Bull. Aust. Math. Soc. **84** (2011), 504–515
doi:10.1017/S0004972711002589

STRICT TOPOLOGY AS A MIXED TOPOLOGY ON LEBESGUE SPACES

SAEID MAGHSOUDI ✉ and RASOUL NASR-ISFAHANI

(Received 14 April 2011)

Abstract

Let X be a locally compact space, and $\mathfrak{L}_0^\infty(X, \iota)$ be the space of all essentially bounded ι-measurable functions f on X vanishing at infinity. We introduce and study a locally convex topology $\beta^1(X, \iota)$ on the Lebesgue space $\mathfrak{L}^1(X, \iota)$ such that the strong dual of $(\mathfrak{L}^1(X, \iota), \beta^1(X, \iota))$ can be identified with $(\mathfrak{L}_0^\infty(X, \iota), \|\cdot\|_\infty)$. Next, by showing that $\beta^1(X, \iota)$ can be considered as a natural mixed topology, we deduce some of its basic properties. Finally, as an application, we prove that $L^1(G)$, the group algebra of a locally compact Hausdorff topological group G, equipped with the convolution multiplication is a complete semitopological algebra under the $\beta^1(G)$ topology.

2010 *Mathematics subject classification*: primary 46A03, 46A70; secondary 46H05, 43A20.

Keywords and phrases: group algebras, Lebesgue spaces, locally compact group, locally convex topology, mixed topology, Radon measure, strict topology.

1. Introduction

Let X be a locally compact space and ι be a positive Radon measure on X; that is, $\iota : \mathcal{B}(X) \to [0, \infty]$ is a regular Borel measure such that it is finite on compacta.

Let $\mathfrak{L}^1(X, \iota)$ be the Lebesgue space as defined in [7]; that is, the Banach space of all ι-measurable functions φ on X such that

$$\|\varphi\|_1 := \int_X |\varphi(x)|\, d\iota(x) < \infty.$$

We denote by $n(X, \iota)$ the topology generated by the norm $\|\cdot\|_1$.

A subset B of X is said to be locally ι-null if $\iota(B \cap C) = 0$ for all compact subsets C of X. A property that holds except for a locally ι-null set is said to hold locally ι-almost everywhere. Let also $\mathfrak{L}^\infty(X, \iota)$ denote the Lebesgue space as defined in [7]; that is, the Banach space of all ι-measurable functions f on X such that

$$\|f\|_\infty := \inf\{c > 0 : |f| \le c \text{ locally } \iota\text{-almost everywhere on } X\} < \infty.$$

Then $\mathfrak{L}^\infty(X,\iota)$ is the dual of $\mathfrak{L}^1(X,\iota)$ equipped with the norm topology $n(X,\iota)$. In fact, the mapping $\Phi : \mathfrak{L}^\infty(X,\iota) \to \mathfrak{L}^1(X,\iota)^*$ defined by

$$\langle \Phi(f), \varphi\rangle = \int_X f(x)\varphi(x)\, d\iota(x)$$

for $f \in \mathfrak{L}^\infty(X,\iota)$ and $\varphi \in \mathfrak{L}^1(X,\iota)$ is an isometric isomorphism. We follow [7] in our definitions and notation for Lebesgue spaces.

We denote by $\mathfrak{L}_0^\infty(X,\iota)$ the subspace of $\mathfrak{L}^\infty(X,\iota)$ consisting of all functions f on X such that for each $\varepsilon > 0$ there is a compact subset C of X for which $\|f\xi_{X\setminus C}\|_\infty < \varepsilon$, where $\xi_{X\setminus C}$ denotes the characteristic function of $X \setminus C$ on X.

In this paper, we introduce and study a locally convex topology $\beta^1(X,\iota)$ on $\mathfrak{L}^1(X,\iota)$ such that $\mathfrak{L}_0^\infty(X,\iota)$ can be identified with the strong dual of $\mathfrak{L}^1(X,\iota)$. We then show that, except for the trivial case where X is finite, there are infinitely many such locally convex topologies $\mathfrak{T}$ on $\mathfrak{L}^1(X,\iota)$, and hence $\mathfrak{L}_0^\infty(X,\iota)^*$ can be considered as the second dual of $(\mathfrak{L}^1(X,\iota),\mathfrak{T})$. We show that $\beta^1(X,\iota)$ can be viewed as a mixed topology. By this observation, we prove some general properties of the locally convex space $(\mathfrak{L}^1(X,\iota),\beta^1(X,\iota))$. This generalizes and places our recent work [9] in a correct frame. In particular, we show that $(\mathfrak{L}^1(X,\iota),\beta^1(X,\iota))$ is bornological, barreled or metrizable space if and only if X is compact. Finally, as an application we completely answer a question raised in [13] about continuity of the convolution in the group algebras of locally compact groups.

2. The space $\mathfrak{L}^1(X,\iota)$ with a locally convex topology

Throughout this paper, let X be a locally compact space and ι be a positive Radon measure on X with $\mathrm{supp}(\iota) = X$. We denote by $\mathcal{C}$ the set of increasing sequences of compact subsets of X and by $\mathcal{R}$ the set of increasing sequences (r_n) of real numbers in $(0,\infty)$ with $r_n \to \infty$. For any $(C_n) \in \mathcal{C}$ and $(r_n) \in \mathcal{R}$, set

$$U((C_n),(r_n)) = \{\varphi \in \mathfrak{L}^1(X,\iota) : \|\varphi\xi_{C_n}\|_1 \le r_n \text{ for all } n \ge 1\},$$

and note that $U((C_n),(r_n))$ is a convex balanced absorbing set in the space $\mathfrak{L}^1(X,\iota)$. It is easy to see that the family $\mathcal{U}$ of all sets $U((C_n),(r_n))$ for $(C_n) \in \mathcal{C}$ and $(r_n) \in \mathcal{R}$ is a base of neighborhoods of zero for a locally convex topology on $\mathfrak{L}^1(X,\iota)$; see for example [14, Theorem 1.18]. We denote this topology by $\beta^1(X,\iota)$ and call it *strict topology*. This topology was introduced in [13] for group algebras. For a recent study of this topology in the semigroup algebra setting see [10, 11]; see also [9].

It is clear that $\beta^1(X,\iota) \subseteq n(X,\iota)$. Our first result shows under which conditions these topologies coincide.

PROPOSITION 2.1. *Let X be a locally compact space and ι be a positive Radon measure on X with $\mathrm{supp}(\iota) = X$. The norm topology $n(X,\iota)$ on $\mathfrak{L}^1(X,\iota)$ coincides with the strict topology $\beta^1(X,\iota)$ if and only if X is compact.*

Proof. Suppose the norm topology and strict topology coincide. Consider the set

$$U := \{\varphi \in \mathfrak{L}^1(X, \iota) : \|\varphi\|_1 < 1\},$$

and note that U is $n(X, \iota)$-open, and thus $\beta^1(X, \iota)$-open. It follows that there is a sequence $((C_n), (r_n))$ in $C \times \mathcal{R}$ such that $U((C_n), (r_n)) \subseteq U$.

We show that $X = C_m$ for some $m \geq 1$. To this end, suppose on the contrary that $X \setminus C_m \neq \emptyset$ for all $m \geq 1$. Fix $m \geq 1$ and apply the assumption to choose a compact subset A of $X \setminus C_m$ such that $\iota(A) > 0$. Set

$$\varphi := \left(\frac{r_m}{\iota(A)}\right) \xi_A.$$

We have

$$\begin{cases} \|\varphi \xi_{C_n}\|_1 = 0 & \text{for all } n < m \\ \|\varphi \xi_{C_n}\|_1 \leq r_m \leq r_n & \text{for all } n \geq m. \end{cases}$$

Thus $\varphi \in U((C_n), (r_n))$ whence $\varphi \in U$; that is, $\|\varphi\|_1 < 1$. On the other hand,

$$r_m = \left\| \frac{r_m}{\iota(A)} \xi_A \right\|_1 = \|\varphi\|_1,$$

which contradicts the fact that $r_n \to \infty$. The converse is clear. □

We denote by $\sigma_0(X, \iota)$ the weak topology $\sigma(\mathfrak{L}^1(X, \iota), \Phi(\mathfrak{L}_0^\infty(X, \iota)))$. Let us remark that $\sigma_0(X, \iota) \subseteq \beta^1(X, \iota) \subseteq n(X, \iota)$. It is clear that $\sigma_0(X, \iota) = \beta^1(X, \iota)$ if X is finite. The next result shows that this is an 'if and only if' statement.

Proposition 2.2. *Let X be a locally compact space and ι be a positive Radon measure on X with* $\operatorname{supp}(\iota) = X$. *The weak topology $\sigma_0(X, \iota)$ on $\mathfrak{L}^1(X, \iota)$ coincides with the strict topology $\beta^1(X, \iota)$ if and only if X is finite.*

Proof. By Proposition 2.1, without loss of generality, we may assume that X is noncompact. In fact, if X is compact, then $\beta^1(X, \iota) = n(X, \iota)$ and $\mathfrak{L}_0^\infty(X, \iota) = \mathfrak{L}^\infty(X, \iota)$. Hence $\mathfrak{L}^1(X, \iota)$ is finite dimensional and, since ι is Radon with full support, X must be finite.

So, suppose that X is not compact. Then there exists a sequence $(C_n) \in C$ such that $\iota(C_n \setminus C_{n-1}) > 0$, where $C_0 = \emptyset$. If $r_n = n^2$ for all $n \geq 1$, then $(r_n) \in \mathcal{R}$ and $U((C_n), (r_n))$ is a $\beta^1(X, \iota)$-neighborhood of zero. Let

$$\varphi_n = \iota(C_n \setminus C_{n-1})^{-1} \xi_{C_n \setminus C_{n-1}}.$$

Then $\varphi_n \in \mathfrak{L}^1(X, \iota)$.

The set $\{\varphi_n : n \geq 1\}$ is a linearly independent set in $\mathfrak{L}^1(X, \iota)$ and $\|\varphi_n \xi_{C_n}\|_1 = 1$. Let E be the subspace of $\mathfrak{L}^1(X, \iota)$ consisting of all $\varphi \in \mathfrak{L}^1(X, \iota)$ with

$$\|\varphi \xi_{C_n}\|_1 = 0 \quad \text{for all } n \geq 1,$$

https://doi.org/10.1017/S0004972711002589 Published online by Cambridge University Press

and note that $\varphi_n \notin E$ for all $n \geq 1$. Then E has an infinite codimension. It follows that any subspace F of $\mathfrak{L}^1(X, \iota)$ contained in $U((C_n), (r_n))$ has an infinite codimension; this is because $F \subset E$. Since any $\sigma_0(X, \iota)$-neighborhood of zero contains a subspace of $\mathfrak{L}^1(X, \iota)$ with a finite codimension, $U((C_n), (r_n))$ is not a $\sigma_0(X, \iota)$-neighborhood of zero, whereas it is a $\beta^1(X, \iota)$-neighborhood. □

We now investigate some properties of the strict topology $\beta^1(X, \iota)$ on $\mathfrak{L}^1(X, \iota)$. We prove that the strict topology is a natural mixed topology. This enables us to establish general properties of $\beta^1(X, \iota)$ topology. Let us, for the convenience of the reader, recall some basic definitions and results concerning mixed topology. For more details see [2]; see also [1, 8] for more recent studies.

Let E be a vector space with two locally convex topologies τ and τ^* satisfying the following conditions.

(i) $\tau^* \subseteq \tau$.
(ii) (E, τ) is a DF-space with a base (B_n) of absolutely convex bounded sets such that $B_n + B_n \subseteq B_{n+1}$ for each n.
(iii) Each B_n is τ^*-closed.

The *mixed topology* $\gamma = \gamma[\tau^*, \gamma]$ on E is that locally convex topology whose base at zero consists of all sets of the form

$$\bigcup_{n=1}^{\infty}(U_1^* \cap B_1 + \cdots + U_n^* \cap B_n),$$

where (U_n^*) is a sequence of absolutely convex τ^*-neighborhoods of zero.

By $\kappa(X, \iota)$ we denote the locally convex topology on $\mathfrak{L}^1(X, \iota)$ generated by the seminorms $\mathcal{P}_C(\varphi) = \int_C |\varphi| \, d\iota$, where C runs through compact subsets of X.

Proposition 2.3. *Let X be a locally compact space and ι be a positive Radon measure on X with* $\operatorname{supp}(\iota) = X$. *The strict topology $\beta^1(X, \iota)$ is the mixed topology* $\gamma[\kappa(X, \iota), n(X, \iota)]$.

Proof. Let (C_n) be an increasing sequence of compact subsets of X and let $\varphi \in \mathfrak{L}^1(X, \iota)$. Since $\varphi = \alpha|\varphi|$ for some function $\alpha \in \mathfrak{L}^\infty(X, \iota)$ with $|\alpha(x)| = 1$ for $x \in X$, without loss of generality we can assume that $\varphi \geq 0$. Take $\varepsilon > 0$ and $k \in \mathbb{N}$. For $n = 1, \ldots, k$, by the regularity of ι, we can find open subsets U_n such that $C_n \subseteq U_n$ and

$$\int_{U_n} |\varphi| \, d\iota \leq \int_{C_n} |\varphi| \, d\iota + \varepsilon.$$

By Urysohn's lemma, one can choose continuous functions φ_n such that

$$\varphi_n(C_n) = \{1\}, \quad \varphi_n(X \setminus U_n) = \{0\}, \quad 0 \leq \varphi_n \leq 1,$$

for $n = 1, \ldots, k$. Now define the functions ϕ_1 and ϕ_2 on X by setting

$$\phi_1 = \max_{1 \leq n \leq k}((1 - \varphi_n)\varphi), \quad \phi_2 = \varphi - \phi_1.$$

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Then $\mathcal{P}_{C_n}(\phi_1) = 0$ for $n = 1, \ldots, k$, and

$$\begin{aligned} \|\phi_2\|_1 &= \int_X \varphi\left(\min_{1\le n\le k}(\varphi_n)\right) d\iota \\ &\le \max\{\mathcal{P}_{U_n}(\varphi) : n = 1, \ldots, k\} \\ &\le \max\{\mathcal{P}_{C_n}(\varphi) : n = 1, \ldots, k\} + \varepsilon. \end{aligned}$$

To complete the proof, we only need to invoke [15, Theorem 3.1.1]. □

This result, together with Cooper [2, Section I.1], gives the following theorem.

Theorem 2.4. *Let X be a locally compact space and ι be a positive Radon measure on X with* $\operatorname{supp}(\iota) = X$. *Then the following properties hold.*

(i) *A subset of $\mathfrak{L}^1(X, \iota)$ is $n(X, \iota)$-bounded if and only if it is $\beta^1(X, \iota)$-bounded.*
(ii) *$\beta^1(X, \iota) = \kappa(X, \iota)$ on $n(X, \iota)$-bounded subsets of $\mathfrak{L}^1(X, \iota)$.*
(iii) *A sequence in $\mathfrak{L}^1(X, \iota)$ is $\beta^1(X, \iota)$-convergent to zero if and only if it is $n(X, \iota)$-bounded and $\kappa(X, \iota)$-convergent to zero.*
(iv) *A linear map from $\mathfrak{L}^1(X, \iota)$ into a locally convex space is $\beta^1(X, \iota)$-continuous if and only if its restriction to $n(X, \iota)$-bounded sets is $\kappa(X, \iota)$-continuous.*
(v) *$\beta^1(X, \iota)$ is the finest locally convex topology on $\mathfrak{L}^1(X, \iota)$ which agrees with $\kappa(X, \iota)$ (or $\beta^1(X, \iota)$) on $n(X, \iota)$-bounded subsets of $\mathfrak{L}^1(X, \iota)$.*
(vi) *A subset of $\mathfrak{L}^1(X, \iota)$ is $\beta^1(X, \iota)$-compact if and only if it is $n(X, \iota)$-bounded and $\kappa(X, \iota)$-compact.*
(vii) *$\mathfrak{L}^1(X, \iota)$ is $\beta^1(X, \iota)$-complete if and only if each $n(X, \iota)$-bounded set is $\kappa(X, \iota)$-complete.*

We recall that a locally convex space (E, τ) is called a *barreled space* if each barrel set (that is, a closed convex balanced absorbing set) in E is a neighborhood of zero; it is called a *bornological space* when every convex balanced subset that absorbs bounded subsets in E is a neighborhood of zero.

Proposition 2.5. *Let X be a locally compact space and ι be a positive Radon measure on X with* $\operatorname{supp}(\iota) = X$. *The following assertions are equivalent.*

(a) *$(\mathfrak{L}^1(X, \iota), \beta^1(X, \iota))$ is barreled.*
(b) *$(\mathfrak{L}^1(X, \iota), \beta^1(X, \iota))$ is bornological.*
(c) *$(\mathfrak{L}^1(X, \iota), \beta^1(X, \iota))$ is metrizable.*
(d) *X is compact.*

Proof. Let us recall from [2] that a mixed topology $\gamma[\tau^*, \tau]$ is a bornological or barreled space if $\tau = \gamma[\tau^*, \tau]$. This, together with the fact that any metrizable space is bornological and Proposition 2.1, gives the result. □

We end the section by the following important result.

Proposition 2.6. *Let X be a locally compact space and ι be a positive Radon measure on X with* $\operatorname{supp}(\iota) = X$. *Then $\mathfrak{L}^1(X, \iota)$ is $\beta^1(X, \iota)$-complete.*

Proof. If (φ_α) is a $n(X,\iota)$-bounded net in $\mathfrak{L}^1(X,\iota)$ which is $\kappa(X,\iota)$-Cauchy, then $(\varphi_{\alpha|C})$ is an $n(X,\iota)$-Cauchy net in $\mathfrak{L}^1(X,\iota)$ for each compact subset $C \subseteq X$. Let φ_C be the limit of $(\varphi_{\alpha|C})$ in the norm topology of $\mathfrak{L}_1(C,\iota_{|C})$. It is obvious that for each two compact subsets C_1 and C_2 with $C_1 \subseteq C_2$

$$(\varphi_{C_2})_{|C_1} = \varphi_{C_1}$$

ι-almost everywhere on C_1. By [4, Lemma B, page 25], there exists an ι-measurable function φ such that $\varphi_{|C} = \varphi_C$ for all compact subsets C. However, $\|\varphi_C\|_1 \le M$ for all compact sets C, where $M > 0$ is a constant with $\|\varphi_\alpha\|_1 \le M$ for all α. Now, the regularity of ι implies that $\varphi \in \mathfrak{L}^1(X,\iota)$ and $\varphi_\alpha \to \varphi$ in the $\kappa(X,\iota)$-topology. In view of Theorem 2.4(vii), the proof is now complete. □

3. The dual of $\mathfrak{L}^1(X,\iota)$ with the strict topology

We start by recalling some definitions and notation. We denote by $\tau_b(X,\iota)$ the strong topology on $(\mathfrak{L}^1(X,\iota),\beta^1(X,\iota))^*$; that is, the topology of uniform convergence on bounded subsets of $\mathfrak{L}^1(X,\iota)$ with respect to the weak topology $\sigma(\mathfrak{L}^1(X,\iota),(\mathfrak{L}^1(X,\iota),\beta^1(X,\iota))^*)$. We also denote by $\tau_n(X,\iota)$ the topology induced by the norm topology of the dual Banach space $(\mathfrak{L}^1(X,\iota),n(X,\iota))^*$ on $(\mathfrak{L}^1(X,\iota),\beta^1(X,\iota))^*$, that is the topology given by the norm defined by

$$\|f\| = \sup\{|f(\varphi)| : \varphi \in \mathfrak{L}^1(X,\iota), \|\varphi\|_1 = 1\},$$

for all $f \in (\mathfrak{L}^1(X,\iota),\beta^1(X,\iota))^*$. As an immediate consequence of part (i) of Theorem 2.4, we obtain the following result.

Proposition 3.1. *Let X be a locally compact space and ι be a positive Radon measure on X with* $\mathrm{supp}(\iota) = X$. *The topologies $\tau_b(X,\iota)$ and $\tau_n(X,\iota)$ are equivalent on* $(\mathfrak{L}^1(X,\iota),\beta^1(X,\iota))^*$.

We now present the key result of this section which is an improvement of [13, Theorem 2] and [9, Theorem 3.1].

Theorem 3.2. *Let X be a locally compact space and ι be a positive Radon measure on X with* $\mathrm{supp}(\iota) = X$. *The dual of $(\mathfrak{L}^1(X,\iota),\beta^1(X,\iota))$ endowed with $\tau_b(X,\iota)$ can be identified with $\mathfrak{L}_0^\infty(X,\iota)$ endowed with $\|\cdot\|_\infty$-topology.*

Proof. We first show that $\Phi(\mathfrak{L}_0^\infty(X,\iota)) \subseteq (\mathfrak{L}^1(X,\iota),\beta^1(X,\iota))^*$. Let g be in $\mathfrak{L}_0^\infty(X,\iota)$ and $\varepsilon > 0$ be given. Choose an element $((C_n),(r_n))$ of $C \times \mathcal{R}$ with $r_1 \ge 2$ such that

$$|g(x)| \le \varepsilon r_n^{-2} \quad (n \ge 1)$$

for locally ι-almost all $x \in X \setminus C_n$. We show that

$$|\langle \Phi(g),\varphi\rangle| \le \varepsilon \quad \text{for all } \varphi \in U((C_n),(r_n))$$

from which it follows that $\Phi(g) \in (\mathfrak{L}^1(X,\iota),\beta^1(X,\iota))^*$.

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To this end, let $\varphi \in U((C_n),(r_n))$, and set $C_0 = \emptyset$ and $r_0 = 2$. Since $g(x) = 0$ for locally ι-almost all $x \in X \setminus \bigcup_{n=2}^{\infty} C_n$, it follows from $\bigcup_{n=2}^{\infty} C_n = \bigcup_{n=0}^{\infty}(C_{n+1} \setminus C_n)$ that

$$\begin{aligned}
|\langle \Phi(g), \varphi\rangle| &= \left|\int_X g(x)\varphi(x)\, d\iota(x)\right| \\
&= \int_{\bigcup_{n=2}^{\infty} C_n} |g(x)|\varphi(x)\, d\iota(x) \\
&\le \sum_{n=0}^{\infty} \int_{C_{n+1}\setminus C_n} |g(x)|\varphi(x)\, d\iota(x) \\
&\le \sum_{n=0}^{\infty} \varepsilon r_n^{-2} \int_{C_{n+1}\setminus C_n} |\varphi(x)|\, d\iota(x).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{n=0}^{m} r_n^{-2} \int_{C_{n+1}\setminus C_n} |\varphi(x)|\, d\iota(x) &= \sum_{n=0}^{m}(r_n^{-2} - r_{n+1}^{-2}) \int_{C_{n+1}\setminus C_1} |\varphi(x)|\, d\iota(x) \\
&\quad + r_{m+1}^{-2} \int_{C_{m+1}\setminus C_1} |\varphi(x)|\, d\iota(x) \\
&\le \sum_{n=0}^{m} 2\,(r_n^{-1} - r_{n+1}^{-1}) r_n^{-1} \int_{C_{n+1}} |\varphi(x)|\, d\iota(x) \\
&\quad + r_{m+1}^{-2} \int_{C_{m+1}} |\varphi(x)|\, d\iota(x) \\
&\le \sum_{n=0}^{m} 2\,(r_n^{-1} - r_{n+1}^{-1}) + r_{m+1}^{-1} \\
&< 2r_0^{-1} - r_{m+1}^{-1} < 1.
\end{aligned}$$

Consequently, $|\langle \Phi(g), \varphi\rangle| \le \varepsilon$ as required.

Now, let F be a $\beta^1(X, \iota)$-continuous functional on $\mathfrak{L}^1(X, \iota)$. Then there is an element $((C_n), (r_n))$ in $\mathcal{C} \times \mathcal{R}$ such that

$$|\langle F, \varphi\rangle| < 1 \quad \text{for all } \varphi \in U((C_n), (r_n)).$$

It is clear that F is also norm continuous on $\mathfrak{L}^1(X, \iota)$. It follows $F = \Phi(g)$ for some $g \in \mathfrak{L}^\infty(X, \iota)$. We show that $g \in \mathfrak{L}_0^\infty(X, \iota)$. It suffices to prove that $|g(x)| \le r_n^{-1}$ for all $n \ge 1$ and locally ι-almost all $x \in X \setminus C_n$.

To this end, suppose on the contrary that there exist $m \ge 1$ such that $\iota(B) > 0$, where

$$B = \{x \in X \setminus C_m : |g(x)| > r_m^{-1}\}.$$

Thus, there is a compact subset K of B such that $0 < \iota(K)$ and also a function $f \in \mathfrak{L}^\infty(X, \iota)$ such that $fg = |g|$ and $\|f\|_\infty \le 1$. Let φ be a function in $\mathfrak{L}^1(X, \iota)$ with

$$\iota(K)\varphi = r_m f \xi_K.$$

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Then

$$\begin{aligned}\iota(K)\left|\int_X g(x)\varphi(x)\,d\iota(x)\right| &= \left|\int_X r_m f g \xi_K\,d\iota\right| \\ &= r_m \int_K |g(x)|\,d\iota(x) \\ &> \iota(K) r_m r_m^{-1} = \iota(K);\end{aligned}$$

that is, $|\langle F, \varphi\rangle| > 1$, which contradicts the fact that $\varphi \in U((C_n), (r_n))$. Therefore

$$\Phi(\mathfrak{L}_0^\infty(X, \iota)) = (\mathfrak{L}^1(X, \iota), \beta^1(X, \iota))^*.$$

Moreover, $\|g\|_\infty = \|\Phi(g)\|$ for all $g \in \mathfrak{L}_0^\infty(X, \iota)$, and so Φ is an identification from $\mathfrak{L}_0^\infty(X, \iota)$ endowed with $\|\cdot\|_\infty$-topology onto $(\mathfrak{L}^1(X, \iota), \beta^1(X, \iota))^*$ endowed with norm topology. Now, invoke Corollary 3.1 to complete the proof. □

The next corollary shows that, except in some trivial cases, there exists uncountably many locally convex topologies compatible with the duality between $\mathfrak{L}^1(X, \iota)$ and $\mathfrak{L}_0^\infty(X, \iota)$.

Corollary 3.3. *Let X be an infinite locally compact space and ι be a positive Radon measure on X with $\operatorname{supp}(\iota) = X$. Then there exist uncountably many locally convex topologies $\mathfrak{T}$ on $\mathfrak{L}^1(X, \iota)$ such that $\sigma_0(X, \iota) \subseteq \mathfrak{T} \subseteq \beta^1(X, \iota)$.*

Proof. Since X is infinite it follows from Proposition 2.2 that $\sigma_0(X, \iota) \subset \beta^1(X, \iota)$. We now only need to recall from [12] that the only case in which the dual pair generates a finite number of polar topologies is when all polar topologies are equal to the weak topology; also, if there is more than one topology for the dual pair then there are uncountably many such topologies. □

As an immediate consequence of Theorem 3.2, we have the following result.

Corollary 3.4. *Let X be an infinite locally compact space and ι be a positive Radon measure on X with $\operatorname{supp}(\iota) = X$. Then there exist uncountably many locally convex topologies $\mathfrak{T}$ on $\mathfrak{L}^1(X, \iota)$ for which the dual of $(\mathfrak{L}^1(X, \iota), \mathfrak{T})$ endowed with the strong topology is $\mathfrak{L}_0^\infty(X, \iota)$ endowed with $\|\cdot\|_\infty$-topology.*

Let us recall that a locally convex space (E, τ) is said to be a *dual space* if there exists a locally convex space (E_0, τ_0) such that (E, τ) coincides with the strong dual of (E_0, τ_0).

Corollary 3.5. *Let X be a locally compact space and ι be a positive Radon measure on X with $\operatorname{supp}(\iota) = X$. Then $(\mathfrak{L}^1(X, \iota), \beta(X, \iota))$ is a dual space if and only if X is compact.*

Proof. If X is compact, then $(\mathfrak{L}^1(X, \iota), \beta(X, \iota))$ is a barreled space by Proposition 2.5. This implies that $(\mathfrak{L}^1(X, \iota), \beta(X, \iota))$ is a dual space; see [5, Lemma 3.1].

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Conversely, suppose that $(\mathfrak{L}^1(X,\iota),\beta(X,\iota))$ is a dual space. Since the dual space of $(\mathfrak{L}^1(X,\iota),\beta(X,\iota))$ is the normed space $\mathfrak{L}_0^\infty(X,\iota)$, it follows that $(\mathfrak{L}^1(X,\iota),\beta(X,\iota))$ is also a normed space; this is because of the fact that if the dual of a dual space (E,τ) is normed space then (E,τ) is a normed space; see [5, Lemma 3.2]. By Proposition 2.5, this can only happen when X is compact. □

Theorem 3.6. *Let X be a locally compact noncompact space and ι be a positive Radon measure on X with* $\operatorname{supp}(\iota) = X$. *Then $\mathfrak{L}_0^\infty(X,\iota)$ is not complemented in $\mathfrak{L}^\infty(X,\iota)$.*

Proof. Since X is a locally compact space, there exists a sequence (U_n) of disjoint compact neighborhoods in X; see [3, Appendix 1.3].

Now, let $I : l^\infty \to \mathfrak{L}^\infty(X,\iota)$ and $R : \mathfrak{L}_0^\infty(X,\iota) \to c_0$ be the linear maps defined by

$$I((\alpha_n)) = \sum_{n=1}^{\infty} \alpha_n \xi_{U_n}$$

for all $(\alpha_n) \in l^\infty$, and

$$R(g) = \left(\frac{1}{\iota(U_n)} \int_{U_n} g(x)\, d\iota(x)\right)_{n\ge1}$$

for all $g \in \mathfrak{L}_0^\infty(X,\iota)$. Clearly both maps are continuous; indeed, $\|I(\alpha_n)\|_\infty = \|(\alpha_n)\|_\infty$ for all $(\alpha_n) \in \ell^\infty$, whence $\|I\| = 1$; moreover, $\|R(g)\|_\infty \le \|g\|_\infty$ for all $g \in \mathfrak{L}_0^\infty(X,\iota)$ and $\|R(\xi_{U_n})\|_\infty = 1$ whence $\|R\| = 1$. Next, suppose on the contrary that there exists a continuous linear projection

$$P : \mathfrak{L}^\infty(X,\iota) \to \mathfrak{L}_0^\infty(X,\iota).$$

If $Q : l^\infty \to c_0$ is the composition $R \circ P \circ I$, then $I((\alpha_n)) \in \mathfrak{L}_0^\infty(X,\iota)$ for all $(\alpha_n) \in c_0$, and we have

$$\begin{aligned} Q((\alpha_n)) &= R\Big(\sum_{m=1}^{\infty} \alpha_m \xi_{U_m}\Big) \\ &= \left(\frac{1}{\iota(U_n)} \int_{U_n} \Big(\sum_{m=1}^{\infty} \alpha_m \xi_{U_m}(x)\Big)\, d\iota(x)\right) \\ &= (\alpha_n). \end{aligned}$$

Hence $Q : l^\infty \to c_0$ is a projection which contradicts the fact that c_0 is not complemented in l^∞; see for example [14, Theorem 27.5]. □

4. An application to group algebras

Let G denote a locally compact group with a fixed left Haar measure λ. The group algebra $L^1(G) := \mathfrak{L}^1(G,\lambda)$ is defined as in [6] equipped with the convolution product $*$ defined by

$$(\varphi * \psi)(s) = \int_G \Delta(t^{-1})\varphi(st^{-1})\psi(t)\, d\lambda(t), \qquad \varphi,\psi \in L^1(G),$$

where Δ denotes the modular function on G.

https://doi.org/10.1017/S0004972711002589 Published online by Cambridge University Press

Now making use of Proposition 2.6 and Theorem 2.4, we are in a position to answer a question raised in Singh [13] about the continuity of the convolution product in $L^1(G)$ under the strict topology $\beta^1(G) := \beta^1(G, \lambda)$.

THEOREM 4.1. *Let G be a locally compact Hausdorff group. Then $(L^1(G), \beta^1(G))$ with the convolution as multiplication is a complete semitopological algebra.*

PROOF. The completeness follows from Proposition 2.6. Now, in view of Theorem 2.4 parts (ii) and (vii), we only need to show that convolution product is $\beta^1(G)$-continuous on norm-bounded subsets of $L^1(G)$. Let $(\varphi_\alpha) \subseteq L^1(G)$ be a norm-bounded net which is $\beta^1(G)$-convergent to zero. Let $\psi \in L^1(G)$ and let $U((C_n), (r_n))$ be an arbitrary $\beta^1(G)$-neighborhood of zero. Choose a compact set $C \subseteq G$ such that

$$\int_{G\backslash C} |\psi(t)|\, d\lambda(t) < \frac{r_1}{2M},$$

where M is a bound for the net (φ_α). Then $C_n C$ is compact. Hence, if we put

$$K_n := C_n C^{-1} \quad \text{and} \quad b_n := \frac{r_n}{2\|\psi\|_1},$$

then $((K_n), (b_n)) \in \mathcal{C} \times \mathcal{R}$, and so there is α_0 such that $\varphi_\alpha \in U((K_n), (b_n))$ for all $\alpha \geq \alpha_0$.

Now, on the one hand,

$$\begin{aligned}\int_{C_n}\int_C |\varphi_\alpha(st^{-1})|\,|\psi(t)|\Delta(t^{-1})\, d\lambda(t)\, d\lambda(s) &= \int_C |\psi(t)|\Delta(t^{-1}) \int_{C_n t^{-1}} \Delta(t)|\varphi_\alpha(s)|\, d\lambda(s)\, d\lambda(t)\\ &\leq \int_C |\psi(t)| \int_{C_n C^{-1}} |\varphi_\alpha(s)|\, d\lambda(s)\, d\lambda(t)\\ &\leq \int_C |\psi(t)|\, \|\varphi_\alpha \xi_{C_n C^{-1}}\|_1\, d\lambda(t)\\ &\leq \|\psi\|_1 \|\varphi_\alpha \xi_{K_n}\|_1.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{C_n}\int_{G\backslash C} |\varphi_\alpha(st^{-1})|\,|\psi(t)|\Delta(t^{-1})\, d\lambda(t)\, d\lambda(s)\\ &\quad= \int_{G\backslash C} |\psi(t)|\Delta(t^{-1}) \int_{C_n t^{-1}} \Delta(t)|\varphi_\alpha(s)|\, d\lambda(s)\, d\lambda(t)\\ &\quad\leq \int_{G\backslash C} |\psi(t)| \int_G |\varphi_\alpha(s)|\, d\lambda(s)\, d\lambda(t)\\ &\quad\leq \|\varphi_\alpha\|_1 \int_{G\backslash C} |\psi(t)|\, d\lambda(t).\end{aligned}$$

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Combining the above estimates, we obtain

$$\begin{aligned}
\|(\varphi_\alpha * \psi)\xi_{C_n}\|_1 &\le \int_{C_n} (|\varphi_\alpha| * |\psi|)(s)\, d\lambda(s) \\
&\le \int_{C_n} \int_G |\varphi_\alpha(st^{-1})|\, |\psi(t)|\Delta(t^{-1})\, d\lambda(t)\, d\lambda(s) \\
&= \int_{C_n} \int_C |\varphi_\alpha(st^{-1})|\, |\psi(t)|\Delta(t^{-1})\, d\lambda(t)\, d\lambda(s) \\
&\quad + \int_{C_n} \int_{G\setminus C} |\varphi_\alpha(st^{-1})|\, |\psi(t)|\Delta(t^{-1})\, d\lambda(t)\, d\lambda(s) \\
&\le \|\psi\|_1 \|\varphi_\alpha \xi_{K_n}\|_1 \\
&\quad + \|\varphi_\alpha\|_1 \int_{G\setminus C} |\psi(t)|\, d\lambda(t) \\
&\le \|\psi\|_1 b_n + M\Big(\frac{r_1}{2M}\Big) \\
&\le \frac{r_n}{2} + \frac{r_n}{2} = r_n
\end{aligned}$$

for all $\alpha \ge \alpha_0$. Hence, $\varphi_\alpha * \psi \longrightarrow 0$ in the $\beta^1(G)$-topology. □

We conclude this work with the following remark.

REMARK 4.2. It is shown in [13, Theorem 5] that convolution considered as a bilinear map from $(L^1(G), \beta^1(G)) \times (L^1(G), \beta^1(G))$ into $(L^1(G), \beta^1(G))$ is hypocontinuous if and only if G is compact.

Acknowledgements

The authors would like to thank the referee for the careful reading of this paper and helpful suggestions. The research of the first-named author was supported by the Research Institute for Fundamental Science, Tabriz, Iran. The second-named author was gratefully acknowledges support by the Center of Excellence for Mathematics at the Isfahan University of Technology.

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SAEID MAGHSOUDI, Department of Mathematics, Zanjan University, Zanjan, Iran
and
Research Institute for Fundamental Science, Tabriz, Iran
e-mail: s_maghsodi@znu.ac.ir

RASOUL NASR-ISFAHANI, Department of Mathematical Sciences,
Isfahan University of Technology, Isfahan 84156-83111, Iran
e-mail: isfahani@cc.iut.ac.ir