathrm{d}t \wedge \mathrm{d}z \wedge J\mathrm{d}z$, we have $\rho = -\frac{1}{2}\mathrm{d}J\mathrm{d}\log v_g/v_0$. Now according to (26),

$$v_0 = -\frac{\xi - \eta}{F(\xi)G(\eta)}\mathrm{d}\xi \wedge \mathrm{d}\eta \wedge \mathrm{d}t \wedge \mathrm{d}z,$$

and, hence, $v_g/v_0 = F(\xi)G(\eta)$; this implies

$$\rho = -\tfrac{1}{2}\mathrm{d}J\mathrm{d}\log|F(\xi)| - \tfrac{1}{2}\mathrm{d}J\mathrm{d}\log|G(\eta)|,$$

from which (31) and (32) follow easily.

From (9) and (30), we get

$$\kappa = s - 2\frac{|\mathrm{d}\xi|^2 + |\mathrm{d}\eta|^2}{(\xi - \eta)^2} - \frac{\Delta(\xi - \eta)}{\xi - \eta};$$

on the other hand, we compute that

$$\Delta\xi = -\frac{F'(\xi)}{\xi - \eta}, \qquad \Delta\eta = \frac{G'(\eta)}{\xi - \eta};$$

and we obtain (33). □

PROPOSITION 10. *An ortho-toric Kähler surface M is extremal if and only if F and G are of the form*

$$F(x) = kx^4 + \ell x^3 + Ax^2 + B_1 x + C_1, \quad G(x) = kx^4 + \ell x^3 + Ax^2 + B_2 x + C_2, \tag{34}$$

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in which case

$$s = -2k(\xi + \eta) - \ell, \tag{35}$$

and $(\bar{g} = (\xi - \eta)^{-2} g, I)$ *is an extremal Kähler metric as well.*

Moreover, M is

- *Bach-flat if and only if* $4k(C_1 - C_2) = (B_1 - B_2)\ell$;
- *of constant scalar curvature if and only if* $k = 0$;
- *scalar-flat (i.e., anti-self-dual) if and only if* $k = \ell = 0$.

Proof. Since the scalar curvature s is a function of ξ and η, $J \operatorname{grad}_g s$ belongs to the span of the Killing vector fields K_1 and K_2 and commutes with them; if it is itself a Killing vector field, it has to be a linear combination of K_1 and K_2 with constant coefficients, i.e. $s = a(\xi + \eta) + b\xi\eta + c$, where a, b, c are constants. By (32), this implies (34). Finally, using (21), we easily compute that the anti-self-dual 2-form associated to the Bach tensor of an ortho-toric extremal Kähler surface is

$$\tilde{B} = \frac{4k(C_1 - C_2) - (B_1 - B_2)\ell}{2(\xi - \eta)^2}\,\omega_I.$$

Since B is also I-invariant, the Kähler metric $(\bar{g}, I)$ is extremal as well (see Section 1.5). □

EXAMPLE 1. For $k \neq 0$ and $4(C_1 - C_2) = (B_1 - B_2)\ell/k$, we obtain explicit Bach-flat Kähler surfaces with nonconstant scalar curvature. These metrics are therefore not anti-self-dual, and for $B_1 \neq B_2$ they are not self-dual either (note that a self-dual Kähler surface is bi-extremal and see the next Proposition). According to Section 1.5, the metric $\tilde{g} = (2k(\xi + \eta) + \ell)^{-2} g$, which is defined on the open subset where $2k(\xi + \eta) + \ell \neq 0$, is Einstein, Hermitian (but non-Kähler) with a locally defined toric isometric action.

PROPOSITION 11. *An ortho-toric Kähler surface M is bi-extremal if and only if F and G are of the form*

$$\begin{aligned} F(x) &= kx^4 + \ell x^3 + Ax^2 + Bx + C_1, \\ G(x) &= kx^4 + \ell x^3 + Ax^2 + Bx + C_2, \end{aligned} \tag{36}$$

in which case the Ricci form is given by $\rho = -2k\varphi - \ell\omega$. *Hence M is weakly self-dual and is*

- *self-dual if and only if* $C_1 = C_2$;
- *Kähler–Einstein if and only if* $k = 0$;
- *Ricci-flat if and only if* $k = \ell = 0$.

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Proof. Since a bi-extremal surface is extremal, we may apply Proposition 10. We then compute

$$\mu = -k(\xi - \eta) + \frac{B_1 - B_2}{2(\xi - \eta)^2}, \tag{37}$$

$$p = 4k^2\xi\eta + k\ell(\xi + \eta) + \frac{\ell^2}{4} - k\frac{B_1 - B_2}{\xi - \eta} + \frac{(B_1 - B_2)^2}{4(\xi - \eta)^4}. \tag{38}$$

As in the proof of Proposition 10, p cannot be a J-holomorphy potential unless it is a linear combination of $\xi + \eta$ and $\xi\eta$ with constant coefficients; this in turn is equivalent to the condition $B_1 = B_2$. Since $\mu = -2k\lambda$ and $s = -2k\sigma - \ell$ it follows that $\rho = -2k\varphi - \ell\omega$. Hence ρ is a Hamiltonian and so M is weakly self-dual by (4).

By substituting in the expression of κ given by (33) we get $\kappa = -(2(C_1 - C_2))/((\xi - \eta)^3)$; since the condition $W^- = 0$ is equivalent to $\kappa = 0$, the characterization of the self-dual case follows. Also M is Einstein if and only if $\mu = 0$, and so the last two assertions are immediate. □

Remark 4. For $k \neq 0$, we can set $k = -\frac{1}{2}$ and $\ell = 0$ by a simultaneous affine change of ξ and η. In this case $\rho = \varphi$, and so the ortho-toric reduction is defined by ρ. However, not all weakly self-dual Kähler surfaces can be put in ortho-toric form; for example weakly self-dual metrics belonging to the general family of cohomogeneity-one extremal metrics considered by E. Calabi [14] are not in general ortho-toric, since K_1 and K_2 are then collinear. We discuss this case in Section 3.

On the other hand, the examples with $k = 0$ in the above Proposition show that among Kähler–Einstein metrics (which are weakly self-dual), there are some which can be put into ortho-toric form, because even though ρ does not define an ortho-toric reduction, there happens to be another Hamiltonian 2-form φ. If additionally $A = \ell = 0$ or $C_1 = C_2$, these examples in fact have cohomogeneity one and their ortho-toric form arises from a choice of maximal torus inside the isometry group. However, the other examples do not have additional symmetries and therefore appear to be new.

PROPOSITION 12. *On an ortho-toric extremal Kähler surface M, the space of infinitesimal symmetries of the Kähler structure is generated by K_1 and K_2 (and infinitesimal rotations in this plane if they are globally defined), except in the following two cases:*

(i) *M is locally a complex space form, i.e., Kähler–Einstein and self-dual; or*
(ii) *M is Ricci-flat (hence, anti-self-dual) and, locally, of cohomogeneity one.*

In terms of F and G, these two cases are respectively described by

$$F(x) = G(x) = \ell x^3 + Ax^2 + Bx + C; \tag{39}$$

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and

$$F(x) = Bx + C_1, \qquad G(x) = Bx + C_2. \tag{40}$$

Proof. Suppose there exist a third infinitesimal symmetry of M, say K_3, which does not lie in the plane spanned by K_1 and K_2; then, we must have $\mathrm{d}s(K_i) = 0$ and $\mathrm{d}\mu(K_i) = 0$, for $i = 1, 2, 3$; this implies that $\mathrm{d}s$ and $\mathrm{d}\mu$ are colinear; we then infer from (35) and (37) that we have $k = 0$ and $B_1 = B_2$ in (34), so that M is Kähler–Einstein by Proposition 11.

If, in addition, $W^- = 0$, we obtain (39) and g is then a self-dual Kähler–Einstein surface, i.e., a complex space form.

If W^- does not vanish identically, on the open set where $W^- \neq 0$, the Hermitian pair (g, I) is invariant under the action of the Killing vector fields K_i's, as I is determined by the eigenform of W^- corresponding to its simple eigenvalue, cf. [21, 7]; then, κ (a constant multiple of the simple eigenvalue of W^-), the square-norm $|\theta|^2$ of the Lee form θ of the pair (g, I) as well as $\delta\theta$ are also invariant under the action of K_i's, $i = 1, 2, 3$; we then have $\mathrm{d}|\theta|^2 \wedge \mathrm{d}\kappa = 0$; by using (30) and Lemma 9, we can check that this implies that F and G satisfy (40); by Proposition 10, the corresponding ortho-toric Kähler surface is Ricci-flat and $\kappa = -(2(C_1 - C_2))/((\xi - \eta)^3)$; it then follows from (30) that $\theta = \frac{1}{3}\mathrm{d}\ln|\kappa|$; in particular, $\mathrm{d}|\theta|^2 \wedge \theta = 0$; by [8, Theorem 1], this implies that the metric is of cohomogeneity one. Since κ is non-zero, it is not constant, and so the metric is not homogeneous.

Evidently the two cases overlap when $C_1 = C_2$ and $A = \ell = 0$, in which case g is flat. □

Remark 5. In fact we do not need to assume *a priori* that M is extremal in the above proof, as long as we assume that the additional symmetry preserves φ, hence $\xi - \eta$. Then $\mathrm{d}s$ and $\mathrm{d}\xi - \mathrm{d}\eta$ are collinear, from which it is easy to deduce that M is extremal.

Let us now collect the results we have established so far about weakly self-dual Kähler surfaces.

THEOREM 2. *Let (M, g, J, ω) be a weakly self-dual Kähler surface. Denote by s, λ, and $p = ((s/2) + \lambda)((s/2) - \lambda)$, the (normalized) scalar curvature, the positive eigenvalue of the trace-free Ricci tensor* Ric_0*, and the Pfaffian of the normalized Ricci tensor, respectively. Then,*

(i) *$K_1 := J\,\mathrm{grad}_g\, s$ and $K_2 := J\,\mathrm{grad}_g p$ are commuting Killing vector fields, and on any simply connected open subset where K_1 and K_2 are linearly independent, the functions $\xi := (s/2) + \lambda$, $\eta := (s/2) - \lambda$, t, z form a globally defined coordinate system with respect to which the Kähler structure (g, J, ω) is*

$$g = (\xi - \eta)\left(\frac{\mathrm{d}\xi^2}{F(\xi)} - \frac{\mathrm{d}\eta^2}{G(\eta)}\right) + \frac{1}{\xi - \eta}(F(\xi)(\mathrm{d}t + \eta\mathrm{d}z)^2 - G(\eta)(\mathrm{d}t + \xi\mathrm{d}z)^2), \tag{41}$$

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$$\begin{aligned} J\mathrm{d}\xi &= \frac{F(\xi)}{\xi-\eta}(\mathrm{d}t+\eta\mathrm{d}z), \qquad & J\mathrm{d}t &= -\frac{\xi\mathrm{d}\xi}{F(\xi)}-\frac{\eta\mathrm{d}\eta}{G(\eta)},\\ J\mathrm{d}\eta &= \frac{G(\eta)}{\eta-\xi}(\mathrm{d}t+\xi\mathrm{d}z), & J\mathrm{d}z &= \frac{\mathrm{d}\xi}{F(\xi)}+\frac{\mathrm{d}\eta}{G(\eta)}, \end{aligned} \tag{42}$$

$$\omega = \mathrm{d}\xi\wedge(\mathrm{d}t+\eta\mathrm{d}z)+\mathrm{d}\eta\wedge(\mathrm{d}t+\xi\mathrm{d}z), \tag{43}$$

where

$$F(x) = kx^4+\ell x^3+Ax^2+Bx+C_1, \tag{44}$$

$$G(x) = kx^4+\ell x^3+Ax^2+Bx+C_2. \tag{45}$$

(ii) *Conversely, each almost Kähler structure* (g, J, ω) *described by* (41)–(45) *is Kähler and weakly self-dual with*

$$s = -2k(\xi+\eta)-\ell, \qquad p = 4k^2\xi\eta+k\ell(\xi+\eta)+\frac{\ell^2}{4},$$

so that $K_1 = \partial/\partial t$ *and* $K_2 = \partial/\partial z$.
(iii) *The Kähler structure described by* (41)--(45) *is self-dual if and only if* $C_1 = C_2$.

In the self-dual case, we recover the general expression found by Bryant [13, Section 4.3.2] depending on an arbitrary polynomial of degree 4.

3. Kähler Surfaces of Calabi Type

3.1. HAMILTONIAN 2-FORMS AND THE CALABI CONSTRUCTION

In this section we classify weakly self-dual Kähler surfaces of nowhere constant scalar curvature s, but for which p and s are not independent. As in the previous section (when we assumed p and s were independent) we do this by finding an explicit formula for a Kähler surface (M, g, J, ω) with a Hamiltonian 2-form $\varphi = \varphi_0 + \frac{3}{2}\sigma\omega$ such that $K_1 := J\,\mathrm{grad}_g\sigma$ has no zero, but $K_2 := J\,\mathrm{grad}_g\pi = bK_1$, where π is the Pfaffian of $\tilde{\varphi}$ and b is (necessarily) constant.

The general theory of Kähler surfaces with a Hamiltonian 2-form, described in Section 1, applies equally to this case. In particular, since K_1 has no zero, we may still write $\pi = \xi\eta$ and $\sigma = \xi+\eta$ for the trace and Pfaffian of $\tilde{\varphi}$, and $\mathrm{d}\xi$ and $\mathrm{d}\eta$ are orthogonal by Proposition 3.

On the other hand, the constructions of Section 2 definitely fail, because K_1 and K_2 are no longer independent: π is an affine function of σ, so ξ and η are not independent functions. Therefore, $\mathrm{d}\xi$ and $\mathrm{d}\eta$, in addition to being orthogonal, are collinear! This is not a contradiction: we deduce that either ξ or η is constant. Since $\pi = \xi(\sigma-\xi) = \eta(\sigma-\eta)$, this constant is the constant b above, so that $\pi = b(\sigma-b)$ and $\lambda = \pm\frac{1}{2}(\sigma-2b)$.

We observe that K_1 is an eigenvector of $-\varphi_0\circ J$, for the eigenvalue λ, and the conformally Kähler anti-self-dual complex structure I is characterized as follows:

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I coincides with J on the distribution generated by K_1 and JK_1, but with $-J$ on the orthogonal distribution. Hence, we are in the following situation.

DEFINITION 8. A Kähler surface (M, g, J, ω) is said to be of Calabi type if it admits a nonvanishing Hamiltonian Killing vector field K such the almost-Hermitian pair (g, I)—with I equal to J on the distribution spanned by K and JK, but $-J$ on the orthogonal distribution—is conformally Kähler.

It is straightforward to obtain an explicit formula for Kähler metrics of Calabi type, using the LeBrun form of a Kähler metric with a Hamiltonian Killing vector field [37].

PROPOSITION 13. *Let (M, g, J, ω) be a Kähler surface of Calabi type. Then the Kähler structure is given locally by*

$$g = (az - b)g_\Sigma + w(z)\mathrm{d}z^2 + w(z)^{-1}(\mathrm{d}t + \alpha)^2, \tag{46}$$

$$\omega = (az - b)\omega_\Sigma + \mathrm{d}z \wedge (\mathrm{d}t + \alpha), \tag{47}$$

where z is the momentum map of the Killing vector field K, t is a function on M with $\mathrm{d}t(K) = 1$, w is a function of one variable, g_Σ is a metric on 2-manifold Σ with area form ω_Σ, α is a 1-form on Σ with $\mathrm{d}\alpha = a\omega_\Sigma$, and a, b are constant. Conversely equations (46)–(47) *define a Kähler structure of Calabi type with $K = \partial/\partial t$, for any g_Σ and V.*

Proof. The proof follows LeBrun's description [37] of Kähler metrics with a Hamiltonian Killing vector field K. Supposing first that (g, J, ω) is only almost Hermitian, with a Killing vector field $K = J\mathrm{grad}z$, and that $K - iJK$ is holomorphic, so that the complex quotient is locally a Riemann surface Σ. Introducing a local holomorphic coordinate $x + iy$ on Σ, we may write

$$g = \mathrm{e}^u w(\mathrm{d}x^2 + \mathrm{d}y^2) + w\,\mathrm{d}z^2 + w^{-1}(\mathrm{d}t + \alpha)^2,$$
$$J\mathrm{d}x = \mathrm{d}y, \qquad J\mathrm{d}z = w^{-1}(\mathrm{d}t + \alpha),$$
$$\omega = \mathrm{e}^u w\,\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}z \wedge (\mathrm{d}t + \alpha).$$

where $\mathrm{d}t(K) = 1$, α is an invariant 1-form with $\alpha(K) = 0$, and u, w are functions of x, y, z. The almost Hermitian structure I is given by

$$I\mathrm{d}x = -\mathrm{d}y, \qquad I\mathrm{d}z = w^{-1}(\mathrm{d}t + \alpha),$$

with Kähler form

$$\omega_I = -\mathrm{e}^u w\,\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}z \wedge (\mathrm{d}t + \alpha).$$

We now impose the condition that (g, J, ω) and $(\bar{g} = \lambda^{-2}g, I, \bar{\omega} = \lambda^{-2}\omega_I)$ are Kähler for some nonvanishing function λ. Now $\mathrm{d}\omega = 0$ if and only if

$$(\mathrm{e}^u w)_z \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}z \wedge \mathrm{d}\alpha, \tag{48}$$

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while $\lambda^{-2}\omega_I$ is closed if and only if

$$\lambda\, \mathrm{d}z \wedge \mathrm{d}\alpha + ((\mathrm{e}^u w)_z \lambda - 2\lambda_z \mathrm{e}^u w)\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y + \\ + 2\lambda_x \mathrm{d}x \wedge \mathrm{d}z \wedge (\mathrm{d}t + \alpha) + 2\lambda_y \mathrm{d}y \wedge \mathrm{d}z \wedge (\mathrm{d}t + \alpha) + 2\lambda_t \mathrm{d}t \wedge \omega_I = 0. \quad (49)$$

In the presence of (48), (49) is equivalent to

$$(\mathrm{e}^u w)_z \lambda = \lambda_z \mathrm{e}^u w, \qquad \lambda_x = \lambda_y = \lambda_t = 0, \quad (50)$$

which holds if and only if $\lambda = \lambda(z)$ and $\mathrm{e}^u w = h\lambda$ for some function $h(x, y)$.

Let θ be the complex 1-form $w\, \mathrm{d}z + i(\mathrm{d}t + \alpha)$. Then, since $\mathrm{d}x \pm i\mathrm{d}y$ is closed, the complex structures I and J are integrable if and only if $\mathrm{d}\theta$ belongs to the ideals generated by $\{\theta, \mathrm{d}x - i\mathrm{d}y\}$ and by $\{\theta, \mathrm{d}x + i\mathrm{d}y\}$ respectively. Since $\mathrm{d}\theta(\partial/\partial t, \cdot) = 0$, these conditions force $(\mathrm{d}x - i\mathrm{d}y) \wedge \mathrm{d}\theta$ and $(\mathrm{d}x + i\mathrm{d}y) \wedge \mathrm{d}\theta$ to vanish. Hence I is integrable if and only if

$$\mathrm{d}\alpha = -w_x \mathrm{d}y \wedge \mathrm{d}z + w_y \mathrm{d}x \wedge \mathrm{d}z + f\mathrm{d}x \wedge \mathrm{d}y \quad (51)$$

while J is integrable if and only if

$$\mathrm{d}\alpha = w_x \mathrm{d}y \wedge \mathrm{d}z - w_y \mathrm{d}x \wedge \mathrm{d}z + f\mathrm{d}x \wedge dy; \quad (52)$$

here f is an arbitrary function. Hence I and J are both integrable if and only if

$$w_x = w_y = 0 \quad \text{and} \quad \mathrm{d}\alpha = f\mathrm{d}x \wedge \mathrm{d}y \quad (53)$$

and f is necessarily a function of x, y only.

Putting together (48), (50), and (53), we see that (g, J, ω) is of Calabi type, with Killing vector field K, if and only if $\mathrm{e}^u w = h(x, y)\lambda(z)$, with $\mathrm{d}\alpha = h(x, y)\lambda_z$, $\lambda_{zz} = 0$ and $w_x = w_y = 0$, so that $\lambda = az - b$ for constants a, b and $w = w(z)$.

Using the freedom in the choice of t, we may then assume α is a 1-form on Σ, while $g_\Sigma = h(x, y)(\mathrm{d}x^2 + \mathrm{d}y^2)$ is a metric on Σ, and the result follows. □

This Proposition shows that Kähler metrics of Calabi type are essentially the same as metrics arising from the well-known *Calabi construction* [14] of metrics on the total space of a Hermitian line bundle over a Riemann surface. In this interpretation, the Killing vector field K generates the natural circle action on the line bundle and the Kähler form is

$$\omega_\Sigma + \mathrm{d}J\mathrm{d}f \quad (54)$$

for a function f of the fibre norm r. Since $-\mathrm{d}J\mathrm{d}\log r$ is the curvature of the line bundle, which is basic, the momentum map z of K is also a function of r. Hence we may locally view f as a function of z, so that, if we write $J\mathrm{d}z = \mathrm{d}t + \alpha$ where $\mathrm{d}t(K) = 1$ and α is basic,

$$\mathrm{d}J\,\mathrm{d}f = \left(\frac{f'(z)}{w(z)}\right)_z \mathrm{d}z \wedge (\mathrm{d}t + \alpha) + \frac{f'(z)}{w(z)}\mathrm{d}\alpha. \quad (55)$$

Therefore, in Proposition 13, setting $b = -1$ without loss of generality, we have $f'(z) = zw(z)$.

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The metrics of Proposition 13 certainly admit a Hamiltonian 2-form, namely $\varphi = (az - b)\omega_I + 3az\omega$. Hence, $\sigma = 2az$ and $\lambda = az - b$, so that $\xi = 2az - b$ and $\eta = b$. The Hamiltonian Killing vector fields associated to φ both vanish when $a = 0$. On the other hand, for a nonzero, we can use the freedom in the choice of z to set $a = 1$ and $b = 0$.

THEOREM 3. *A Kähler surface is of Calabi type if and only if either:*

(i) *it is locally a Kähler product of two Riemann surfaces, one of which admits a Killing vector field; or*
(ii) *it admits a Hamiltonian 2-form whose associated Killing vector fields are dependent but not both zero.*

The Kähler structure is then given explicitly by the Calabi construction (46)–(47)*: in case* (i) $a = 0$*, while in case* (ii) *we may take* $a = 1$*,* $b = 0$ *without loss of generality.*

3.2. WEAKLY SELF-DUAL KÄHLER SURFACES OF CALABI TYPE

We begin this section by computing the curvature of a Kähler surface of Calabi type which is not a local Kähler product. Therefore we set $a = 1$, $b = 0$, and write $w(z) = z/V(z)$, so that the Kähler structure is

$$g = zg_\Sigma + \frac{z}{V(z)}\mathrm{d}z^2 + \frac{V(z)}{z}(\mathrm{d}t + \alpha)^2, \tag{56}$$

$$\omega = z\,\omega_\Sigma + \mathrm{d}z \wedge (\mathrm{d}t + \alpha), \tag{57}$$

As with ortho-toric Kähler surfaces, the curvature is entirely determined by the scalar curvature s of g, the conformal scalar curvature κ of the Hermitian pair (g, I), and the trace-free part ρ_0 of the Ricci form of (g, J).

LEMMA 10. *For a nonproduct Kähler surface* (M, g, J, ω) *of Calabi type, given by* (56)–(57)*,* ρ_0 *is a multiple* μ *of the Kähler form* ω_I *of the Hermitian pair* (g, I)*, and* μ, s, κ *are given by*

$$\mu = -\frac{1}{4z}\left(s_\Sigma + \left(\frac{V_z}{z^2}\right)_z z^2\right), \tag{58}$$

$$s = \frac{s_\Sigma - V_{zz}}{6z}, \tag{59}$$

$$\kappa = \frac{1}{6z}\left(s_\Sigma - z^2\left(z^2\left(\frac{V}{z^4}\right)_z\right)_z\right), \tag{60}$$

where s_Σ *is the scalar curvature of* Σ*. In particular, on the open subset of* M *where* μ *has no zero, the anti-self-dual almost-complex structure determined by* ρ_0 *is equal to* I*.*

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Proof. The Ricci form ρ is given by $\rho = \rho_\Sigma - \frac{1}{2}\mathrm{d}J\mathrm{d}\log V$. The first term is $\frac{1}{2}s_\Sigma\omega_\Sigma$, and we compute the second term as follows:

$$\mathrm{d}J\,\mathrm{d}\log V = \mathrm{d}\left(\frac{V_z}{V}J\mathrm{d}z\right) = \mathrm{d}\left(\frac{V_z}{z}(\mathrm{d}t+\alpha)\right)$$
$$= \frac{V_{zz}}{z}\mathrm{d}z\wedge(\mathrm{d}t+\alpha) + V_z\left(\frac{1}{z}\omega_\Sigma - \frac{1}{z^2}\mathrm{d}z\wedge(\mathrm{d}t+\alpha)\right).$$

Evidently we may write this as a linear combination of ω and ω_I, and we readily obtain (58) and (59).

The conformal scalar curvature is most easily computed by noticing that the conformal Kähler metric $\bar{g} = z^{-2}g$ is also of Calabi type, with $\bar{z} = 1/z$ and $\bar{V}(\bar{z}) = \bar{z}^4V(1/\bar{z}) = V(z)/z^4$. Hence, its scalar curvatures are

$$\frac{s_\Sigma - \bar{V}_{\bar{z}\bar{z}}}{6\bar{z}} = \frac{z}{6}\left(s_\Sigma - z^2\left(z^2\left(\frac{V}{z^4}\right)_z\right)_z\right)$$

from which (60) follows, since $\kappa = z^{-2}\bar{s}$. □

PROPOSITION 14. *Let M be a nonproduct Kähler surface M of Calabi type, with Killing vector field K. Then the scalar curvature of M is a momentum map for a multiple of K if and only if g_Σ has constant curvature k and V is of the form*

$$V(z) = A_1z^4 + A_2z^3 + kz^2 + A_3z + A_4. \tag{61}$$

Any Kähler surface given by (56)–(57) *is extremal, with Ricci form $\mu\omega_I + \frac{3}{2}s\omega$, where*

$$\mu = -A_1z + \frac{A_3}{2z^2}, \tag{62}$$

$$s = -2A_1z - A_2.; \tag{63}$$

also the conformal scalar curvature of (g, I) is

$$\kappa = -\frac{A_3}{z^2} - \frac{2A_4}{z^3}. \tag{64}$$

Hence:

(i) *g has constant scalar curvature if and only if $A_1 = 0$;*
(ii) *g is scalar-flat (i.e., anti-self-dual) if and only if $A_1 = A_2 = 0$.*
(iii) *(g, J) is Kähler–Einstein if and only if $A_1 = A_3 = 0$;*
(iv) *g is weakly self-dual if and only if $A_3 = 0$.*
(v) *g is self-dual if and only if $A_3 = A_4 = 0$.*
(vi) *(g, J) is bi-extremal if and only if g is weakly self-dual;*
(vii) *the Bach tensor of g vanishes if and only if $4A_1A_4 - A_2A_3 = 0$.*

Proof. From (59), we have

$$6zs + V_{zz} = s_\Sigma.$$

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If s is an affine function of z, then both sides of this equation must be constant. Hence $s_\Sigma = 2k$ and V must be a quartic in z with quadratic term kz^2. The formulae for μ, s and κ are immediate, as are (i)–(v). (For (iv) we use the fact that I is integrable and $z^{-2}\omega_I$ is closed, so that weak self-duality is equivalent to the equation $\mathrm{d}s = 2\mathrm{d}\mu$.)

(vi) The Pfaffian p of the normalized Ricci form is given by

$$p = \left(-2A_1 z - \tfrac{1}{2}A_2 + \frac{A_3}{2z^2}\right)\left(-\tfrac{1}{2}A_2 - \frac{A_3}{2z^2}\right) \tag{65}$$

in particular, p is a rational function of z; since z is a holomorphy potential, p is a holomorphy potential if and only if it is an affine function of z, if and only if $A_3 = 0$; p is then equal to $-\frac{1}{2}A_2(s + \frac{1}{2}A_2)$.

(vii) For any extremal Kähler surface the Bach tensor is J-invariant and the corresponding anti-self-dual 2-form, is expressed by (21) which yields

$$\tilde{\boldsymbol{B}} = \frac{(4A_1A_4 - A_2A_3)}{z^2}\omega_I. \qquad \square$$

This family of extremal Kähler metrics has been considered in many places. In particular, it includes the extremal Kähler metrics of cohomogeneity one under $U(2)$ constructed by Calabi in [14]; more generally, it turns out that these metrics all have cohomogeneity one under a (local) action of a four-dimensional Lie group, locally isomorphic to a central extension of the isometry group of a surface of constant curvature k. We refer to these metrics as *extremal Kähler surfaces of Calabi type* and briefly recall how they may be realized as diagonal Bianchi metrics, of class IX, VIII or II, according to whether k is positive, negative or zero.

Up to rescaling, we can—and will—assume that $k = \varepsilon$, with $\varepsilon = 1, 0$ or -1. As is well known, we can now write $\mathrm{d}t + \alpha = \sigma_3$ and introduce t-dependent 1-forms σ_1, σ_2 on Σ with $g_\Sigma = \sigma_1^2 + \sigma_2^2$, $\sigma_1 \wedge \sigma_2 = \omega_\Sigma$ and

$$\mathrm{d}\sigma_1 = \varepsilon\sigma_2 \wedge \sigma_3, \quad \mathrm{d}\sigma_2 = \varepsilon\sigma_3 \wedge \sigma_1, \quad \mathrm{d}\sigma_3 = \sigma_1 \wedge \sigma_2. \tag{66}$$

By substituting $\sigma_1, \sigma_2, \sigma_3$ in (56)

$$g = \frac{z}{V(z)}\mathrm{d}z^2 + z(\sigma_1^2 + \sigma_2^2) + \frac{V(z)}{z}\sigma_3^2; \tag{67}$$

the complex structure J is determined by

$$J\sigma_1 = \sigma_2, \qquad J\,\mathrm{d}z = \frac{V(z)}{z}\sigma_3 \tag{68}$$

while the Kähler form ω is

$$\omega = \mathrm{d}z \wedge \sigma_3 + z\sigma_1 \wedge \sigma_2 = \mathrm{d}(z\sigma_3). \tag{69}$$

We here recognize *bi-axial diagonal Bianchi metrics* of class IX, VIII or II, according as ε is equal 1, -1 or 0; these admit a cohomogeneity one local action of SU(2) if $\varepsilon = 1$, of SU(1, 1) if $\varepsilon = -1$, of the Heisenberg group Nil if $\varepsilon = 0$, and the orbits are level sets of z. This can be seen as follows: denote by $(Z_1, Z_2, Z_3 = K_1)$ the triplet of

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vector fields determined by $\sigma_i(Z_j) = \delta_{ij}$ and $\mathrm{d}z(Z_i) = 0$, $i, j = 1, 2, 3$, where δ_{ij} is the Kronecker symbol. Then, for each value of z, Z_1, Z_2, Z_3 are tangent to the corresponding orbit M_z and generate a Lie algebra isomorphic to $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$ or $\mathfrak{nil}^3$ according as ε is equal to 1, -1 or 0; therefore, each orbit can be locally identified to the corresponding Lie group and we can locally construct a new triple of independent vector fields, $(\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3)$, such that $[\tilde{Z}_i, Z_j] = 0$ and $\mathrm{d}z(\tilde{Z}_i) = 0$, $i, j = 1, 2, 3$ (for each orbit, if (Z_1, Z_2, Z_3) is a basis of left-invariant vector fields, $(\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3)$ is a basis of right-invariant vector fields); $\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$ are clearly Killing with respect to g and all commute with K_1.

If $\varepsilon \neq 0$, $K_1, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$ generate a four-dimensional Lie algebra, corresponding to a local action of $U(2)$ if $\varepsilon = 1$, of $U(1,1)$ if $\varepsilon = -1$. If $\varepsilon = 0$, $\tilde{Z}_3$ equals K_1, up to a constant factor; on the other hand, we get an additional Killing vector field $\tilde{K}_1$ generated by the rotations around the origin in the Euclidean 2-plane $\mathbb{E}^2$ of x, y; then, $\tilde{K}_1, K_1, \tilde{Z}_1, \tilde{Z}_2$ generate a four-dimensional Lie algebra, say $\mathfrak{g}$, corresponding to a local action of the group, G, obtained by forming the semi-direct product of Nil by S^1 for the natural action of S^1 on Nil by (outer) automorphisms; the centre of $G = \mathrm{Nil} \ltimes S^1$ coincides with the centre of Nil; the latter is one-dimensional again and the quotient of G by its center is isomorphic to $\mathrm{Isom}(\mathbb{E}^2)$; in other words, G is isomorphic to a (nontrivial) one-dimensional central extension of $\mathrm{Isom}(\mathbb{E}^2)$.

If we concentrate our attention on weakly self-dual Kähler surfaces, we readily infer from the foregoing:

THEOREM 4. *Let (M, g, J) be a weakly self-dual Kähler surfaces. Denote by s the scalar curvature and by p the Pfaffian of the normalized Ricci form; let $K_1 = J\,\mathrm{grad}_g s$ and $K_2 = J\,\mathrm{grad}_g p$ be the associated Killing vector fields and assume that K_1 has no zero and that $K_2 = bK_1$, where b is a real constant (possibly zero).*

(i) *Then (M, g, J) admits a local action of cohomogeneity one of $G = U(2), U(1,1)$ or $\mathrm{Nil} \ltimes S^1$ and is locally isomorphic to a diagonal Bianchi metric of class IX, VIII or II respectively.*

More precisely, let $\sigma_1, \sigma_2, \sigma_3$ denote the (local) 1*-forms on M induced by this action, corresponding to a triple of G-invariant* 1*-forms on G, so that $\mathrm{d}\sigma_3 = \sigma_1 \wedge \sigma_2$, $\mathrm{d}\sigma_2 = \varepsilon\sigma_3 \wedge \sigma_1$, $\mathrm{d}\sigma_1 = \varepsilon\sigma_2 \wedge \sigma_3$, where $\varepsilon = 1, -1$ or 0 according as $G = U(2)$, $U(1,1)$ or $\mathrm{Nil} \ltimes S^1$; then, the Kähler structure (g, J) can be put in the form* (67)–(68)*, where z is an affine function of s and $V(z)$ is of the form*

$$V(z) = A_1 z^4 + A_2 z^3 + \varepsilon z^2 + A_4. \tag{70}$$

(ii) *Conversely, each Kähler surface of the form* (67)–(68)*, with V given by* (70) *is weakly self-dual.*

(iii) *This Kähler structure is self-dual if and only if $A_4 = 0$.*

Remark 6. By substituting $A_3 = 0$ in (62) and (64), we readily infer that A_4 and the constant c appearing in (23) and (24) are linked together by

$$c = 2A_1^3 A_4; \tag{71}$$

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moreover, $A_1 \neq 0$, as s is nonconstant, so that $A_4 = 0$ if and only if $c = 0$; as we already know, both conditions are equivalent to g being self-dual. □

3.3. THE WEAKLY SELF-DUAL KÄHLER METRIC ON THE FIRST HIRZEBRUCH SURFACE

We close this section by providing an example of a *compact* weakly self-dual Kähler surface (M, g, J); this belongs to the family of extremal metrics constructed in [14] by E. Calabi on the first *Hirzebruch surface* F_1, viewed as the compactification of the total space of the tautological line bundle $L = \mathcal{O}(-1)$ over the complex projective line $\mathbb{C}P^1 = \mathbb{P}(\mathbb{C}^2)$ obtained by adding a *section at infinity*, say C_∞; then, the zero section, C_0, has self-intersection -1 and F_1 can also be considered as a blown up of the projective plane $\mathbb{C}P^2$ at some point, with exceptional divisor C_0, and $F_1 - C_0 = \mathbb{C}^2 - \{(0,0)\}$.

We here recall the main features of Calabi's construction from [14]. Let r be the usual norm in $\mathbb{C}^2$ and, for convenience, introduce the function t defined on $\mathbb{C}^2 - \{(0,0)\}$ by $\mathrm{e}^t = r^2$; the group $U(2)$ naturally acts on F_1, by preserving C_0 and C_∞; by [15], the connected group of isometries, $\mathrm{Isom}_0(M, g)$, of any compact extremal Kähler surface (M, g, J) is a maximal compact subgroup of the connected group of holomorphic transformations $\mathrm{Aut}_0(M, J)$; it follows that any extremal metric on F_1 is isometric to a U(2)-invariant metric, in particular has a globally defined potential on the open set $M_0 = F_1 - C_0 = \mathbb{C}^2 - \{(0,0)\}$ of the form $u = u(t)$; the Kähler form is thus given by $\omega = \psi(t)\mathrm{d}\mathrm{d}^c t + \psi'(t)\mathrm{d}t \wedge \mathrm{d}^c t$, where $\psi(t) := u'(t)$; conversely, any such Kähler metric extends to F_1 if and only if $\psi(t)$ extends to a C^∞ function of e^t in the neighbourhood of $t = -\infty$ and to a C^∞ function of e^{-t} in the neighbourhood of $t = \infty$; in particular, ψ has a limit a when $t \to -\infty$ and a limit, b, when $t \to \infty$,with $0 < a < b$; the corresponding Kähler class $[\omega]$ is then equal to $4\pi(-a[C_0] + b[C_\infty])$, where $[C_0]$ and $[C_\infty]$ denotes the Poincaré dual of C_0 and similarly for $[C_\infty]$; conversely, each Kähler class on F_1 is of this form, for some pair $0 < a < b$; it is easily checked that the Ricci for is given by $\rho = \mathrm{d}\mathrm{d}^c v$, with $v = t - \frac{1}{2}\log\psi - \frac{1}{2}\log\psi'$, so that $s = 2((v'/\psi) + (v''/\psi'))$; on the other hand, such a metric is extremal if and only if s is an affine function of ψ; by easy successive partial integrations, we infer that this metric is extremal if and only if ψ satisfies the following differential relation: $\psi\psi' = V(\psi)$, where V is a polynomial of the form (61); moreover, the extremal metric is actually defined on F_1 and has its Kähler class parameterized by the pair (a, b) as above if and only if the coefficients of V are given by

$$\begin{aligned} A_1 &= \frac{-2a}{(b-a)(a^2+4ab+b^2)}, & A_2 &= \frac{(3a^2-b^2)}{(b-a)(a^2+4ab+b^2)}, \\ A_3 &= \frac{ab(3a^2-b^2)}{(b-a)(a^2+4ab+b^2)}, & A_4 &= \frac{-2a^3b^2}{(b-a)(a^2+4ab+b^2)}; \end{aligned} \tag{72}$$

we thus get an extremal Kähler metric, say $g_{(a,b)}$, in each Kähler class of F_1 (we have a similar construction for the other Hirzebruch surfaces F_k [14]; the coefficients of V are then given by (76) below).

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All these metrics are of cohomogeneity one under the action of $U(2)$ and can also be put in the form (67)–(68), with the same parameter ψ, the same polynomial $V(\psi)$, $\mathrm{d}^c t = \sigma_3$ and $\mathrm{d}\mathrm{d}^c t = \sigma_1 \wedge \sigma_2$; in particular, according to Proposition 14, $g_{(a,b)}$ if weakly self-dual if and only if $A_3 = 0$, and this happens if and only if a and b are related by

$$b^2 = 3a^2; \tag{73}$$

in this case, $A_2 = 0$ as well, meaning that $p = 0$, and $V(\psi) = A_1\psi^4 + \psi^2 + A_4$ is a function of ψ^2; in particular, $\psi\psi' = V(\psi)$ is easily integrated into

$$\psi = a\left(\frac{1 + 3\mathrm{e}^{t+t_0}}{1 + \mathrm{e}^{t+t_0}}\right)^{\frac{1}{2}} = a\left(\frac{1 + 3\mathrm{e}^{t_0} r^2}{1 + \mathrm{e}^{t_0} r^2}\right)^{\frac{1}{2}}, \tag{74}$$

where t_0 is a constant; the latter can be made equal to zero without loss of generality by a mere translation of the parameter t, i.e. a rescaling of r; then, up to rescaling, the Kähler form is given by

$$\omega = \frac{(1 + 3\mathrm{e}^t)^{\frac{1}{2}}}{(1 + \mathrm{e}^t)^{\frac{1}{2}}} \mathrm{d}\mathrm{d}^c t + \frac{\mathrm{e}^t}{(1 + \mathrm{e}^t)^{\frac{3}{2}}(1 + 3\mathrm{e}^t)^{\frac{1}{2}}} \mathrm{d}t \wedge \mathrm{d}^c t. \tag{75}$$

In the sequel, the Kähler metric given by (75) will be referred to as the *Calabi weakly self-dual Kähler metric* of F_1. It may be noticed that, according to (72), the scalar curvature $s = -2A_1\psi$ is nonconstant and (strictly) positive.

In the next section we show that, conversely, each compact weakly self-dual Kähler surface with nonconstant scalar curvature is, up to rescaling, isomorphic to the first Hirzebruch surface equipped with the Calabi weakly self-dual Kähler metric.

4. Compact Weakly Self-dual Kähler Surfaces

Compact *self-dual* Kähler surfaces have been described by B.-Y. Chen in [20]: these are locally symmetric, hence of constant holomorphic sectional curvature or, locally the product of Riemann surfaces of opposite constant curvature (see [13] for a higher-dimensional generalization).

In [32], W. Jelonek proved that compact real analytic weakly self-dual Kähler surfaces are either Kähler–Einstein, or locally the product of two Riemann surfaces of constant Gauss curvatures, or biholomorphic to a ruled surface.

We show that the hypothesis of real analyticity can actually be removed and that, except in the case when the scalar curvature is constant, the only weakly self-dual Kähler ruled surface is the first Hirzebruch surface F_1 equipped with the Calabi weakly self-dual Kähler metric, as described in the preceding section (up to rescaling). More precisely, we have the following result.

THEOREM 5. *Let (M, g, J) be a compact weakly self-dual Kähler surface. Then (M, g, J) is either*

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(i) *Kähler–Einstein; or*
(ii) *locally isomorphic to the product of two Riemann surfaces of constant Gauss curvatures, or*
(iii) *up to rescaling, isomorphic to the first Hirzebruch surface F_1 equipped with a Calabi weakly self-dual Kähler metric.*

Proof. By Proposition 6, we know that g is either self-dual, hence, by the above mentioned result of B.-Y. Chen, described by (i) or (ii), or of constant scalar curvature, hence, again, described by (i) or (ii), or of nonconstant scalar curvature. In the latter case, the (negative) Kähler structure $(\bar{g}, I)$ is globally defined; it then follows from a result of Kotschick [36] that the signature of M is zero [36]; moreover, since the (real) holomorphic field $K_1 = J\,\mathrm{grad}_g\, s$ has nonempty zero set, we know by [19] that the Kodaira dimension of the Kähler surface (M, J) is $-\infty$, hence (M, J) is a ruled surface which is the projectivization $\mathbb{P}(E)$ of a rank 2 holomorphic vector bundle E over a compact complex curve Σ [11].

If $\Sigma = \mathbb{C}P^1$, (M, J) is a *Hirzebruch surface* $F_k = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-k))$, where k is a positive integer, or the product $\mathbb{C}P^1 \times \mathbb{C}P^1$; the only extremal Kähler metrics of $\mathbb{C}P^1 \times \mathbb{C}P^1$ are the (symmetric) product metrics, which are of constant scalar curvature; on the other hand, any maximal compact subgroup of $\mathrm{Aut}_0(M, J)$ is conjugate to U(2), and therefore any extremal Kähler metric must be a cohomogeneity-one U(2) metric [15], hence, locally of the form (67) with $\varepsilon = 1$ (cf. the end of the preceding section for the case when $k = 1$); as shown by E. Calabi in [14], for each $k > 0$, any Kähler class of F_k carries a unique extremal Kähler metric (up to a reparameterization); each one can be put in the form (67), where the polynomial V, in the notation of (61), is determined by

$$\begin{aligned} A_1 &= \frac{(k+1)a + (k-1)b}{(b-a)(a^2+b^2+4ab)}, & A_2 &= \frac{(2-k)b^2 - (k+2)a^2}{(b-a)(a^2+b^2+4ab)}, \\ A_3 &= \frac{ab((2-k)b^2 - (k+2)a^2)}{(b-a)(a^2+b^2+4ab)}, & A_4 &= \frac{a^2b^2((k+1)a + (k-1)b)}{(b-a)(a^2+b^2+4ab)}, \end{aligned} \tag{76}$$

where $0 < a < b$ are the parameters of the Kähler class; according to Proposition 14, g is weakly self-dual precisely when $A_3 = 0$; in the present situation, this is equivalent to $A_2 = 0$ and happens if and only if $k = 1$ and $\mu := a/b = 1/\sqrt{3}$, i.e. if (M, g, J) is the first Hirzebruch surface equipped with a Calabi weakly self-dual Kähler metric.

We now show that a compact ruled surface (M, J) whose base Σ is a compact complex curve of genus $\mathbf{g}(\Sigma)$ at least 1 does not carry weakly self-dual Kähler metrics of non-constant scalar curvature. We thus assume that $(M, J) = \mathbb{P}(E)$ carries a weakly self-dual Kähler metric of non-constant scalar curvature to get a contradiction.

Using an argument from [38, 49], we first observe that the rank two vector bundle E splits as $E = \mathcal{O} \oplus L$, where $\mathcal{O}$ stands for the trivial holomorphic line bundle and L is a holomorphic line bundle L of degree $\deg(L) > 0$. Indeed, recall the already mentioned result of E. Calabi [15] that the connected component of the isometry group $\mathrm{Isom}_0(M, g)$ is a maximal compact subgroup in $\mathrm{Aut}_0(M, J)$; according to

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M. Maruyama [41], the group of automorphisms of ruled surfaces can be described as follows: If $\mathbf{g}(\Sigma) \geqslant 1$, there exists an exact sequence

$$\{\mathbf{1}\} \to \mathrm{Aut}_\Sigma(\mathbb{P}(E)) \to \mathrm{Aut}(\mathbb{P}(E)) \to \mathrm{Aut}(\Sigma), \tag{77}$$

where $\mathrm{Aut}_\Sigma(\mathbb{P}(E))$ denotes the group of relative automorphisms of the bundle $\mathbb{P}(E) \to \Sigma$, and $\mathrm{Aut}(\Sigma)$ is the group of automorphisms of Σ (of course, $\mathrm{Aut}(\Sigma)$ is finite if $\mathbf{g}(\Sigma) \geqslant 2$); on the other hand, the (nontrivial) homomorphic vector field $\Xi_1 = K_1 - iJK_1$ whose real part is the Killing vector field $K_1 = J\,\mathrm{grad}_g\, s$ has a nonempty zero set; since Ξ_1 preserves the (unique) ruling $M = \mathbb{P}(E) \to \Sigma$, it projects onto a holomorphic vector field on the base Σ; since Ξ_1 has at least one zero, the induced vector field on Σ vanishes; it follows that Ξ_1 is tangent to the $\mathbb{C}P^1$ fibers (equivalently, Ξ_1 belongs to the Lie algebra of $\mathrm{Aut}_\Sigma(\mathbb{P}(E))$); this shows that the kernel of the group homomorphism f: $\mathrm{Isom}_0(M, g) \to \mathrm{Aut}_0(\Sigma)$ induced by the exact sequence (77) is a nontrivial compact subgroup of $\mathrm{Aut}_\Sigma(\mathbb{P}(E))$; one can therefore find an S^1 in the connected component of the identity of $\mathrm{Aut}_\Sigma(\mathbb{P}(E))$; denote by Ξ_0 the induced holomorphic vector field, such that the imaginary part $K_0 = \mathrm{Im}(\Xi_0)$ generates the S^1-action, whereas Ξ_0 itself generates a $\mathbb{C}^*$-action; as a matter of fact, Ξ_0 can be identified to a traceless holomorphic section of $\mathrm{End}(E)$, say $\mathbf{s}$; note that $\mathbf{s}$ is of constant determinant; since $K_0 = \mathrm{Im}(\Xi_0)$ generates a periodic S^1-action, $\mathbf{s}$ must be diagonalizable; this shows that we have a holomorphic splitting of E into eigensubbundles of $\mathbf{s}$; by twisting by a line bundle, we obtain the splitting $E = \mathcal{O} \oplus L$, where $\deg(L) \geqslant 0$; then, Ξ_0 is nothing but the Euler vector field of L. If the degree of L is zero, then any Kähler class contains a locally symmetric Kähler metric [51], so that any extremal Kähler metric on (M, J) is of constant scalar curvature [14], a contradiction; we thus obtain a splitting $E = \mathcal{O} \oplus L$ where L is a holomorphic line bundle of $\deg(L) > 0$.

As $\mathrm{Isom}_0(M, g)$ is a maximal compact subgroup in $\mathrm{Aut}_0(M, J)$, we may assume [49] that (up to a biholomorphism) the metric g is invariant under the fixed S^1 action generated by $K_0 = \mathrm{Im}(\Xi_0)$. For any nontrivial Killing vector field, K, which arises from a real holomorphy potential, the argument already used above shows that $\Xi = K - iJK$ must be tangent to the fibers, and therefore $\Xi \wedge \Xi_0 = 0$. In particular, we get that $K_0 = fK_1 + hJK_1$, where f, h are smooth functions defined on an open dense subset of M where $K_1 = J\,\mathrm{grad}_g s \neq 0$. But $\langle K_0, JK_1\rangle = -\mathrm{d}s(K_0) = -\mathcal{L}_{K_0} s = 0$, i.e. $h = 0$, and therefore f is a constant. By rescaling the metric if necessary we may assume therefore that $K_1 = K_0 = \mathrm{Im}(\Xi_0)$. Similarly, $K_2 = J\,\mathrm{grad}_g p$ must be a constant multiple of K_1 and by Theorem 4, g must be locally of cohomogeneity one, i.e., g can be written of the form (67) on an open dense subset of M.

Note that M contains exactly two curves fixed by the $\mathbb{C}^*$-action generated by Ξ_0, corresponding to the zero and infinity sections, C_0 and C_∞, of $M = \mathbb{P}(\mathcal{O} \oplus L)$; moreover, the function z appearing in (67) makes sense on the whole of M as being a momentum map of the corresponding S^1-action (up to multiplication by a nonzero constant); it then follows that z: $M \to \mathbb{R}$ maps M onto an interval $[a, b]$, such that z is regular on $M - (z^{-1}(a) \cup z^{-1}(b))$; therefore, for any $t_0 \in (a, b)$, $\Sigma = z^{-1}(t_0)/S^1$,

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whereas $C_0 = z^{-1}(a)$ and $C_\infty = z^{-1}(b)$ (see [38]). By using an argument from [38, p. 42], it is shown that $q = |K_1|^2$ is a smooth function on $\Sigma \times [a, b]$, which satisfies the boundary conditions

$$q(., a) = q(., b) = 0; \qquad \left(\frac{\partial}{\partial z} q\right)(., a) = -\left(\frac{\partial}{\partial z} q\right)(., b) = k, \tag{78}$$

where k is a real constant; for Calabi's metrics (67) one has $q = (V(z))/z$, so that the Equations (78) read

$$V(a) = V(b) = 0; \qquad V'(a) = ka; \qquad V'(b) = -kb;$$

we thus obtain the following values for the coefficients A_1, A_2, A_3, A_4 of V (notations of (61)):

$$\begin{aligned} A_1 &= \frac{k(a+b)+\varepsilon(a-b)}{(b-a)(a^2+4ab+b^2)}, & A_2 &= \frac{-k(a^2+b^2)+2\varepsilon(b^2-a^2)}{(b-a)(a^2+4ab+b^2)}, \\ A_3 &= \frac{ab(-k(a^2+b^2)+2\varepsilon(b^2-a^2))}{(b-a)(a^2+4ab+b^2)}, & A_4 &= \frac{a^2b^2(k(a+b)+\varepsilon(a-b))}{(b-a)(a^2+4ab+b^2)}; \end{aligned} \tag{79}$$

according to Proposition 14 we have $A_3 = 0$, and from (79) we also get $A_2 = 0$; by (63) and (62) it follows that the Ricci tensor of g has two distinct eigenvalues, equal to $s/6$ and $s/3$, respectively; by Proposition 6, Ric_0 nowhere vanishes on M, meaning that the scalar curvature s nowhere vanishes as well; since K_1 is a non-trivial Killing vector field, s must be everywhere positive; this shows that the first Chern class $c_1(M)$ of (M, J) is positive, and therefore $c_1^2(M) > 0$, a contradiction [11].

Remark 7. The case (ii) of Theorem 5 includes in particular ruled surfaces $\mathbb{P}(E)$ over a Riemann surface Σ of genus $g \geqslant 1$ when the holomorphic vector bundle E is stable or polystable, cf., e.g., [51].

5. Weakly Self-dual Almost Kähler Manifolds

5.1. THE MATSUMOTO–TANNO IDENTITY FOR ALMOST KÄHLER 4-MANIFOLDS

Recall that an *almost Kähler* manifold is an almost Hermitian manifold (M, g, J, ω) for which the Kähler 2-form ω is closed. The almost complex structure J of an almost Kähler manifold is not integrable in general; if it is, we obtain a Kähler manifold.

We would like to identify the Ricci tensor Ric of an almost Kähler manifold with a 2-form ρ, the Ricci form, as in the Kähler case. However, only the J-invariant part of Ric defines a 2-form, whereas on a (nonintegrable) almost Kähler manifold, the Ricci tensor is not in general J-invariant. We shall therefore impose J-invariance as an extra requirement.

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Throughout this section we will always assume that (M, g, J) is an almost Kähler 4-manifold whose Ricci tensor is J-invariant, i.e., $\mathrm{Ric}(J\cdot, J\cdot) = \mathrm{Ric}(\cdot, \cdot)$. We then adopt the notations of Section 1, and, in analogy with the Kähler case, we consider the type (1,1) *Ricci form*, ρ, of (M, g, J) defined by $\rho(\cdot, \cdot) = \mathrm{Ric}(J\cdot, \cdot)$; the anti-self-dual part of ρ is denoted by ρ_0. It is a remarkable fact [22] that even though J is not integrable, ρ is still a closed (1,1)-form (although it is no longer a representative of $1/2\pi c_1^{\mathbb{R}}$). Using this observation, the proof of Lemma 1 easily extends to the case of almost Kähler 4-manifolds with J-invariant Ricci tensor.

LEMMA 11. *For any almost Kähler 4-manifold with J-invariant Ricci tensor the identity (1) is satisfied. In particular, the anti-self-dual Weyl tensor W^- of such a manifold is harmonic if and only if the Matsumoto-Tanno identity (2) is satisfied.*

Proof. The proof follows the one given in the Kähler case, with slight modifications in places where the nonintegrability of J must be taken into account: the Cotton–York tensor is now written as

$$C_{X,Y}(Z) = -\tfrac{1}{2}((\nabla_X\rho)(Y, JZ) - (\nabla_Y\rho)(X, JZ)) - \tfrac{1}{2}(\rho(Y, (\nabla_X J)(Z)) - \\ - \rho(X, (\nabla_Y J)(Z))) + \tfrac{1}{2}(ds(X)\langle Y, Z\rangle - ds(Y)\langle X, Z\rangle). \tag{80}$$

Since ρ closed [22], we have

$$(\nabla_X\rho)(Y, JZ) - (\nabla_Y\rho)(X, JZ) = -(\nabla_{JZ}\rho)(X, Y). \tag{81}$$

As an algebraic object, $\nabla_X J$ is a skew-symmetric endomorphism of TM, associated (by g-duality) to the section $\nabla_X\omega$ of the bundle of J-anti-invariant 2-forms; it then anti-commutes with J, and commutes with any skew-symmetric endomorphism associated to a section of $\Lambda^- M$; in particular, $\nabla_X J$ commutes with the endomorphism corresponding to ρ_0 via the metric (which will be still denoted by ρ_0). We thus obtain

$$\rho(Y, (\nabla_X J)(Z)) - \rho(X, (\nabla_Y J)(Z)) \\ = \tfrac{3s}{2}((\nabla_Y\omega)(X, JZ) - (\nabla_X\omega)(Y, JZ)) + (\nabla_X\omega)(Y, \rho_0(Z)) - (\nabla_Y\omega)(X, \rho_0(Z)).$$

By using the closedness of ω we derive

$$\rho(Y, (\nabla_X J)(Z)) - \rho(X, (\nabla_Y J)(Z)) = \tfrac{3s}{2}(\nabla_{JZ}\omega)(X, Y) - (\nabla_{\rho_0(Z)}\omega)(X, Y). \tag{82}$$

Substituting (81) and (82) in (80), we finally get

$$\nabla_Z\rho_0 = -\tfrac{3s}{2}\mathrm{d}s(Z)\omega - 2C(JZ) - \nabla_{\mathrm{Ric}_0(Z)}\omega + \mathrm{d}s \wedge JZ^\flat, \tag{83}$$

The $\Lambda^- M$-component of (83) gives the identity (1); the last part of the lemma is immediate. □

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It follows that Lemma 4 and, hence, Proposition 1 remain true for weakly self-dual Kähler surfaces, so that on the open set M_0 where $\rho_0 \neq 0$ the almost Hermitian structure $(\bar{g} = \lambda^{-2}g, I)$ defined on M_0 by $\rho_0 = \lambda\omega_I$ (see Lemma 2) is Kähler.

The theory of Hamiltonian 2-forms φ does not extend automatically to the almost Kähler case: there is no reason, in general, to suppose that the trace and Pfaffian of φ are Poisson-commuting holomorphy potentials, nor can we appeal to the open mapping theorem when J is not integrable. On the other hand, Proposition 3 does generalize in the following sense: if we write $\sigma = \xi + \eta$ and $\pi = \xi\eta$, then $d\xi$ and $d\eta$ are orthogonal and $J\mathrm{d}\sigma = 2I\mathrm{d}\lambda$ on the closure of M_0.

Fortunately, when φ is the Ricci form, we can show more.

LEMMA 12. *On a weakly self-dual almost Kähler 4-manifold with J-invariant Ricci tensor, $K = J\mathrm{grad}_g s$ is a Killing vector field.*

Proof. On M_0, the Ricci tensors of both $\bar{g} = \lambda^{-2}g$ and g are I-invariant, and therefore $I\mathrm{grad}_g\lambda$ is Killing vector field (with respect to both metrics) [7]. Since $I\mathrm{d}s = 2J\mathrm{d}\lambda$, $J\mathrm{d}s = 2I\mathrm{d}\lambda$ and K is a Killing vector field on M_0. On the other hand if λ vanishes identically on an open set U then g is Einstein on U, so that s is constant, and K is a trivial Killing vector field. Hence, by continuity, K is a Killing vector field everywhere. □

Because of this observation, it is natural to strengthen the definition of Hamiltonian 2-forms in the almost Kähler case: we say that a closed J-invariant 2-form φ on an almost Kähler 4-manifold is *Hamiltonian* if its trace-free part φ_0 is a twistor 2-form and its trace σ is a momentum map for a Killing vector field.

Note that for any Killing vector field K, $\nabla_X(\nabla K) = R^g_{K,X}$ and so the 1-jet $\{K, \nabla K\}$ is parallel with respect to a globally defined connection, cf. [35]. Hence if K vanishes on an open set so does ∇K, and therefore K vanishes on any connected component meeting that open set. It follows that if φ is Hamiltonian, the open set M_0 where $\varphi_0 \neq 0$ is dense or empty in each connected component.

In particular, we obtain the following generalization of Proposition 6.

PROPOSITION 15. *Let (M, g, J, ω) be a connected weakly self-dual almost Kähler 4-manifold with J-invariant Ricci tensor. Then one of the following holds:*

(i) *ρ_0 is identically zero; then, (g, J, ω) is an Einstein almost Kähler 4-manifold; or*
(ii) *the scalar curvature s of g is constant, but ρ_0 is not identically zero; then, (g, J) is obtained from a Kähler surface (g, I) with two distinct constant principal Ricci curvatures, λ and μ, in the following manner: J equals to I on the λ-eigenspace of the Ricci tensor, but to $-I$ on the μ-eigenspace; hence I is compatible with the opposite orientation of (M, J); or*
(iii) *s is not constant and g is self-dual; or*

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(iv) *W^- and ρ_0 have no zero; then, $(\bar{g} = \lambda^{-2}g, I)$ is a globally defined extremal Kähler metric of non-constant scalar curvature, which is compatible with the opposite orientation of (M, J).*

5.2. WEAK SELF-DUALITY, HAMILTONIAN 2-FORMS, AND A CONJECTURE OF GOLDBERG

The existence of *nonintegrable* almost Kähler 4-manifolds listed in (i)–(iv) of Proposition 15 appears to be a nontrivial problem. We collect below some remarks and known results on this issue:

- A long-standing conjecture of Goldberg [28, 47] states that a compact Einstein almost Kähler manifold must be a Kähler–Einstein manifold. The first *local* examples of nonintegrable Einstein almost Kähler 4-manifolds have been recently discovered in [44, 9]; we shall provide new examples (see Example 2 below), but for all these examples the Ricci tensor and the anti-self-dual Weyl tensor identically vanish.
- The almost Kähler 4-manifolds described in Proposition 15(ii) have been recently studied in [6]. It is known that there are essentially two examples of *homogeneous* Kähler surfaces (M, g, I) which give rise to homogeneous non-integrable almost Kähler 4-manifolds described in (ii) of Proposition 15; specifically, (M, g, I) is either isomorphic to $(\mathrm{SU}(2) \ltimes \mathrm{Sol}_2)/\mathrm{U}(1)$ (in the case when the signature of the Hermitian form is (2,2), cf. [48]), or to $(\mathrm{Isom}(\mathbb{E}^2) \ltimes \mathrm{Sol}_2)/\mathrm{SO}(2)$(in the case when the signature of the Hermitian form is (0,2), cf. [51]); see also Example 3 below. However, there are also many nonhomogeneous examples [4].
- We believe that there should exist nonintegrable examples of almost Kähler 4-manifolds described in Proposition 15(iii) and (iv). However, as we discuss below, it is not clear how to generalize the constructions of Sections 2 and 3 to obtain such examples.

According to Proposition 4, for a weakly self-dual Kähler surface, not only the scalar curvature, but also the Pfaffian of the normalized Ricci form is a momentum map for a Killing vector field. Indeed, this much holds for the trace and the Pfaffian of the normalized 2-form associated to any Hamiltonian 2-form, by Proposition 2.

Hence, one approach to generalize the constructions of Sections 2 and 3 to the almost Kähler case is to study Hamiltonian 2-forms such that the Pfaffian is also a momentum map for a Killing vector field. The following Lemma shows that there are no nonintegrable examples with linearly independent Killing vector fields, which means that there is no direct generalization of the constructions of Section 2.

LEMMA 13. *For an almost Kähler 4-manifold with a Hamiltonian 2-form φ, the Pfaffian π of the associated normalized 2-form $\tilde{\varphi}$ is a momentum map for a Killing*

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vector field $\tilde{K} = J\mathrm{grad}_g\pi$ *if and only if either* (g, J) *is a Kähler or* $\tilde{K}$ *is a constant multiple of* $K = J\mathrm{grad}_g\sigma$.

Proof. Since $\pi = \frac{1}{4}\sigma^2 - \lambda^2$ we have $\mathrm{d}\pi = \frac{1}{2}\sigma\mathrm{d}\sigma + \varphi_0(J\mathrm{d}\sigma)$. Straightforward calculation gives

$$\nabla(Jd\pi) = \mathrm{d}\sigma \wedge J\mathrm{d}\sigma - \frac{1}{2}|\mathrm{d}\sigma|^2\omega + \frac{1}{2}\sigma\nabla(J\mathrm{d}\sigma) - \varphi_0 \circ \nabla\mathrm{d}\sigma.$$

Since $K = J\mathrm{grad}_g\sigma$ is Killing by assumption, it follows that $J\mathrm{grad}_g\pi$ is Killing if and only if $\varphi_0 \circ \nabla\mathrm{d}\sigma$ is skew. This is automatic on the open set where φ_0 vanishes, where $d\sigma = 0$ and hence $d\pi = 0$. Therefore we can assume φ_0 is nonvanishing and write $\varphi_0 = \lambda\omega_I$, where I is a complex structure of the opposite orientation to J.

Now $I \circ \nabla\mathrm{d}\sigma$ is skew if and only if $\nabla\mathrm{d}\sigma$ is I-invariant; since J and I commute, this means that the J-anti-invariant part of $\nabla\mathrm{d}\sigma$ must be I-invariant. But $K = J\mathrm{grad}_g\sigma$ is Killing, so the J-anti-invariant part of $(\nabla_X\mathrm{d}\sigma)(Y)$ is equal to $2(\nabla_X\omega)(K, Y) + (\nabla_K\omega)(X, Y)$. The latter is I-invariant if and only if for any vector field X we have

$$(\nabla_{IX}J)(K) = I(\nabla_X J)(K). \tag{84}$$

Suppose now that $\nabla_X J = A \neq 0$ for a vector X at some point. Since $\nabla_{IX}J$, like A, is a J-anti-invariant endomorphism, we can write $\nabla_{IX}J = bA + cJA$ for some $b, c \in \mathbb{R}$. Equation (84) now reads $bA(K) + cJA(K) = IA(K)$. Since A commutes with I and anticommutes with J, by applying A to the both sides we obtain: $-bK + cJK = -IK$. However K is orthogonal to both JK and IK, and so $b = 0$ and $c = \pm 1$. Thus, on the open set where J is nonintegrable and $\varphi_0 \neq 0$, we have $J\mathrm{d}\sigma = \pm I\mathrm{d}\sigma = \pm 2J\mathrm{d}\lambda$, so $\mathrm{d}\pi$ and $\mathrm{d}\sigma$ are linearly dependent. □

The Calabi construction in Section 3 does generalize to the almost Kähler case and generates some new examples of self-dual Ricci-flat almost Kähler 4-manifolds. However, we shall see that there are no nonintegrable examples of nonconstant scalar curvature.

PROPOSITION 16. *Let* (M, g, J, ω) *be an almost Kähler* 4*-manifold with* J*-invariant Ricci tensor and a nonvanishing Hamiltonian Killing vector field* K*. Suppose that the pair* $(\bar{g} = \lambda^{-2}g, I)$ *is Kähler, where* λ *is a momentum map for a nonzero multiple of* K*, and* I *is equal to* J *on* $\mathrm{span}(K, JK)$*, but to* $-J$ *on the orthogonal complement of* $\mathrm{span}(K, JK)$.

Then either J *is integrable, or* (g, ω) *is given explicitly by*

$$g = \frac{W}{z}(z^2 g_\Sigma + \mathrm{d}z^2) + \frac{z}{W}\left(\mathrm{d}t + \frac{V}{z}\mathrm{d}z + \beta\right)^2, \tag{85}$$

$$\omega = zW\omega_\Sigma + \mathrm{d}z \wedge \left(\mathrm{d}t + \frac{V}{z}\mathrm{d}z + \beta\right), \tag{86}$$

where g_Σ *is a metric on* 2*-manifold* Σ *with area form* ω_Σ*,* β *is a* 1*-form on* Σ *with* $d\beta = W\omega_\Sigma$*, and* $V + iW$ *is an arbitrary holomorphic function on* Σ.

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Conversely any such metric satisfies the above hypotheses, and J is integrable if and only if the function W is constant.

Proof. Without loss of generality, we take $\lambda = z$ to be a momentum map for K (with respect to ω). As in Proposition 13, cf. LeBrun [37], we may introduce coordinates such that

$$g = \mathrm{e}^u w(\mathrm{d}x^2 + \mathrm{d}y^2) + w\,\mathrm{d}z^2 + w^{-1}(\mathrm{d}t + \alpha)^2$$
$$\omega = \mathrm{e}^u w \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}z \wedge (\mathrm{d}t + \alpha)$$
$$\bar{\omega} = z^{-2}(-\mathrm{e}^u w\,\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}z \wedge (\mathrm{d}t + \alpha))$$

where $\bar{\omega}$ is the Kähler form of I with respect to $\bar{g} = z^{-2}g$. The integrability of I together with the closedness of ω and $\bar{\omega}$ yields

$$\mathrm{e}^u w = h(x, y)z, \qquad \mathrm{d}\alpha = -w_x \mathrm{d}y \wedge \mathrm{d}z + w_y \mathrm{d}x \wedge \mathrm{d}z + h(x, y)\mathrm{d}x \wedge \mathrm{d}y$$

from which we obtain the integrability condition

$$w_{xx} + w_{yy} = 0. \tag{87}$$

The Ricci tensor of g is J-invariant if and only if

$$\left(\frac{zu_z - 2}{zw}\right)_x = 0 \quad \text{and} \quad \left(\frac{zu_z - 2}{zw}\right)_y = 0$$

so that we can write $zu_z = f(z)\,zw - 2$. Since $e^u w = h(x, y)z$, we obtain $(zw)_z + f(z)/z(zw)^2 = 0$.
The latter is explicitly integrated, and we get

$$z\,w(x, y, z) = \frac{1}{F(z) + G(x, y)}$$

for some functions $F(z)$ and $G(x, y)$. By substituting into (87) we discover that either F or G must be constant.

If G is constant, then $w_x = w_y = 0$, i.e., g is of Calabi type and J is integrable.

Consider now the case when F is a constant; then, $w = W/z$ and $\mathrm{e}^u = z^2\mathrm{e}^U$, where $W(x, y)$ is a positive harmonic function and $\mathrm{U}(x, y)$ is an arbitrary function of (x, y); the almost Kähler structure (g, ω) takes the form (85)–(86) where:

- $g_\Sigma = \mathrm{e}^U(\mathrm{d}x^2 + \mathrm{d}y^2)$;
- $\omega_\Sigma = \mathrm{e}^U \mathrm{d}x \wedge \mathrm{d}y$;
- W is a positive harmonic function on Σ;
- α satisfies $\mathrm{d}\alpha = -W_y \mathrm{d}x \wedge \mathrm{d}z/z + W_x \mathrm{d}y \wedge \mathrm{d}z/z + W\omega_\Sigma$ and we can locally choose t so that $\alpha = V\mathrm{d}z/z + \beta$, where V is a harmonic conjugate of W and $\mathrm{d}\beta = W\sigma_\Sigma$.

This almost Hermitian structure (g, J, ω) is almost Kähler with J-invariant Ricci tensor, since w, e^u and α solve the required equations. □

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One directly calculates the normalized scalar curvature s of the metric (85): it is given by

$$s = \frac{1}{6zW\mathrm{e}^U}\Big(U_{xx} + U_{yy} + 2\mathrm{e}^U\Big).$$

On a weakly self-dual almost Kähler 4-manifold with J-invariant Ricci tensor, we have seen that s is a momentum map for a Killing vector field. However, s cannot be a multiple of z unless it vanishes. Hence, this construction does not yield any nonintegrable weakly self-dual almost Kähler metrics of Calabi type with *nonconstant* scalar curvature. However, it does provide the following new examples of self-dual *Ricci-flat* strictly almost Kähler 4-manifolds.

EXAMPLE 2. Let (g, J, ω) be given by (85) and suppose that $s = 0$; this means that U is a solution of the Liouville equation, i.e., that g_Σ is the standard metric on an open subset of S^2, while $H = W + iV$ is a nonconstant holomorphic function on Σ with positive real part. If we write $z = r$, we see that the metric

$$g = \frac{W}{r}(\mathrm{d}r^2 + r^2 g_{S^2}) + \frac{r}{W}(\mathrm{d}t + \alpha)^2$$

is given by applying the Gibbons–Hawking Ansatz using the harmonic function W/r, which is invariant under dilation with weight -1 in the sense that

$$r\frac{\partial}{\partial r}\left(\frac{W}{r}\right) = -\frac{W}{r}.$$

(In fact this is the natural scaling weight for W/r, since it is the Higgs field of an abelian monopole on $\mathbb{R}^3$.)

This class of Gibbons–Hawking metrics has been studied before in [16] and [18]. In addition to the triholomorphic Killing vector field $\partial/\partial t$, these metrics also admit a triholomorphic homothetic vector field $r\,\partial/\partial r$. Therefore, by [27], the local quotient by $r\,\partial/\partial r$ is a *hyperCR Einstein–Weyl space*. In this case the quotient Einstein–Weyl structure was obtained explicitly in [16] and is an Einstein–Weyl space *with a geodesic symmetry*.

The reader is referred to these references for more information. However, to the best of our knowledge, the observation that these metrics are almost Kähler is new. Note that the Kähler form is not an eigenform of the Weyl tensor, showing that the solutions are different from the previously known examples of Nurowski–Przanowski [44] and Tod, which were obtained by applying the Gibbons–Hawking Ansatz to a translation-invariant harmonic function.

We next use the rough classification given by Proposition 15 to obtain the following partial result which motivates the further study of compact weakly self-dual almost Kähler 4-manifolds with J-invariant Ricci tensor:

THEOREM 6. *Suppose that there exists a compact weakly self-dual almost Kähler 4-manifold (M, g, J, ω) with J-invariant Ricci tensor, for which the almost complex structure J is not integrable. Then one of the following two alternatives holds:*

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(i) *The scalar curvature of g is a negative constant; then M admits an Einstein, non-integrable almost Kähler structure; or*

(ii) *(M, g, J, ω) belongs to case* (iv) *of Proposition 15, and the globally defined Kähler structure $(\bar{g}, I)$ is isomorphic to an extremal Kähler metric which is not locally of cohomogeneity one, on a minimal ruled surface $S = \mathbb{P}(\mathcal{O} \oplus L) \to \Sigma_{\mathbf{g}}$ with $\mathbf{g} \geqslant 1$ and* $\deg L > 0$.

Proof. We inspect the possible compact *nonintegrable* almost Kähler 4-manifolds given by (i)–(iv) of Proposition 15.

The case of constant scalar curvature is described by Proposition 15(i), (ii). Our claim then follows by [47] and [6, Th.2].

Suppose that s is not constant, i.e., that $K = J\mathrm{grad}_g s$ is a nontrivial Killing vector field by Lemma 12. Let $x_0 \in M$ be a zero of K; then, the isotropy subgroup $H(x_0)$ of the connected group of isometries of (M, g) is a compact group of dimension at least one; one can therefore take an S^1 in $H(x_0)$. Hodge theory implies that on a compact almost Kähler manifold any isometry which is homotopic to the identity inside the group of diffeomorphisms is a symplectomorphism (see e.g. [39]); hence the chosen isometric S^1-action is *symplectic* with respect to ω. Since x_0 is a fixed point of the S^1-action, we obtain a Hamiltonian S^1-action on (M, ω) [40]. The manifold is then equivariantly (and orientedly) diffeomorphic to a rational or a ruled complex surface endowed with a holomorphic circle action [10, 2, 34]. Moreover, in this case (M, g, J, ω) is given by Proposition 15 (iii) or (iv).

Consider first the case (iii). Since s is not constant, the self-dual Weyl tensor W^+ does not vanish [5, Cor.1]. By the Chern–Weil formulae, the signature of M is strictly positive and, therefore, M is diffeomorphic to $\mathbb{CP}^2$ [11]. Combining the results of [30] and [46], one sees that on $\mathbb{CP}^2$ the only self-dual conformal structure with nontrivial (conformal) Killing vector field is the standard one. Thus, modulo diffeomorphisms, we may assume that g is conformal to the standard Kähler metric (g_0, ω_0). Since ω and ω_0 are both harmonic self-dual 2-forms on $(\mathbb{CP}^2, g_0)$, and since $b^+(\mathbb{CP}^2) = 1$, we conclude that $\omega = \mathrm{const}.\omega_0$, showing that J is integrable, a contradiction.

Suppose (M, g, J, ω) is as in Proposition 15(iv). Nowρ_0 determines an integrable almost complex structure I compatible with g and with the opposite orientation of (M, J), such that $(\bar{g} = \lambda^{-2} g, I)$ is an *extremal* Kähler metric with $I\mathrm{grad}_{\bar{g}} \bar{s} = const.K$. Denote by $\overline{M}$ the smooth manifold M endowed with the orientation induced by I. Thus, the oriented smooth 4-manifolds M and $\overline{M}$ both admit complex structures. Since M is the underlying smooth manifold of a rational or a ruled complex surface, we conclude as in the proof of Theorem 5 that the complex surface $(\overline{M}, I)$ is a ruled surface of the form $\mathbb{P}(E)$, where $E \to \Sigma_{\mathbf{g}}$ is a holomorphic rank 2 bundle over a compact Riemann surface $\Sigma_{\mathbf{g}}$ of genus $\mathbf{g}$, and which splits as $E = \mathcal{O} \oplus L$ for a holomorphic line bundle L of degree $\deg(L) > 0$.

We have to prove that $\mathbf{g} \geqslant 1$. Indeed, if $\mathbf{g} = 0$, we obtain the Hirzebruch surface $F_k = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-k))$, where k is a positive integer. As we have already observed in the proof of Theorem 5, the extremal Kähler metrics on these surfaces are the Calabi

cohomogeneity-one U(2)-metrics, i.e. $\bar{g}$ is given by the Calabi construction (67)–(68); since $g = \text{const.}\bar{s}^2\bar{g}$, it follows that g is a cohomogeneity-one metric as well and therefore $\text{grad}_{\bar{g}}\bar{s} = \text{const.}\text{grad}_g s$, showing that $JK = IK$. Since s is not constant, by Lemma 13, J is integrable on the open dense subset where $ds \neq 0$, hence everywhere. We thus conclude that $\mathbf{g} \geq 1$. Note that the above local argument applies to any extremal Kähler $\bar{g}$ which is locally of cohomogeneity one, so that the last part of the theorem also follows. □

We do not have any examples in case (ii) of the above theorem. Indeed, the only examples we know of extremal Kähler metrics on the minimal ruled surfaces in (ii) are locally cohomogeneity-one Calabi-type metrics [49,50].

5.3. ALMOST KÄHLER 4-MANIFOLDS OF CONSTANT LAGRANGIAN SECTIONAL CURVATURE

In this section we deduce another global result from Theorem 6. An almost Kähler 4-manifold (M, g, ω) is said to have (pointwise) *constant Lagrangian sectional curvature* if the sectional curvature of g, at each point of M, is constant on the set of Lagrangian 2-planes at that point-recall that the latter are the planes $X \wedge Y$ with $\omega(X, Y) = 0$. One can make the same definition for almost Kähler manifolds of any dimension, but for $2n > 4$, any almost Kähler $2n$-manifold of constant Lagrangian sectional curvature is in fact Kähler with constant holomorphic sectional curvature [23, 24]. Conversely, it is easy to see that any complex space form has constant Lagrangian sectional curvature (see, e.g., [25]).

In four dimensions, the situation is more interesting. A simple local calculation (cf. e.g. [5]) shows that an almost Kähler 4-manifold has constant Lagrangian sectional curvature if and only if the Ricci tensor is J-invariant, the Weyl tensor is self-dual, and the Kähler form is one of its roots (i.e., M has *Hermitian Weyl tensor* in the sense of [3]).

The following homogeneous example shows that the integrability for almost Kähler 4-manifolds with constant Lagrangian sectional curvature does not follow locally (nor even for complete metrics).

EXAMPLE 3. Consider the homogeneous Kähler surface $M = (\text{SU}(2) \ltimes \text{Sol}_2)/\text{U}(1)$, where Sol_2 denotes the real two-dimensional solvable subgroup of upper triangular matrices in $\text{SL}_2(\mathbb{R})$.

We take the unique left-invariant Kähler structure (g, I) on M, determined by the property that the constant principal Ricci curvatures are equal to $(-1, +1)$, cf. [48]. According to Proposition 15(ii), (M, g) admits an almost Kähler structure J with J-invariant Ricci tensor. Since the scalar curvature of g is zero, g is self-dual (with the orientation opposite to I) [26]; furthermore, J is not integrable [6], and by using the general formulae in [12, Ch. 7] one easily checks that (M, g, J) has constant Lagrangian sectional curvature.

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In contrast to this example, there were a number of reasons [3, 5] to believe that a *compact* almost Kähler 4-manifold of constant Lagrangian curvature must be a self-dual Kähler metric. The conjectured integrability of the almost complex structure has been proved under some additional assumptions on curvature [5] or the topology [3] of the manifold, but the general question was left open. As a consequence of Theorem 6 we are now able to give a positive answer.

COROLLARY 1. *A compact, four-dimensional, almost Kähler manifold has constant Lagrangian sectional curvature if and only if it is a Kähler self-dual surface.*

Proof. Suppose (M, g, J, ω) is a compact almost Kähler 4-manifold of constant Lagrangian sectional curvature, but for which J is not integrable. According to [5, Th. 2] the scalar curvature s of g is not constant; then, by Theorem 6, the smooth manifold M is diffeomorphic to a minimal ruled surface. Since any such surface admits an orientation reversing involution, we conclude that M carries a complex structure which is compatible with the orientation induced by ω. Then, by [3, Cor. 2], the almost complex structure J must be integrable, contradicting our assumption. □

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