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Portfolio Optimization with Mental Accounts

Sanjiv Das, Harry Markowitz, Jonathan Scheid, and Meir Statman*

Abstract

We integrate appealing features of Markowitz's mean-variance portfolio theory (MVT) and Shefrin and Statman's behavioral portfolio theory (BPT) into a new mental accounting (MA) framework. Features of the MA framework include an MA structure of portfolios, a definition of risk as the probability of failing to reach the threshold level in each mental account, and attitudes toward risk that vary by account. We demonstrate a mathematical equivalence between MVT, MA, and risk management using value at risk (VaR). The aggregate allocation across MA subportfolios is mean-variance efficient with short selling. Short-selling constraints on mental accounts impose very minor reductions in certainty equivalents, only if binding for the aggregate portfolio, offsetting utility losses from errors in specifying risk-aversion coefficients in MVT applications. These generalizations of MVT and BPT via a unified MA framework result in a fruitful connection between investor consumption goals and portfolio production.

I. Introduction

Economic analysis regularly separates consumption decisions from production decisions. This separation underlies the insight of comparative advantage. In Ricardo's famous example, Portugal has a comparative advantage in the production of wine while England has a comparative advantage in the production of cloth. People in each country are made better off by producing according to their relative advantage, whether wine or cloth, and trading what they produce for the combination of wine and cloth that maximizes their consumption utility.

Separation of production from consumption also underlies Markowitz's (1952) mean-variance portfolio theory (MVT). Each mean-variance investor has a consumption utility function that depends on the expected return of her overall

*Das, srdas@scu.edu, Leavey School of Business, Santa Clara University, 500 El Camino Real, Santa Clara, CA 95053; Markowitz, harryhmm@aol.com, Rady School of Management, University of California San Diego, 9500 Gilman Dr., La Jolla, CA 92093; Scheid, jscheid@bellatore.com, Bellatore, 333 W. San Carlos St., San Jose, CA 95110; and Statman, mstatman@scu.edu, Leavey School of Business, Santa Clara University, 500 El Camino Real, Santa Clara, CA 95053 and Tilburg University. We thank George Chacko, Tricia Doerksen, Hersh Shefrin, Al Steele, and Raman Uppal for helpful insights and comments. We are also grateful for the comments of Stephen Brown (the editor) and Martin Gruber (the referee). Das and Statman are grateful for support from a Dean Witter Foundation Fellowship, and Das has also been supported by a Breetwor Fellowship.

portfolio and its standard deviation, the measure of risk in MVT. Each mean-variance investor faces a *production function* in the form of the mean-variance efficient frontier and each chooses to *consume* from among the efficient portfolios the one that maximizes her utility, combining expected returns and risk in the optimal proportions.

Placing the consumption function of wine and cloth next to the mean-variance consumption function of expected returns and risk highlights the difference between the two. Wine and cloth are consumption goods, but expected returns and risk are only stations on the way to ultimate consumption goals. Ultimately, individual investors want their portfolios to satisfy goals such as a secure retirement, college education for the children, or being rich enough to hop on a cruise ship whenever they please. Institutional investors want to satisfy goals such as paying promised benefits to beneficiaries and adding new benefits.

Investors are attracted to MVT by its logic and practical application. It seems logical to choose portfolios based on their overall expected return and risk, and the mean-variance optimizer is a practical tool, quick at drawing the efficient frontier. But MVT does not answer many of investors' questions. How does one create a portfolio that is best at satisfying one's goals? Is such a portfolio on the mean-variance efficient frontier? What is one's attitude toward risk? How does one apply MVT if one has many attitudes toward risk that vary by goal? For instance, one is very averse to risk with the portion of one's portfolio devoted to the retirement goal, but one is much less averse to risk with the portion devoted to the college education goal, and one is willing to take any risk, even be risk seeking, with the portion devoted to getting rich. And what is one's optimal portfolio if one perceives risk not as the standard deviation of the return of the overall portfolio but as the probability of not reaching the threshold of each particular goal? Our purpose in this paper is to answer these questions.

While MVT is silent about ultimate portfolio consumption goals, such goals are central in the behavioral portfolio theory (BPT) of Shefrin and Statman (2000). These investors do not consider their portfolios as a whole. Instead, investors consider their portfolios as collections of mental accounting (MA) subportfolios where each subportfolio is associated with a goal and each goal has a threshold level. BPT investors care about the expected return of each subportfolio and its risk, measured by the probability of failing to reach the threshold level of return. Each mental account has an efficient frontier that reflects the trade-off between expected returns and the probability of failing to reach the threshold level of that mental account. A BPT subportfolio is dominated when there is another subportfolio with the same expected return and a lower probability of failing to reach the threshold level. Investors choose subportfolios on the efficient frontier by their trade-off between expected returns and the probability of failing to reach the threshold level. It is important to note that risk seeking can be optimal for BPT investors, while MVT investors are always risk averse.

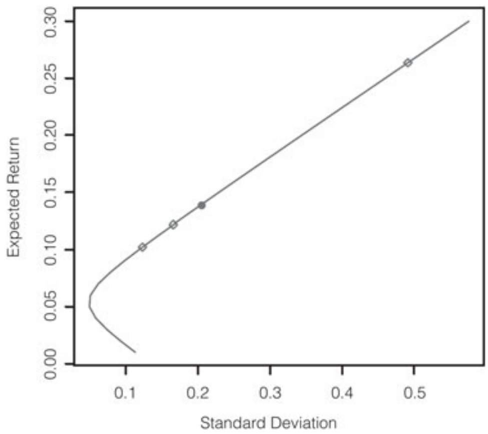
Much work on portfolio optimization is devoted to attempts to maximize out-of-sample performance. For example, DeMiguel, Garlappi, and Uppal (2009) show that a $1/n$ rule yields an ex post efficiency level higher than that obtained by conventional MVT techniques. Our work is different—we integrate appealing features of MVT and BPT into a new framework. We call that framework the MA framework to distinguish it from both the MVT and the BPT frameworks. Features of the MA framework include an MA structure of portfolios, a definition of risk as the probability of failing to reach the threshold level in each mental account, and attitudes toward risk that vary by account. We do not integrate into MA the BPT feature where investors might be risk seeking in their mental accounts.[1] This extension is left for future research.

The canonical MVT optimization comprises minimizing the variance of a portfolio, $\min_w w'\Sigma w$, subject to i) achieving a specified level of expected return $\mathrm{E} = w'\mu$ and ii) being fully invested (i.e., $w'\mathbf{1} = 1$), where $w \in R^n$ is a vector of portfolio weights for n assets, $\Sigma \in R^{n\times n}$ is the covariance matrix of returns of the choice assets, and $\mu \in R^n$ is the vector of n expected returns. The unit vector is denoted $\mathbf{1}$. Varying E results in a set of solutions $\{w(\mathrm{E})\}$ to this problem, delivering portfolios that are mean-variance efficient. Represented graphically in mean-variance space, this set $\{w(\mathrm{E})\}$ traces out the MVT "efficient frontier" (see Figure 1).[2]

FIGURE 1

The MVT Efficient Frontier and Mental Account Portfolios

The curve in Figure 1 is the MVT efficient frontier when there are no short-selling constraints. The three diamond-shaped points on the line correspond to the three mental account portfolios presented in Table 1. The dot on the line (third point from the left) comprises a portfolio that mixes 60% of the first portfolio and 20% each of the second and third. This aggregate of three mental account portfolios is also mean-variance efficient and resides on the frontier.



In MA the threshold return is denoted as H, and the canonical problem is to maximize expected return, $\max_w w'\mu$, subject to a specified maximum probability of failing to reach the threshold (i.e., $\mathrm{Prob}[r(p) < H] < \alpha$). Here $r(p)$ denotes the portfolio's return, and α is the maximum probability of failing to reach the

[1] See work by De Giorgi, Hens, and Mayer (2005) for the relation of this problem to prospect theory and two-fund separation. The mean-variance problem with expected shortfall constraints has been analyzed in Jaeger, Rudolf, and Zimmermann (1995).

[2] A sizable literature related to MVT succeeds these main results of Markowitz (1952). For a small sampling relevant to this paper, see Markowitz (1976), Levy and Markowitz (1979), Markowitz (1983), Markowitz (1991), Basak and Shapiro (2001), DeMiguel et al. (2009), Alexander and Baptista (2008), and Alexander, Baptista, and Yan (2007).

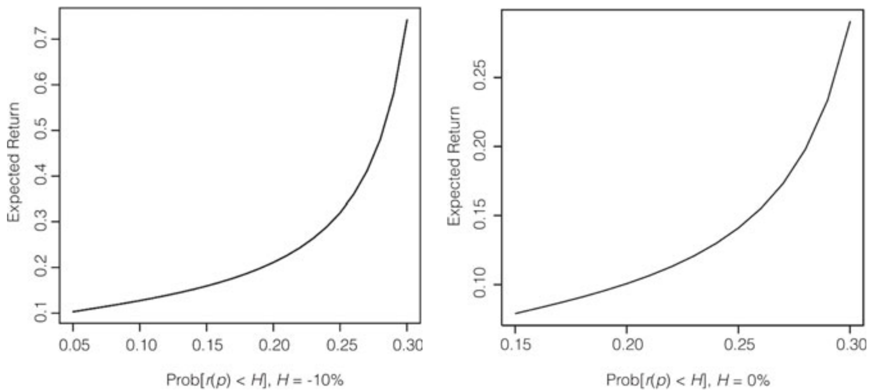
threshold. In MA, overall investor goals are subdivided into subportfolio goals. An investor following MA might specify that she would like her retirement portfolio, currently worth P_0, to accumulate to a threshold dollar amount P_T after T years, implying a threshold return per year of $\{[P_t/P_0]^{1/T} - 1\} \equiv H$, and failing to meet this threshold with probability α. Keeping threshold H fixed, and solving the problem repeatedly for different levels of α, gives corresponding maximized expected return levels $(w'\mu)$; the plot of expected return against α for fixed H results in the MA portfolio frontier. We obtain one frontier for every threshold level H (see Figure 2). As we will see, there is a mathematical connection between these two problems.

FIGURE 2

Efficient Frontiers in MA

Figure 2 presents derived MA frontiers for the inputs chosen in Table 1. This frontier is generated by solving equations (7) and (8) for changing levels of probability (α) of failing to reach the threshold H. In Graph A, $H = -10\%$. A higher expected return comes with a higher probability of not reaching this threshold. In Graph B, $H = 0\%$. Similar features are evident.

Graph A. $H = -10\%$ *Graph B.* $H = 0\%$



The focus of this paper is the integration of portfolio production and consumption by combining the features of MVT and MA into a unified framework. The main results are as follows:

i) *Problem Equivalence*

a) We show that portfolio optimization over two moment distributions where wealth is maximized subject to reaching a threshold level of return with a given level of probability (i.e., the MA problem) is mathematically equivalent to MVT optimization. This is also equivalent to optimization under a safety-first criterion as in Telser (1956).

b) This equivalence has three consequences: First, that MA optimal portfolios always lie on the MVT efficient frontier. Second, that each MA problem's constraint specifies a mapping into an "implied" risk-aversion coefficient in the MVT problem. Third, as we will see, a many-to-one mapping where many MA portfolios may map into a single mean-variance efficient portfolio.

c) The trade-off between risk and return in MA embodies a value-at-risk (VaR) type constraint. We show how MVT, MA, and risk management using VaR are connected, providing an analytic mapping between the different problem formulations.[3] We also show that the VaR-analogous representation provides an analytical approach to check for feasibility of MA portfolios, because a combination of high threshold levels and low maximum probabilities of failing to reach them might not be feasible with an available set of assets.

ii) *Mental Account Subportfolios*

a) The framework is predicated on two assumptions. First, that investors are better at stating their goal thresholds and probabilities of reaching thresholds in MA (the consumption view) than their risk-aversion coefficients in MVT (the production view). Second, that investors are better able to state thresholds and probabilities for subportfolios (e.g., retirement, bequest, education, etc.) than for an aggregate portfolio. We present simulations to show that better problem specification delivers superior portfolios.[4]

b) The MA framework results in no loss in MVT efficiency when short selling is permitted. As is known, combinations of MVT-efficient subportfolios result in an efficient aggregate portfolio (see Sharpe, Alexander, and Bailey (1999), Huang and Litzenberger (1988) for a proof).[5] Since MA portfolios are mathematically equivalent to MVT portfolios, combining optimal MA subportfolios also results in an aggregate portfolio that is on the MVT frontier.

c) If no short sales are allowed, subportfolio optimization results in a few basis points (bp) loss in efficiency relative to optimizing a single aggregate portfolio (see also Brunel (2006)). However, this loss is small compared to the loss that occurs from investors inaccurately specifying their risk aversions. We present simulations and robustness checks that show little or no degradation in Sharpe ratios when MA optimization is applied. We show that the efficiency loss declines as investors become increasingly risk averse.

The paper proceeds as follows. Section II presents the MVT setting in a form where risk aversion is explicitly specified. An example used throughout the paper is set up. Section III derives the explicit relationship between MVT and

[3]Our alternative representation complements work examining biases in optimization with VaR constraints, discussed in Basak and Shapiro (2001), Alexander and Baptista (2008), and Alexander et al. (2007).

[4]Barsky, Juster, Kimball, and Shapiro (1997) find that eliciting preference parameters over specific behaviors yields risk tolerance levels that are borne out in tangible behavior. They also find substantial preference heterogeneity across individuals, thereby emphasizing that the need to better understand individual preferences is crucial in portfolio formation.

[5]This was pointed out in the original text by William Sharpe. We cite the 6th edition here. The point is made on p. 194.

MA portfolios and also makes clear the relation of these problems to VaR. Feasibility restrictions in MA are also analyzed. Section IV analyzes the loss in efficiency from misspecification of risk aversion in a standard mean-variance set up. Section V specifically examines portfolio efficiency in the MA framework when short-selling restrictions are imposed. We conclude in Section VI.

II. Mean-Variance Optimization of Mental Account Portfolios

We first present an alternate solution to the mean-variance optimization problem that will help make the connection between MVT and MA more explicit, analytically derived in Section III. In the canonical MVT problem, stated somewhat more generally, the objective function is to minimize $(\gamma/2)\text{Var}(p)$, subject to the constraint $\text{E}[r(p)] = \text{E}$, where the risk-aversion coefficient $\gamma \equiv 1$; $\text{Var}(p)$ stands for the variance of portfolio p's return, and the expected return on the portfolio $\text{E}[r(p)]$ is set equal to a fixed level E. We solve instead an analogous restated problem where we maximize $\text{E}[r(p)] - (\gamma/2)\text{Var}(p)$, with different $\gamma > 0$, each solution corresponding to a portfolio on the efficient frontier.

Many forms of mean-variance optimization exist, but in general they all offer good approximations to most common utility functions (see Levy and Markowitz (1979), Markowitz (1991)). Here our mean-variance utility function contains a single parameter for risk aversion, denoted γ, balancing investor trade-offs in mean-variance space.

Investors choose portfolio weights $w = [w_1, \ldots, w_n]'$ for n assets, where the assets have a mean return vector $\mu \in R^n$ and a return covariance matrix $\Sigma \in R^{n \times n}$. The full statement of the MVT problem is as follows:

$$\max_{w} \quad w' \, \mu - \frac{\gamma}{2} \, w' \, \Sigma \, w, \tag{1}$$

subject to the fully invested constraint

$$w' \, \mathbf{1} \quad = \quad 1, \tag{2}$$

where $\mathbf{1} = [1, 1, \ldots, 1]' \in R^n$.

The solution to this optimization problem in closed form is (see the Appendix for the full derivation)

$$w \quad = \quad \frac{1}{\gamma} \Sigma^{-1} \left[\mu - \left(\frac{\mathbf{1}' \Sigma^{-1} \mu - \gamma}{\mathbf{1}' \Sigma^{-1} \mathbf{1}} \right) \mathbf{1} \right] \quad \in \quad R^n. \tag{3}$$

This optimal solution w is an n-vector and is easily implemented given it is analytical. In this version of the Markowitz problem, we specify γ, while in the standard problem we specify expected return. Of course, there is a mapping from one to the other.

Note that we may trace out the mean-variance efficient frontier by choosing different values for $\gamma > 0$ and resolving the problem.[6] Knowing the risk-aversion

[6] Alternatively, we may solve the original Markowitz problem to generate the frontier as well. The

coefficient γ for each investor implies the point on the frontier that maximizes mean-variance investor utility.

Example. We introduce a numerical example that will be used to illustrate the results in the rest of the paper. Suppose we have three assets with mean vector and covariance matrix of returns

$$(4) \qquad \mu = \begin{bmatrix} 0.05 \\ 0.10 \\ 0.25 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 0.0025 & 0.0000 & 0.0000 \\ 0.0000 & 0.0400 & 0.0200 \\ 0.0000 & 0.0200 & 0.2500 \end{bmatrix}.$$

The first asset is a low-risk asset, analogous to a bond. It has low return and low variance compared to the other two more "risky" assets, analogous to a low-risk stock and a high-risk stock. In an MA framework, investors choose to divide their aggregate portfolios into subportfolios. To make this more specific, suppose an investor divides an aggregate portfolio into three subportfolios: retirement, education, and bequest. She is risk averse in the retirement subportfolio and her risk-aversion coefficient for this subportfolio is $\gamma = 3.7950$. She is somewhat less risk averse in the education subportfolio and her risk-aversion coefficient for it is $\gamma = 2.7063$. She is even less risk averse in the bequest subportfolio, with risk-aversion coefficient $\gamma = 0.8773$.

Table 1 shows the optimal portfolio weights for the three subportfolios computed using equation (3). The standard deviation of the retirement subportfolio is 12.30%, and that of the education subportfolio is 16.57%. The standard deviation of the bequest subportfolio is highest, 49.13%. The aggregate portfolio based on a 60:20:20 division of investable wealth across the three subportfolios is also shown in the table, with a standard deviation of 20.32%.

As the coefficient of risk aversion γ declines, less is invested in the bond and more in the two stocks. In the bequest subportfolio, risk aversion has dropped very low and now the investor leverages her portfolio by taking a short position in the bond and increasing the long positions in the two stocks. Note that the aggregate portfolio is still unlevered.

Markowitz problem is

$$\min_w \frac{1}{2} w' \Sigma w \qquad \text{s.t.} \qquad \begin{aligned} w'\mu &= \mathrm{E}(R), \\ w'\mathbf{1} &= 1. \end{aligned}$$

The first constraint requires that a fixed level of expected return be met for the minimized level of portfolio variance. This constraint mimics the effect of choosing γ in our modified formulation of the problem. The second constraint requires that all moneys be invested, also known as the fully invested constraint. The well-known solution to this problem is (see Huang and Litzenberger (1988) for one source):

$$\begin{aligned} w &= \lambda \Sigma^{-1}\mu + \gamma \Sigma^{-1}\mathbf{1}, \\ \lambda &= \frac{CE - A}{D}, \qquad \gamma = \frac{B - AE}{D}, \\ A &= \mathbf{1}'\Sigma^{-1}\mu, \qquad B = \mu'\Sigma^{-1}\mu, \\ C &= \mathbf{1}'\Sigma^{-1}\mathbf{1}, \qquad D = BC - A^2. \end{aligned}$$

The frontier is traced by repeatedly solving this problem for different values of $\mathrm{E}(R)$.

TABLE 1

Holdings of Mean-Variance Efficient Portfolios for Varying Risk Aversion

In Table 1, the portfolio weights are provided for three assets computed using the solution in equation (3). Risk aversion is decreasing as γ decreases. We also show the aggregate portfolio comprising a 60:20:20 mix of the three subportfolios. The three subportfolios correspond to the retirement, education, and bequest accounts. The expected returns m of each individual subportfolios, as well as the standard deviations s, are also shown.

Risk Aversion:	$\gamma = 3.7950$	$\gamma = 2.7063$	$\gamma = 0.8773$	60:20:20 Mix
Assets	Retirement Subportfolio	Education Subportfolio	Bequest Subportfolio	Aggregate Portfolio
1 (bond)	0.53943	0.37873	–0.78904	0.2416
2 (low-risk stock)	0.26562	0.34986	0.96200	0.4217
3 (high-risk stock)	0.19495	0.27141	0.82704	0.3367
Total weights	1	1	1	1
Expected return (r_i)	10.23%	12.18%	26.35%	13.84%
Std. dev. (s_i)	12.30%	16.57%	49.13%	20.32%

This section presented the MVT problem in a form that is necessary for the analysis of the main contributions of the paper in subsequent sections. We next examine the linkage between the MVT problem and other problem formulations.

III. Mental Accounts, VaR, and Mean-Variance Optimal Portfolios

In this section, we derive the equivalence between MVT and MA, and we demonstrate the linkage to VaR.

It is easy to chart the mean-variance frontier of MVT. However, stating one's γ with precision is difficult. Therefore, financial planning firms elicit risk attitudes using questionnaires that do not ask about γ directly. Two refinements of this approach are of interest. First, eliciting risk attitudes for MA subportfolios is easier than eliciting the risk attitude for an aggregate portfolio. Second, investors are better at stating their threshold levels for each goal and maximum probabilities of failing to reach them than stating their risk-aversion coefficients.

For instance, in MA an investor may specify that the return on a portfolio should not fall below a level H with more than α probability. This is equivalent to optimizing portfolios using Telser's (1956) criterion. For normal distributions, this is connected to VaR and is the same as saying that the $\text{VaR}_\alpha = H$ in the language of risk managers. In Shefrin and Statman's (2000) BPT, investors maximize expected returns subject to a constraint that the probability of failing to reach a threshold level H not exceed a specified maximum probability α. This is the same as expected wealth optimization with a VaR constraint. In this section, we will see that when investors are quadratic utility maximizers or returns are multivariate normal, this problem maps directly into the Markowitz mean-variance problem, thereby resulting in optimal portfolios that are mean-variance efficient. Thus there is a mathematical mapping between investor goals (consumption) and mean-variance portfolios (production) via the medium of mental accounts.

Consider an investor with threshold level $H = -10\%$ in one of her mental accounts. She wants to maximize expected return subject to the constraint that the maximum probability of failing to reach this level of return does not exceed $\alpha = 0.05$. Or consider the same investor in a different mental account, where

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$H = 10\%$ and she wishes to maximize return subject to the constraint that the probability of failing to reach H not exceed 0.05. Hence, the investor acts as if she has different risk preferences in each of the mental accounts. We will show here how these separate specifications of the MA constraint imply a mapping into different risk preference parameters under MVT.

Before proceeding to the technical specifics, we highlight two theoretical features of the problem in this section. First, we show that portfolio optimization in the MA framework with VaR constraints yields an optimal portfolio that resides on the MVT-efficient frontier, consistent with the results in Telser (1956). We extend this result by showing that each VaR constraint in the MA framework corresponds to a particular implied risk-aversion coefficient in the MVT framework. Alexander et al. (2007) solve a mean-variance problem with VaR and CVaR (conditional VaR) constraints and find that the frontier is impacted inwards with the constraint. In their model, $\gamma = 1$. What we show instead is that the same problem with the VaR constraint may be translated into an unconstrained problem with an implicitly higher γ. There is no inconsistency between the models, for the former one keeps γ fixed and shows that the mean-variance trade-off is impacted adversely with the imposition of the VaR constraint. The latter model imposes the mean-variance impact by altering γ. Thus, in our paper, the VaR-constrained problem has an alternate representation.[7]

Second, since an MA investor divides her portfolio optimization into subportfolio optimizations, there is the natural question of the efficiency of the aggregate of the subportfolios. We show that the aggregated portfolio is also analogous to a mean-variance portfolio with a risk-translated γ coefficient with short selling, resulting in no loss of mean-variance efficiency, even after imposing the MA structure. This is true when short selling in the aggregate portfolio is permitted. The case with short-selling constraints is solved in Section V.

We now show that solving the MA problem is analogous to solving a standard mean-variance problem with a specific "implied" risk-aversion coefficient. Consider a threshold level of return H for portfolio p, and the maximum probability of the portfolio failing to reach return $r(p)$ as α. In other words,

$$\text{Prob}[r(p) \leq H] \quad \leq \quad \alpha. \tag{5}$$

If we assume that portfolio returns are normally distributed, then this statement implies the following inequality:

$$H \quad \leq \quad w'\mu + \Phi^{-1}(\alpha)[w'\Sigma w]^{1/2}, \tag{6}$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution function. We note that the assumption of normality is without loss of generality. We impose normality for convenience because it is a common practical choice. Since this optimization problem may be infeasible, we provide a full discussion of feasibility in Section III.B.

[7]See Basak and Shapiro (2001) for continuous-time dynamic portfolio problems with VaR constraints.

The goal of the investor is to choose the best portfolio (in mean-variance space) that satisfies the constraint in equation (6). Recalling that the optimal weights w are given by equation (3), these may be substituted into equation (6), and we obtain an equation that we can solve for the "implied" risk aversion γ of the investor for this particular mental account. Noting that the constraint in equation (6) is an equality when optimality is achieved, we have the solution to the investor's implied risk aversion γ and the optimal weights $w(\gamma)$ embedded in the following equations:

$$H = w(\gamma)'\mu + \Phi^{-1}(\alpha)[w(\gamma)'\Sigma w(\gamma)]^{1/2}, \tag{7}$$

where

$$w(\gamma) = \frac{1}{\gamma}\Sigma^{-1}\left[\mu - \left(\frac{\mathbf{1}'\Sigma^{-1}\mu - \gamma}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\right)\mathbf{1}\right]. \tag{8}$$

The solution to equation (7) is easily obtained to find γ, the *implied* risk aversion for the mental account, after equation (8) has been substituted into equation (7). Note that in equation (8) the portfolio weights are nonlinear in the risk-aversion coefficient.

Hence, the portfolio optimization problem for an MA investor is specified by a threshold level of return H and a probability level α. There is a semianalytic solution to the MA portfolio problem that uses the MVT formulation. When an investor specifies her MA preferences for each subportfolio through the parameter pair (H, α), she is implicitly stating what her risk preferences (γ) are over the given portfolio choice set (μ, Σ). We may thus write the implied risk aversion for each mental account as a mapping function $\gamma(\mu, \Sigma; H, \alpha)$. We illustrate these results by returning to our numerical example.

Example. Assume an investor with three mental accounts as before. The portfolio choice set is the same as in equation (4) that provides the input values of μ, Σ. In the first mental account, suppose we have that $H = -0.10$ and $\alpha = 0.05$. That is, the investor stipulates that she does not want the probability of failing to reach $H = -10\%$ to exceed $\alpha = 0.05$. Then solving equation (7) results in an implied risk aversion of $\gamma_1 = 3.9750$. When we change these values to $H = -5\%$ with $\alpha = 0.15$, then we get $\gamma_2 = 2.7063$ as the solution to equation (7). If we choose $H = -15\%$ and $\alpha = 0.20$, then the implied risk-aversion coefficient is $\gamma = 0.8773$. (Recall that these were the three values of γ for which we reported MVT weights in Table 1.) This illustrates the mapping from MA parameters into MVT risk-aversion coefficients. The portfolio weights in Table 1 are exactly those obtained here in the three mental accounts we optimized in MA.

Therefore, an MA investor behaves in a compartmentalized manner, where mental accounts are associated with varying levels of risk aversion. However, this is not a departure from optimality within the MA framework. It is important to note that because of the mapping from the MA constraint into MVT risk aversion, *each* portfolio is mean-variance efficient and resides on the portfolio frontier. We show this graphically in Figure 1.

As a corollary, the aggregate portfolio of MA subportfolios is also mean-variance efficient, because combinations of portfolios on the efficient frontier are mean-variance efficient. As in Table 1, suppose the first portfolio is allocated 60%, the second 20%, and the third 20% of the investor's wealth. The aggregate portfolio has a mean return of 13.84% with a standard deviation of 20.32%. This portfolio lies on the efficient frontier and is depicted in Figure 1 as a small dot (the third portfolio from the left of the graph). Hence, we see that even when the investor divides her aggregate portfolio *production* problem into three separate MA problems based on *consumption* characteristics, the component mental account portfolios and aggregated portfolio are all on the MVT efficient frontier.

We note further that the investor can look at the probability of failing to reach various thresholds once the subportfolio has been optimized. This is also true in the mean-variance setting. Table 2 shows the combinations of threshold return levels and probabilities of failing to reach them for the three subportfolios in Table 1. We see that the probability that the investor would have negative returns in the three portfolios is 20%, 23%, and 30%, respectively. These correspond to decreasing risk aversions in the three subportfolios. The probability of a negative return in the aggregate portfolio is 25%. Since the portfolio weights are not linearly proportional to the risk-aversion coefficient γ (see equation (3)), the risk-aversion coefficient implied in the aggregate portfolio is *different* from the weighted average of the risk-aversion coefficients of the three subportfolios.

TABLE 2

Threshold Return Levels and Corresponding Probabilities of Not Reaching Them

Table 2 is based on the three portfolios shown in Table 1. Each portfolio is the one that maximizes the expected return while attaining the threshold H with the specified level of probability α.

Risk Aversion:	$\gamma = 3.7950$	$\gamma = 2.7063$	$\gamma = 0.8773$	60:20:20 Mix
Threshold (H)	Retirement Subportfolio Prob[$r < H$]	Education Subportfolio Prob[$r < H$]	Bequest Subportfolio Prob[$r < H$]	Aggregate Portfolio Prob[$r < H$]
–25.00%	0.00	0.01	0.15	0.03
–20.00%	0.01	0.03	0.17	0.05
–15.00%	0.02	0.05	0.20	0.08
–10.00%	0.05	0.09	0.23	0.12
–5.00%	0.11	0.15	0.26	0.18
0.00%	0.20	0.23	0.30	0.25
5.00%	0.34	0.33	0.33	0.33
10.00%	0.49	0.45	0.37	0.42
15.00%	0.65	0.57	0.41	0.52
20.00%	0.79	0.68	0.45	0.62
25.00%	0.89	0.78	0.49	0.71
Mean return	10.23%	12.18%	26.35%	13.84%
Std. dev.	12.30%	16.57%	49.13%	20.32%

A. Trading Off Thresholds and Probabilities

BPT emphasizes the trade-off between thresholds and the probability of failing to reach them. In MA, as in BPT, investors maximize expected wealth subject to a maximum probability of failing to reach a threshold level of return, while in MVT they minimize variance, subject to a level of return.

Efficient frontiers in MA have expected returns on the y-axis and probabilities of not reaching a specified threshold on the x-axis. Graph A of Figure 2 presents the derived MA frontier for the retirement subportfolio in Table 1. The frontier is generated by solving equations (7) and (8) for various levels of probability of failing to reach the threshold H. For example, we set $H = -10\%$. The expected return increases as we increase the maximum probability of failing to reach this threshold. We see that expected return is convex in the probability of failing to reach the threshold, the MA measure of risk.

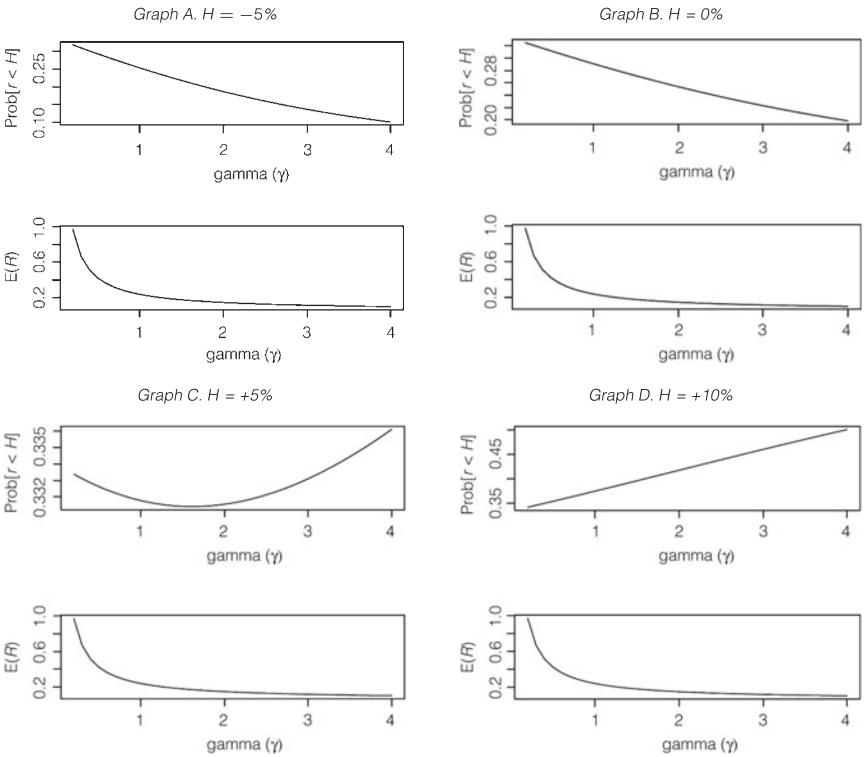
Graph B of Figure 2 shows the frontier generated by MA portfolios where $H = 0\%$, a common threshold because it is the dividing line between gains and losses. Again, we see that increasing probabilities of failing to gain allow higher expected return.

Figure 3 shows the probabilities of failing to reach a threshold H and the expected returns when risk aversion (γ) varies, for different threshold levels. The figure comprises four panels of two plots each, one for the probability of failing to reach thresholds and the other for the expected return. We see how the probability of failing to reach the threshold (upper plot) and the expected return (lower plot) change as risk aversion increases. Graph A presents the case when $H = -5\%$.

FIGURE 3

Probability of Failing to Reach Threshold Returns H and Expected Returns E(R) when Risk Aversion (γ) Varies

Figure 3 comprises four graphs of two plots each; each graph is comprised of a plot for the Prob$[r < H]$ and one of the expected return E(R) below it. Each set relates to a threshold level in the set $H = \{-5\%, 0\%, +5\%, +10\%\}$. Risk aversion γ is varied on the x-axis.



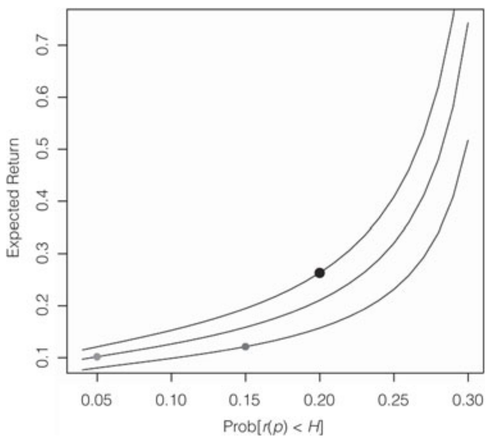
An increase in risk aversion leads to a lowering of the probability of failing to reach a threshold but also to a lower expected return. Graph B presents the case where $H = 0\%$ and has the same inference. In Graph D, $H = 10\%$. Here, the probability of failing to reach the threshold is increasing in risk aversion, while the expected return is decreasing. MA investors require compensation in the form of higher expected returns for a higher probability of failing to reach a threshold return. Hence the portfolio that is most to the left of the plot dominates all other portfolios, and the efficient frontier consists of only that portfolio. Finally, Graph C is for $H = 5\%$. The probability of failing to reach this threshold declines at first as risk aversion increases, but then it increases. Efficient portfolios lie in the range where the probability of failing to reach the threshold is declining, and all portfolios in the range beyond this point are dominated.

MA frontiers are plotted with fixed threshold levels with the probability of failing to reach the threshold on the *x*-axis, and the expected return on the *y*-axis. For each level of the threshold H we obtain a different MA frontier. Figure 4 shows our three MA portfolios, which have different thresholds. As thresholds increase, we shift from the lowest frontier to the highest one. Our three mental accounts reside on separate MA frontiers because they are optimized for different thresholds, but all portfolios on these frontiers reside on the mean-variance efficient frontier (in standard deviation and expected return space). Hence, there is a one-to-many mapping from a single MVT frontier to a set of MA frontiers and vice versa. For example, the retirement portfolio occupies a single point on the MVT frontier but corresponds to many sets of (H, α), as seen in Table 2.

FIGURE 4

MA Frontiers as Thresholds *H* Are Varied

Figure 4 presents derived MA frontiers for the inputs chosen in Table 1. This frontier is generated by solving equations (7) and (8) for changing levels of the probability of failing to reach the threshold (α). H is set, in turn, to -5%, -10%, and -20%. Expected return rises with the probability of failing to reach the threshold. Hence, the frontier moves to the right as H declines. The three portfolios are for three mental accounts with the following thresholds (H) and probabilities of failing to reach the threshold (α): The left-most point on the middle frontier above is for $(H, \alpha) = (-10\%, 0.05)$, the second point from the left on the lowest frontier is for $(H, \alpha) = (-5\%, 0.15)$, and the right-most point on the highest frontier is for $(H, \alpha) = (-15\%, 0.20)$.



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B. Feasibility in MA

Achieving particular combinations of thresholds, probabilities of failing to reach them, and expected returns may not always be feasible with a given set of assets. The MA problem has a feasible solution when

$$(9) \qquad H \leq w'\mu + \Phi^{-1}(\alpha)[w'\Sigma w]^{1/2}.$$

The problem has no *feasible* solution when $H > w'\mu + \Phi^{-1}(\alpha)[w'\Sigma w]^{1/2}$. One way to find if the problem has a feasible solution is to maximize the value of the right-hand side of equation (9) and check if it is greater than H. This results in the following optimization program:

$$(10) \qquad \max_{w} Q \equiv w'\mu + \Phi^{-1}(\alpha)[w'\Sigma w]^{1/2},$$

subject to

$$(11) \qquad w'\mathbf{1} = 1.$$

The Lagrangian for this problem is

$$(12) \qquad \max_{w,\lambda} Q = w'\mu + \Phi^{-1}(\alpha)[w'\Sigma w]^{1/2} + \lambda[1 - w'\mathbf{1}].$$

The first-order conditions are

$$(13) \qquad \frac{\partial Q}{\partial w} = \mu + \Phi^{-1}(\alpha)[w'\Sigma w]^{-1/2}\Sigma w - \lambda\mathbf{1} = 0,$$

$$(14) \qquad \frac{\partial Q}{\partial \lambda} = 1 - w'\mathbf{1} = 0.$$

We premultiply all terms in equation (13) by Σ^{-1} and, defining $[w'\Sigma w]^{-1/2} \equiv M$, we get, after rearranging,

$$(15) \qquad \lambda\Sigma^{-1}\mathbf{1} = \Sigma^{-1}\mu + \Phi^{-1}(\alpha)Mw,$$

$$(16) \qquad \lambda\mathbf{1}'\Sigma^{-1}\mathbf{1} = \mathbf{1}'\Sigma^{-1}\mu + \Phi^{-1}(\alpha)M\mathbf{1}'w.$$

Noting that $\mathbf{1}'w = 1$, we get

$$(17) \qquad \lambda = \frac{\mathbf{1}'\Sigma^{-1}\mu + \Phi^{-1}(\alpha)M}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}.$$

Substitute the solution for λ into equation (15) and rearrange to get the equation for portfolio weights:

$$(18) \qquad w = \frac{1}{\Phi^{-1}(\alpha)M}\Sigma^{-1}\left[\mu - \left(\frac{\mathbf{1}'\Sigma^{-1}\mu + \Phi^{-1}(\alpha)M}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\right)\mathbf{1}\right].$$

Note however, we have eliminated λ but we have obtained an equation with $w \in R^n$ on both sides, since $M = [w'\Sigma w]^{-1/2}$, giving us

$$(19) \qquad w = \frac{1}{\Phi^{-1}(\alpha)[w'\Sigma w]^{-1/2}}\Sigma^{-1}\left[\mu - \left(\frac{\mathbf{1}'\Sigma^{-1}\mu + \Phi^{-1}(\alpha)[w'\Sigma w]^{-1/2}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\right)\mathbf{1}\right].$$

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This is a system of n implicit equations, best solved numerically. Once we get the solution and plug it back into the objective function $Q \equiv w'\mu + \Phi^{-1}(\alpha)[w'\Sigma w]^{1/2}$ to get the maximized value, we can check if $H < Q$. If not, then the problem is infeasible with the current portfolio choice set, and other assets need to be considered or H reduced.

IV. Efficiency Loss from Misspecification of Risk Aversion

Mean-variance investors are advised to determine their optimal aggregate portfolio on the efficient frontier by balancing their aversion to risk with their preference for high returns. But investors find it difficult to specify their optimal aggregate portfolio for two reasons. First, investors have more than one level of risk aversion. In our example the level of risk aversion associated with money dedicated to retirement is high but the level of risk aversion associated with money dedicated to education is lower and their risk aversion with money dedicated to bequest is even lower. Investors who are asked for their level of risk aversion in the aggregate portfolio must weight their three levels of risk aversion by the proportion dedicated to each in the aggregate portfolio. This is a difficult task unless investors are guided to begin by breaking down the aggregate portfolio into the three MA subportfolios and determine the proportions of each in the aggregate portfolio. Investors who skip the mental accounts subportfolios stage are likely to misspecify their level of risk aversion in the aggregate portfolio.

Moreover, investors find it difficult to specify their levels of risk aversion even in the mental accounts subportfolios when they are asked to specify their level of risk aversion in units of variance, since variance offers investors little intuitive meaning. This adds to the likelihood of misspecification of optimal aggregate portfolios. Investors are better able to specify their level of risk aversion in units of thresholds for each of the MA subportfolio and the probabilities of failing to reach them.

One of the benefits of the mental account framework is that risk preferences are specified better. It is nevertheless useful to examine the loss in mean-variance efficiency that occurs when our investor misspecifies her risk aversion (γ). Using the same example as before, we present in Table 3 (and depict geometrically in

TABLE 3

Degradation in Expected Return When Risk-Aversion Coefficients Are Misspecified by Investors

Table 3 reports the basis points loss from using a misspecified γ. The comparison is made using the utility loss translated into basis points as shown in Figure 5. Numbers in the table are computed as the average of loss from an upward and downward perturbation of the risk-aversion coefficient by the percentage error. The three risk-aversion levels correspond to those in Table 1. We can see that losses increase as investors become less risk averse.

	Misspecification of the Risk-Aversion Coefficient		
Risk Aversion (γ)	10%	20%	30%
3.7950	2.50	10.94	28.72
2.7063	3.50	15.34	40.27
0.8773	5.29	23.15	44.22

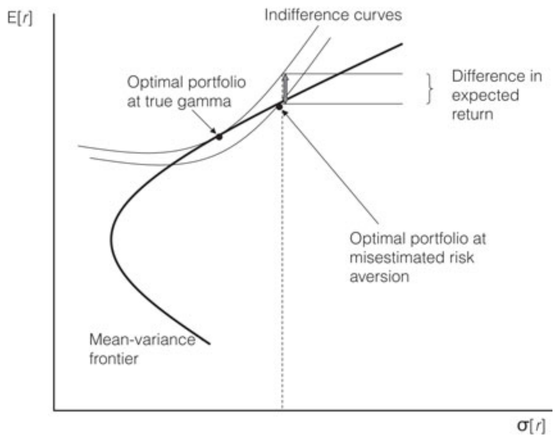
(numbers in basis points)

Figure 5) the loss in aggregate utility (translated into basis points of return) when the investor misspecifies her risk aversion. Losses are in the range of 5 bp to 40 bp. Losses are higher for investors who are less risk averse.

FIGURE 5

Degradation in Expected Return when Risk Aversion is Misspecified by an Investor

Figure 5 shows the MVT frontier and the point on the frontier that the investor would choose if he could specify correctly his risk-aversion coefficient (γ). The tangential indifference curve is also presented. The point to the right is efficient, but it lies on a lower indifference curve. On the *y*-axis, we see the difference between the indifference curves with respect to the expected return.



Since it is easier for investors to specify their risk-return trade-offs in specific goal-based subportfolios than in a single aggregate portfolio, they do not incur the costs of misspecification, as is shown in this numerical example. Hence, MA portfolios do better if they result in better specification of investor goals. However, they result in a loss in portfolio efficiency because the aggregate of optimized subportfolios is not always mean-variance efficient. In the next section we show that this loss of efficiency is very small.

V. Loss of Efficiency with Short-Selling Constraints

So far we have assumed that short sales are allowed with no constraints. We found that MA portfolios reside on the mean-variance efficient frontier. Hence, solutions to the MVT and MA problems coincide. In this section, we assess the MVT and MA problems when short selling is constrained.

Portfolio optimization in practice entails a quadratic objective function involving expected returns and constraints that are linear in portfolio weights for being fully invested. The MA problem adds a nonlinear constraint, namely, that the probability of failing to reach threshold H not be greater than α. However, with no short-sale constraints, we demonstrated a tractable representation of the problem and a simple solution procedure (as shown in equations (3) and (7); an alternate solution was provided in equation (19)). In short, our previous results developed full or semianalytic solutions that did not call for quadratic programming (QP) optimizers.

Short-selling constraints complicate the QP problem when taken in conjunction with the nonlinear MA constraint in addition to short-selling constraints. While standard QP software usually requires linear constraints, the addition of the MA constraint did not hamper us in the previous sections, since we bypassed the use of QP with semianalytic solutions. This was possible partly because all the constraints we imposed were equality constraints. With the introduction of inequality (short-selling) constraints on the portfolio weights in this section, we have to resort to QP numerical approaches and also deal with the nonlinear MA constraint. We show how a reformulation of the problem allows us to obtain the efficiencies of standard QP algorithms with linear constraints.

The MA portfolio optimization problem with additional short-selling constraints is as follows:

$$\max_{w} \quad w'\mu \qquad \text{s.t.} \tag{20}$$

$$w'\mu + \Phi^{-1}(\alpha)\sqrt{w'\Sigma w} \geq H, \tag{21}$$

$$w'\mathbf{1} = 1, \tag{22}$$

$$w \geq L, \tag{23}$$

$$w \leq U. \tag{24}$$

Here $w \in R^n$ is the vector of constrained portfolio weights. The upper and lower bound vectors are $\{U, L\} \in R^n$, such that $L \leq w \leq U$. In order to employ powerful QP routines, we recast the problem above into the program below, where we embed the nonlinear constraints into a subsidiary objective function, resulting in a QP with only linear constraints. The full problem statement is as follows:

$$\text{Solve}_{\gamma} \quad w(\gamma)'\mu + \Phi^{-1}(\alpha)\sqrt{w(\gamma)'\Sigma w(\gamma)} = H, \tag{25}$$

where $w(\gamma)$ is the solution to the following optimization program:

$$\max_{w} \quad w'\mu - \frac{\gamma}{2} w'\Sigma w \qquad \text{s.t.} \tag{26}$$

$$w'\mathbf{1} = 1, \tag{27}$$

$$w \geq L, \tag{28}$$

$$w \leq U. \tag{29}$$

Hence, we solve nonlinear equation (25) in variable γ containing function $w(\gamma)$ that comes from a numerical solution to the subsidiary maximization problem in equations (26)–(29). We fix γ and solve the QP in equations (26)–(29). Then we check if equation (25) holds. If not, we move γ in the appropriate direction and resolve the QP. We search efficiently over γ, and convergence is achieved rapidly. If there is no convergence, then it also implies that the program in equations (20)–(21) is infeasible.[8] The solution delivers the risk-aversion coefficient γ implied by the MA parameters (H, α).

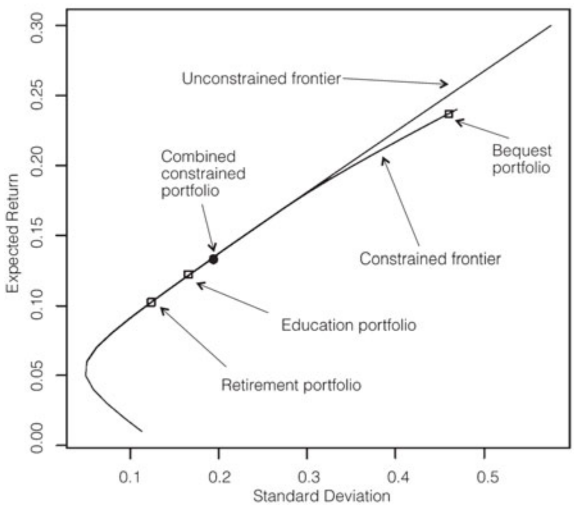
[8]We undertake optimization using the R computing package (see http://www.r-project.org/). This contains the minpack.lm and quadprog libraries, which we applied to this problem.

We solved the short-selling constrained problem with the same inputs as before. First, we generated the portfolio frontier for the short-selling constrained Markowitz problem. The frontier is plotted in Figure 6 and is the right most of the two frontiers. This frontier is enveloped by the unconstrained portfolio frontier (the left-most frontier), and the two frontiers coincide in the region where the short-selling constraint is not binding. In the case of the retirement and education mental accounts, the constraint is not binding, as may be seen from Table 1. In the same table, since the optimal solution to the third mental account entails short selling the first asset, we know that the solution will differ when the short-sell constraint is imposed. We see this point as the right-most point in Figure 6. This portfolio does not reside on the unconstrained efficient frontier but lies on the constrained frontier and is efficient in the mean-variance space limited to portfolios in which short selling is not permitted.

FIGURE 6

The MVT Frontier and Mental Account Portfolios (without and with short-selling constraints)

In Figure 6 the upper curve is the MVT frontier when there are no short-selling constraints. The lower curve is the MVT frontier with short-selling constraints (see Section V), and hence, lies in a *y*-axis range that is bounded below by the asset with the lowest return and bounded above by the asset with the highest return. The three squares correspond to the three constrained mental account portfolios that lie on the constrained MVT frontier. The dot (third point from the left) comprises a portfolio that mixes 60% of the first portfolio and 20% each of the second and third. One can see that the aggregate portfolio is very close to the frontier, and the loss of MVT efficiency is small.



In the presence of short-selling constraints, the aggregate portfolio is not necessarily on the constrained portfolio frontier. Figure 6 shows the three short-selling constrained mental account portfolios as well as the aggregate portfolio formed from the weighted average of the three portfolios. The aggregate portfolio has a mean return of 13.31% with a standard deviation of 19.89% and lies just below the frontier. If the same portfolio were to lie on the constrained frontier at the same standard deviation, it would return 13.43%. The loss of mean-variance efficiency because of the short-selling constraint is 12 bp.[9] Alexander and Baptista (2008) show that VaR constraints mitigate the adoption of an inefficient portfolio. The presence of such constraints in our model may explain why the loss in efficiency amounts to only a few basis points.

At a practical level, we need only impose the short-selling constraint at the aggregate portfolio level. Often the aggregate portfolio does not entail short selling even when some subportfolios do, as is evidenced in the portfolios in Table 1. Imposition of the aggregate short-selling constraint results in an inefficient portfolio relative to one in which no short-selling constraints are imposed only when the unconstrained optimization of each subportfolio results in an aggregate portfolio that entails short selling. Solving the MA portfolio problem with an aggregate short-selling constraint is undertaken in the following manner: Optimize all the individual subportfolios with no short-selling constraints. Check if the aggregate portfolio entails short selling. If not, the process is complete. If the aggregate portfolio has some securities that are in short positions, accept the holdings in all portfolios that have no short sales as they are. These then provide the limits on positions for the remaining subportfolios, which are optimized subject to residual position limits.

Specifically, let subportfolios be indexed by k and assets by j. Subportfolio asset weights are denoted w_{kj}. The set of subportfolios that have no short sales is Ω_0, and the set with short sales is Ω_1. The aggregate short-sales constraint is given by $\Sigma_k w_{kj} \geq 0, \forall j$, where $\Sigma_k \Sigma_j w_{kj} = 1$. We break down the problem into two steps. First, optimize each subportfolio with no constraints and identify the set Ω_0, Compute residual positions $R_j = -\Sigma_{k \in \Omega_0} w_{kj}, \forall j$. Second, optimize the portfolios $k \in \Omega_1$ such that $\Sigma_{k \in \Omega_1} w_{kj} \geq R_j, \forall j$. This two-step approach uses the same technology as before.

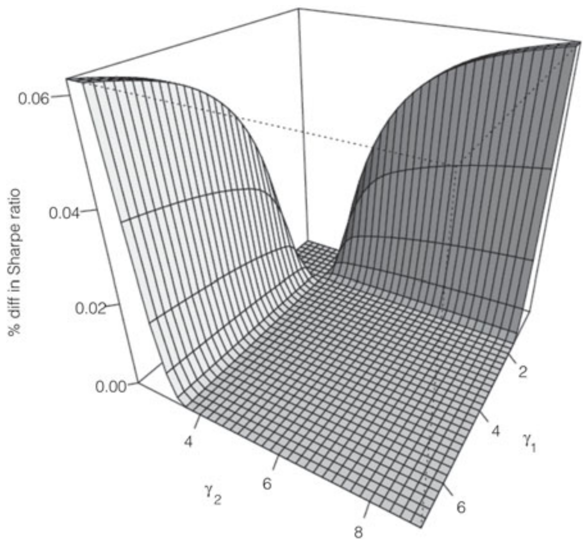
Even if we impose short-selling constraints at the level of each individual subportfolio, we will see that the loss in efficiency is very small. To explore this, we examine a simple case with two subportfolios, each of equal weight in the aggregate portfolio. For each subportfolio, we vary the risk aversion from 0 to 10. This is the same range used by Mehra and Prescott (1985). However, we note that most studies find that relative risk aversion lies in the range of 0 to 3. For all combinations of risk aversion in the subportfolios, we compare the Sharpe ratio of the subportfolio strategy against the Sharpe ratio of a single aggregate optimized portfolio with the same average return as the subportfolio approach. We report the percentage difference in Sharpe ratios in Figure 7. It is easy to see that investors with low risk aversion will suffer more efficiency loss than investors with high risk aversion if they invest using the MA framework. Nevertheless, the worst case percentage reduction in Sharpe ratio is very small, under 6% of the original Sharpe ratio. And the mean reduction in Sharpe ratio is much less than 1%. And when risk aversion is high and leverage is not sought, there is no loss in efficiency at all, since the short-selling constraint is not binding.

[9]Brunel (2006) conducts an analysis of goal-based portfolios and finds that the risk-adjusted loss is 8 bp. One may trace the origins of this idea to Markowitz (1983), who argues that a similar wedge invalidates the capital asset pricing model theoretically but not materially. Even naive allocation ($1/N$) strategies do almost as well as optimal allocations (see DeMiguel et al. (2009)).

FIGURE 7

Loss of Efficiency in Terms of Percentage Decline in Sharpe Ratios for Differing Levels of Risk Aversion

For two subportfolios, we varied the risk aversions (γ_1, γ_2) and obtained the subportfolio weights, assuming equal weights for each subportfolio. We then solved for the single aggregate portfolio that delivered the same expected return as the weighted average return of the subportfolios. The 3D plot presents the percentage difference in Sharpe ratios between the Markowitz and mental account approaches. The worst case of efficiency loss occurs when the two subportfolios have very low risk aversion. This is intuitive, since the tendency to want leverage is highest when risk aversion is low. If leverage is not desired, then the short-sale constraint is not binding and in this case mental account optimization results in no loss of efficiency. Nevertheless, the worst case percentage reduction in the Sharpe ratio can be seen to be around only 6%. The graph also makes clear that the level of risk aversion is relevant and not the difference in risk aversions of the subportfolios. The parameters are the same as those in Figure 4.



To stress the same example further, we induce greater leverage by reducing the return on the risk-free asset and increasing the returns on the risky assets. The results are shown in Figure 8. The magnitude of the efficiency loss remains small and is slightly smaller now that the baseline Sharpe ratios have increased.

As a final robustness test, we optimized portfolios on the following choice set of assets: the market return, a value portfolio (HML), and the risk-free asset. Therefore, we employed real data in the numerical simulations. Using annual data from 1927 to 2007, we computed the mean returns and covariance matrix and then repeated the comparison of mental accounts versus aggregate portfolio optimization. The results are in Figure 9 and show even smaller losses in efficiency than in the previous examples.

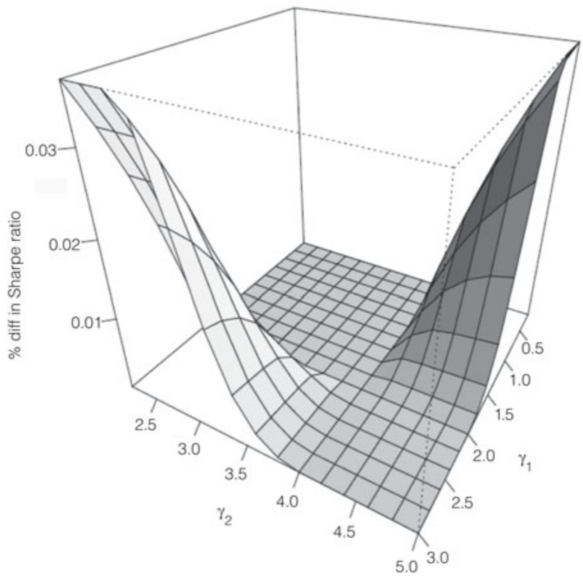
VI. Summary and Conclusions

While mean-variance portfolio theory (MVT) of Markowitz (1952) is silent about ultimate portfolio consumption goals, such goals are central in the behavioral portfolio theory (BPT) of Shefrin and Statman (2000). BPT investors do not consider their portfolios as a whole. Instead, BPT investors consider their portfolios as collections of mental account subportfolios where each subportfolio is

FIGURE 8

Loss of Efficiency in Terms of Percentage Decline in Sharpe Ratios for Differing Levels of Risk Aversion

For two subportfolios, we varied the risk aversions (γ_1, γ_2) and obtained the subportfolio weights, assuming equal weights for each subportfolio. We then solved for the single aggregate portfolio that delivered the same expected return as the weighted average return of the subportfolios. The 3D plot presents the percentage difference in Sharpe ratios between the Markowitz and mental account approaches. The worst case percentage reduction in the Sharpe ratio can be seen to be around only 3.5%. The parameters are as follows: the mean returns are $\{0.04, 0.11, 0.30\}$. The covariances are (row by row) $\{[0.00225, 0, 0], [0, 0.044, 0.022], [0, 0.022, 0.275]\}$. Compared to the parameters in Figure 7, here the return on the low-risk asset has been reduced and that on the risky assets has been increased. The variance of the low-risk asset has been reduced and that on the risky assets has been increased by 10% of the previous values to correspond to the changes in return.



associated with a goal and each goal has a threshold level. BPT investors care about the expected return of each subportfolio and its risk, measured by the probability of failing to reach the threshold level of return.

We integrate appealing features of MVT and BPT into a new mental accounting (MA) framework. Features of the MA framework include an MA structure of portfolios, a definition of risk as the probability of failing to reach the threshold level in each mental account, and attitudes toward risk that vary by account.

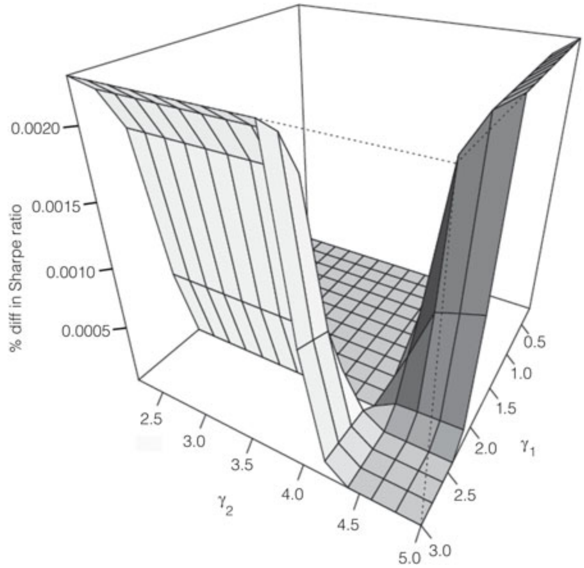
Once the investor specifies her subportfolio threshold levels and probabilities, the problem may be translated into a standard mean-variance problem with an *implied* risk-aversion coefficient. Aggregate portfolios composed of mean-variance efficient subportfolios are also mean-variance efficient. However, these portfolios are not identical to portfolios that are optimized by the rules of MVT with a weighted average of risk-aversion coefficients across mental accounts. When constraints are placed on short selling, aggregates of subportfolios are inefficient in comparison to a single optimal portfolio by only a few basis points. Portfolio inefficiency that arises from investors' inability to specify accurate mean-variance trade-offs in the aggregate portfolio level could be much larger.

The MA framework developed here provides a problem equivalence among MVT, MA, and risk management using VaR. This offers a basis for sharpening

FIGURE 9

Loss of Efficiency in Terms of Percentage Decline in Sharpe Ratios for Differing Levels of Risk Aversion

For two subportfolios, we varied the risk aversions (γ_1, γ_2) and obtained the subportfolio weights, assuming equal weights for each subportfolio. We then solved for the single aggregate portfolio that delivered the same expected return as the weighted average return of the subportfolios. The 3D plot presents the percentage difference in Sharpe ratios between the Markowitz and mental account approaches. The worst case percentage reduction in the Sharpe ratio can be seen to be around 0.25% (very small). The choice assets are the market return, the value (HML) portfolio, and the risk-free asset. The parameters are as follows: the mean returns are $\{0.1200, 0.0515, 0.0378\}$. The covariances are (row by row) $\{[0.0405, 0.0033, -0.0002], [0.0033, 0.0199, 0.0001], [-0.0002, 0.0001, 0.0009]\}$. These are based on annual data from 1927 to 2007.



aggregate portfolio choice through subportfolio optimization with preferences expressed in the intuitive language of thresholds and probabilities of failing to reach them. These generalizations of MVT and BPT via a unified MA framework result in a fruitful connection between investor consumption goals and portfolio production.

Extensions to this work involve extending the mental account optimization framework to nonnormal multivariate distributions of asset returns, to introducing products that have option-like features, and to optimization with background risks such as labor and real estate.[10]

Appendix. Derivation of Equation (3)

To solve this maximization problem, we set up the Lagrangian with coefficient λ:

$$\text{(A-1)} \qquad \max_{w,\lambda} L \quad = \quad w'\,\mu - \frac{\gamma}{2}\,w'\,\Sigma\,w + \lambda[1 - w'\mathbf{1}].$$

[10]For the latter extension, see related work by Baptista (2008).

The first-order conditions are

$$\text{(A-2)} \qquad \frac{\partial L}{\partial w} = \mu - \gamma \Sigma w - \lambda \mathbf{1} = 0,$$

$$\text{(A-3)} \qquad \frac{\partial L}{\partial \lambda} = 1 - w'\mathbf{1} = 0.$$

Note that equation (A-2) is a system of n equations. Rearranging equation (A-2) gives

$$\text{(A-4)} \qquad \Sigma w = \frac{1}{\gamma}[\mu - \lambda \mathbf{1}],$$

and premultiplying both sides of this equation by Σ^{-1} gives

$$\text{(A-5)} \qquad w = \frac{1}{\gamma}\Sigma^{-1}[\mu - \lambda \mathbf{1}].$$

The solution here for portfolio weights is not yet complete, as the equation contains λ, which we still need to solve for. Premultiplying equation (A-5) by $\mathbf{1}'$ gives

$$\text{(A-6)} \qquad \mathbf{1}'w = \frac{1}{\gamma}\mathbf{1}'\Sigma^{-1}[\mu - \lambda \mathbf{1}],$$

$$\text{(A-7)} \qquad 1 = \frac{1}{\gamma}[\mathbf{1}'\Sigma^{-1}\mu - \lambda \mathbf{1}'\Sigma^{-1}\mathbf{1}],$$

which can now be solved for λ to get

$$\text{(A-8)} \qquad \lambda = \frac{\mathbf{1}'\Sigma^{-1}\mu - \gamma}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}.$$

Plugging λ back into equation (A-5) gives the closed-form solution for the optimal portfolio weights:

$$w = \frac{1}{\gamma}\Sigma^{-1}\left[\mu - \left(\frac{\mathbf{1}'\Sigma^{-1}\mu - \gamma}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\right)\mathbf{1}\right] \in R^n.$$

This optimal solution w is an n-vector and is easily implemented, given that it is analytical.

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