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NEIGHBORLY 4-POLYTOPES AND NEIGHBORLY COMBINATORIAL 3-MANIFOLDS WITH TEN VERTICES

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1. Introduction. A *combinatorial n-sphere* is a simplicial n-complex whose body (i.e., the union of its members) is homeomorphic to the topological n-sphere S^n. A *combinatorial n-manifold* is a simplicial n-complex M such that M is connected, and for every vertex x in M the complex link(x, M), the link of x in M, is a combinatorial $(n - 1)$-sphere. For more details the reader should consult Alexander [**1**] and Grünbaum [**16**]. All the spheres and manifolds to which we refer are combinatorial.

A priori it is not clear whether or not every n-sphere is an n-manifold. However, an affirmative answer to this question was given by Alexander [**1**, Theorem 12.2.a]. In the same paper Alexander proved also that the boundary complex of every simplicial $(n + 1)$-polytope is an n-sphere. It is well known (see e.g. [**14**, Chapter 13]) that every 2-sphere is polytopal, i.e., can be realized as the boundary complex of some simplicial (3-dimensional) polytope. Mani [**18**] proved that a similar result holds for every n-sphere with up to $n + 4$ vertices, while for every $n \geqq 3$ there exists an n-sphere with $n + 5$ vertices that is not polytopal. A first example of a 3-sphere with 8 vertices that is not polytopal was discovered by Grünbaum [**15**], and a second such example was discovered by Barnette [**9**].

A complete enumeration of all the (types of) 3-manifolds with up to nine vertices has been carried out in [**3**], [**5**] and [**6**], and a brief report and references on these results can be found in [**6**]. We mention here only that except for one, all those manifolds were shown to be spheres. The exception is a 3-manifold with 9 vertices which, following [**5**], we denote by $N_{51}{}^9$. $N_{51}{}^9$ was shown in [**5**] to be non-orientable, and therefore it is not a sphere. We used (unpublished) Corollary 6.3.9 of [**17**] to calculate the fundamental group of $N_{51}{}^9$, and found it to be Z, the free group on one variable.

A particular and important case of a 3-manifold is the neighborly 3-manifold. A *neighborly* 3-manifold (3-sphere, 4-polytope) is a 3-manifold (3-sphere, 4-polytope) in which every two vertices are joined by an edge. The 3-manifold $N_{51}{}^9$ is neighborly.

In the present work we investigate and enumerate, using a *CDC* 6400 computer, all the neighborly 3-manifolds with 10 vertices. Thus, this is an extension

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of the work carried out in [**3**; **4**; **5**; **6**]. However, there is a significant difference in the present work and that of the above references, which manifests itself in two ways. First of all, the technical aspects of finding all of the cases is much more complicated here. This together with the large number of cases and the amount of time needed to find them, in particular the checking of the isomorphic cases, necessitated a need to develop special and sophisticated methods for checking the isomorphisms. This has been carried out here although we have not entered into a description of these methods. Secondly, and most important, in the above works enumeration of the cases played the main role whereas here classification of the cases plays the central role, and enumeration occupies a small part (Section 2) of this work.

In the present work we strive to achieve the following:

1. To obtain a complete catalogue of all the neighborly 3-spheres and 3-manifolds with 10 vertices;
2. to classify those 3-spheres, as far as possible, into polytopal and non-polytopal spheres, and to classify the non-spherical 3-manifolds according to their topological type;
3. to get answers to at least some of the problems posed in [**4**] and [**5**];
4. to find 4-polytopes, 3-spheres and 3-manifolds which have a particularly interesting structure; and in particular,
5. to find an orientable 3-manifold which is not a sphere and is minimal with respect to the number of vertices.

We have succeeded in achieving all of the above goals.

The complete catalogue of the neighborly 3-manifolds with 10 vertices contains 3677 types. The full description of all of those types is beyond the scope of a short paper, and we give here (in Table 1) a detailed description of only nine of those cases, chosen for representing particular phenomena. The complete catalogue can be obtained upon request from the author.

The main result to be established in the present paper is:

THEOREM 1. *There are precisely* 3677 *different combinatorial types of neighborly* 3*-manifolds with* 10 *vertices. They split into* 3540 *spheres*, (*of which at least* 333 *and at most* 432 *are polytopal*) 83 *non-orientable* 3*-manifolds which are homeomorphic to each other and* 54 *orientable non-spherical* 3*-manifolds which are homeomorphic to each other. Each of the* 137 *non-spherical cases has the free group on one variable as its fundamental group.*

In view of [**3**], [**5**] and [**6**], where it is shown that every 3-manifold with up to nine vertices is either a sphere or a non-orientable 3-manifold, Theorem 1 implies (compare [**5**, Theorem 1.2]):

THEOREM 2. *The minimal n such that there exists an orientable* 3*-manifold with n vertices that is not a sphere is* $n = 10$.

In Section 2 we describe the construction of the neighborly 3-manifolds with 10 vertices (briefly: N^{10}'s). In Section 3 we classify the 3677 N^{10}'s found in

Section 2 into 3573 *simply obtainable* and 104 *non-simply-obtainable* N^{10}'s. In Section 4 we classify the 3573 simply obtainable N^{10}'s into four classes, two of which contain all the polytopal cases. In Section 5 we classify the 104 non-simply-obtainable N^{10}'s into three classes. While carrying out the work in Section 5 we define a certain transformation τ on 3-manifolds which appears to be of importance beyond the scope of the present work (see Remark 5, Section 6 and [**8**]). The classification given in Sections 4 and 5 yields the proof of Theorem 1. We conclude in Section 6 with some remarks.

Our terminology follows [**14**; **3**; **4**] and in particular [**5**] and [**6**].

2. Construction of the N^{10}**'s.** Our method for constructing all the neighborly 3-manifolds with 10 vertices is a simple and natural modification of the method described in detail in [**3**], and resembles [**5**]. Therefore we briefly describe here only the main idea. For further details the reader should consult [**3**].

Let N^{10} be the general name for a neighborly 3-manifold with 10 vertices, and let the vertices of each N^{10} be labeled $0, 1, 2, \ldots, 9$. Because of the neighborliness of N^{10}, the complex $S = \text{link}(0, N^{10})$ is a triangulation of the 2-sphere, with 9 vertices.

In the process of construction of the N^{10}'s we have to let S run over all the possibilities of a 2-sphere with 9 vertices. Algorithms for finding all the 2-spheres with k vertices from those with $k - 1$ vertices are well known in the literature (see, e.g., [**12**]), but we prefer to use a simple modification of the algorithm described in [**3**] for constructing all 3-manifolds with k vertices, to find all the 2-spheres with k vertices (Euler's formula is needed here to exclude 2-manifolds which are not spheres). Thus our algorithm generates directly the 2-spheres with k vertices, without passing through those with $k - 1$ vertices. We programmed this algorithm to yield all the 2-spheres with 9 vertices (and also with 10 vertices; see Remark 6, Section 6) finding that there are 50 such cases (which agrees with [**12**] and with [**20**, p. 86]) which are denoted by S_i, $1 \leqq i \leqq 50$. Of those 50 S_i's precisely 24 are *stacked* (which agrees with [**11**, Table 2]); they are $S_{27}, S_{28}, \ldots, S_{50}$. (For the definition of a stacked 2-sphere see [**5**, Definition 2.1] and, for a more general treatment, see [**2**]. A stacked 2-sphere is essentially a dissection of a 3-ball as defined in [**11**]). A few of the S_i's are shown in Figure 1, and the detailed list of all 50 S_i's is included in the final catalogue. (The author wishes to thank the referee for pointing out that the duals of all 50 S_i's are shown in [**20**, Plate III].)

Let $f_i(C)$ denote the number of i-simplices in a simplicial complex C. For every 3-manifold M, the well known Euler's formula

$$\sum_{i=0}^{3} (-1)^i f_i(M) = 0$$

holds, and yields easily that each N^{10} contains precisely 35 3-simplices. Since for every N^{10} the 2-sphere $S = \text{link}(0, N^{10})$ contains 9 vertices, S contains

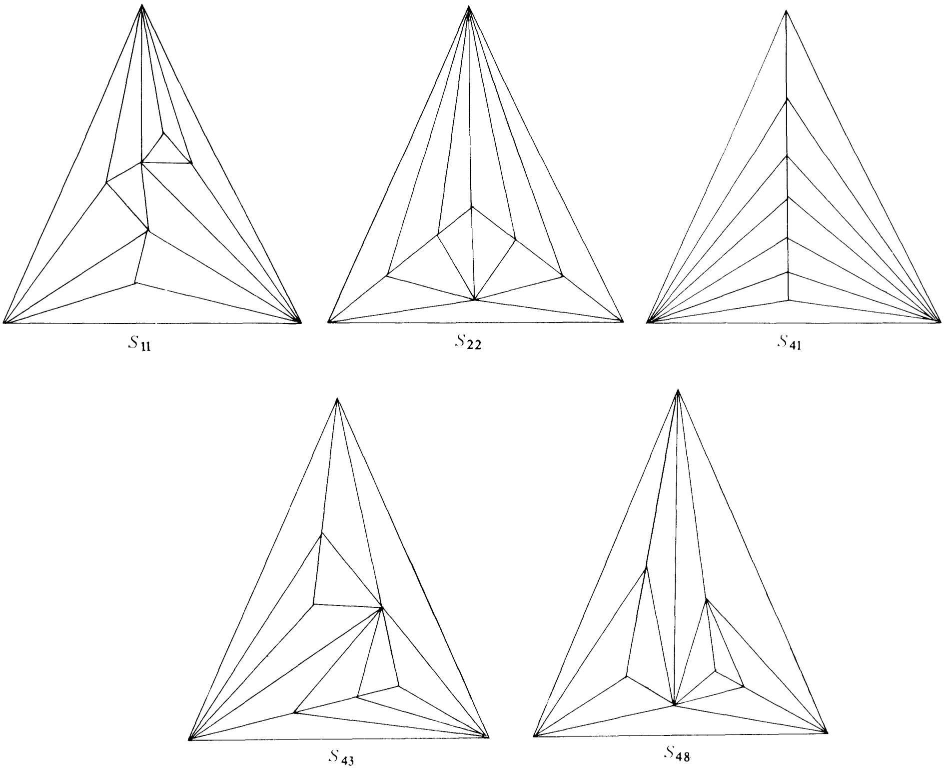

FIGURE 1

precisely 14 triangles (2-simplices). Therefore it follows that star(0, N^{10}) contains 14 3-simplices, and hence R = antistar(0, N^{10}) contains 21 3-simplices. Notice that star(0, N^{10}) depends on S only and not on N^{10}; namely, it is $\{0 \vee \Delta : \Delta \in S\}$. Here $0 \vee \Delta$ is the simplex of dimension dim $\Delta + 1$ whose vertices are 0 and those of Δ. We denote star(0, N^{10}) by S'.

Now, for each of the 50 possibilities for S, we label the vertices of S as 1, 2, . . . , 9, and we find all possible 3-complexes R with the same nine vertices such that $R \cup S'$ is a neighborly 3-manifold (necessarily with 10 vertices). Following [**3**, Section 3], R is easily seen to satisfy the following conditions:

(1) $f_3(R) = 21$.

(2) Each 2-simplex $\Delta \in R$ belongs to precisely two 3-simplices in R if $\Delta \notin S$ and to precisely one 3-simplex in R if $\Delta \in S$.

(3) For each vertex $x \in S$, if the valence of x in S is j, then x belongs to precisely $14 - j$ 3-simplices in R.

(4) $S \subset R$.

(5) For every two vertices $x, y \in R$, there is in R a 3 simplex that contains both x and y.

After obtaining all such complexes R (again, consult [**3**] for details) a check was made to verify that each $S' \cup R$ was indeed a manifold. For each of the manifolds thus obtained we calculated the *edge-valence matrix* and its determinant. The last two concepts are very helpful in identifying our manifolds, and for their definition and significance we refer to [**4**]. (Although the edge-valence matrix was originally defined in [**4**] for 3-spheres, the same definition holds for every 3-manifold.) All the N^{10}'s thus obtained were classified into isomorphism classes, and a representative for each class was chosen. Altogether we found 3677 different N^{10}'s.

3. First classification. The next step after finding the 3677 N^{10}'s was to classify them into *directly obtainable* and *non directly obtainable* manifolds. Those concepts are of main importance in the present work and therefore, though already defined in [**5**] and [**6**], we briefly repeat here their definition.

Let M be a 3-manifold, and let x be a vertex of M. The 'hole' created in M by removing star(x, M) can sometimes be refilled by some 3-element C (i.e., a simplicial 3-complex whose body is homeomorphic to a topological 3-cell) such that bd$C$ = link(x, M), all the vertices of C are in bd$C$, and M' = antistar$(x, M) \cup C$ is a 3-manifold. In this case we say ([**5**, Section 2]) that M is *directly obtainable* at the vertex x from M', and that C is the *refill*. For given M and x it is in general not easy to find an appropriate refill C, or to prove its non-existence. However, if the 2-sphere link(x, M) is stacked (see [**5**, Definition 2.1]) then there exists a natural candidate for such a refill, namely, the unique simplicial 3-complex C with bd$C$ = link(x, M), all of whose vertices *and edges* are in its boundary (this is essentially the *dissection of the* 3-*ball* with boundary link(x, M), as defined in [**11**]). If this particular 3-element C is indeed a refill, i.e., M' = antistar$(x, M) \cup C$ is a 3-manifold, we call it a *simple refill*, and say that M is *simply obtainable at x from M'*, or, if M' is immaterial, that M is *simply obtainable at* x. We say that M is *simply obtainable* if it is simply obtainable in at least one of its vertices (which, of course, must then have a stacked link).

LEMMA 3. *Let M be a* 3-*manifold which is directly obtainable at a vertex $x \in M$ from M'. If M is neighborly, then M' is also neighborly,* link(x, M) *is a stacked* 2-*sphere and M is simply obtainable at x from M'.*

Proof. Assume that M is neighborly and that C is the refill with which M is directly obtainable at x from M'. Since all the edges of M which are not in antistar(x, M) contain the vertex x and M is neighborly, we have that for every two vertices y, z in M' the edge yz is in antistar(x, M) and therefore in M'. Thus M' is neighborly and C does not contain any inner edge, i.e., all the edges of C are in bd$C$ = link(x, M). C is therefore a simple refill. It follows that link(x, M) is a stacked 2-sphere (see [**5**, page 117; **2**, Section 1; **4**, Section 2]) and M is simply obtainable at x from M'.

An immediate corollary of Lemma 3 is that if some N^{10} is directly obtainable at a vertex $x \in N^{10}$ from a manifold M', then link(x, N^{10}) is one of the 24 stacked 2-spheres S_i $(27 \leqq i \leqq 50)$, and M' is one of the 51 neighborly 3-manifolds with 9 vertices which were found in [**5**] and have been denoted there by N_i^9 $(1 \leqq i \leqq 51)$.

For each of the 3677 N^{10}'s and for each vertex $x \in N^{10}$ such that link$(x, N^{10}) = S_i$, for some $27 \leqq i \leqq 70$, we checked whether or not N^{10} is simply obtainable at x, and we calculated the N_i^9 $(1 \leqq i \leqq 51)$ from which N^{10} is simply obtainable at x. Altogether we found that the 3677 N^{10}'s split into 3573 cases which are simply obtainable, and 104 cases which are not simply obtainable and therefore, because of Lemma 3, also not directly obtainable.

4. Classification of the 3573 simply obtainable N^{10}'s. In order to classify the 3573 simply obtainable N^{10}'s into spheres and non-spheres, we use the following lemma, the proof of which is not difficult (it follows easily from the Auxiliary Theorem in the next section) and we omit it. (Compare [**4**, Theorem 2.5 and Remark 2.6]). Usually, the notation $|C|$ is used to denote the body, i.e., the union of the simplices, of a simplicial complex C. However, in the present and in particular in the next section, we find it often more convenient to use C for denoting both the complex C and its body, and it will not be difficult for the reader to decide in each case which of the two possibilities is meant.

LEMMA 4. *If a 3-manifold M is directly obtainable from a 3-manifold M', then $|M|$ and $|M'|$ are homeomorphic to each other.*

The 51 neighborly 3-manifolds N_i^9 $(1 \leqq i \leqq 51)$ were shown in [**5**] to split into 50 N_i^9's with $1 \leqq i \leqq 50$ which are spheres and a unique N_{51}^9 which is a non-orientable manifold. Of our 3573 simply obtainable N^{10}'s precisely 34 cases are simply obtainable from N_{51}^9. Thus, by Lemma 4, those 34 cases are all homeomorphic to N_{51}^9, therefore they all are homeomorphic to each other and are not orientable. By the same reasoning, each of the remaining 3539 N^{10}'s is simply obtainable from some sphere N_i^9 $(1 \leqq i \leqq 50)$ and is therefore a sphere.

Next we try to classify these 3539 spheres into polytopal and non-polytopal spheres. By [**6**, Theorem 4], if a 3-sphere M (other than the boundary complex of the 4-simplex) has a vertex x such that M is not directly obtainable at x or is directly obtainable at x from a non-polytopal sphere, then M is not polytopal. This excludes the possibility that any of the 104 non-directly obtainable N^{10}'s be a polytopal sphere, and helps to classify the 3539 directly obtainable spheres, as follows:

Among the 50 spheres N_i^9 $(1 \leqq i \leqq 50)$, there are precisely 23 polytopal spheres. They are the N_i^9's with $1 \leqq i \leqq 23$ (see [**4**]). Therefore, each of our 3539 spheres which is simply obtainable at at least one of its vertices from some N_i^9 with $24 \leqq i \leqq 50$, cannot be polytopal. Also, if a sphere N^{10} has a vertex x at which it is not directly obtainable, then that sphere is not polytopal. This last phenomenon occurs, because of Lemma 3, always when the 2-sphere link(x, N^{10}) is not stacked, i.e., link$(x, N^{10}) = S_i$ for some $1 \leqq i \leqq 26$. However, it can also happen that link(x, N^{10}) is stacked, and nevertheless N^{10} is not simply obtainable at x. This happens when the 3-element C, which is the natural and the only candidate for a refill (in the terminology of [**11**], C is the dissection of the 3-ball with boundary link(x, N^{10})), shares some 3-simplex with antistar(x, N^{10}) and therefore $C \cup$ antistar(x, N^{10}) is not a 3-manifold. It is interesting to note that this last possibility did not happen with any of the 3539 spheres under discussion (see Remark 7, Section 6).

Now, the above phenomena were found in precisely 3107 of our 3539 simply obtainable spheres, and therefore they are not polytopal.

We do not know yet whether or not the remaining 432 simply obtainable spheres are polytopal. Let N be any of those spheres and let x be a vertex of N. Then link(x, N) is a stacked 2-sphere (i.e., it is S_i for some $27 \leqq i \leqq 50$) and N is simply obtainable at x from some polytopal neighborly 3-sphere N' with 9 vertices, which is N_j^9 for some $1 \leqq j \leqq 23$. Assume for the moment that N is polytopal, and let P be a 4-polytope which realizes N and has the same labeling of the vertices. Then the 4-polytope $P' = \text{conv}(P \backslash \{x\})$ necessarily realizes N'. However, starting with some 4-polytope P' which realizes N' there is, in general, no guarantee for the existence of a point x such that the polytope $P = \text{conv}(P' \cup \{x\})$ realizes N. The existence of such a point x may depend on the particular polytope P' chosen to realize N'.

An argument very similar to the argument given in [**4**, page 282] shows, however, that if link(x, N) is S_{41} (see Figure 1), then such a point x exists for *every* polytope P' chosen to realize N' (see also Remark 8 in Section 6). Therefore each of the 432 N^{10}'s under consideration which has S_{41} as a link of at least one of its vertices must be polytopal. There are precisely 333 such cases. The remaining 99 N^{10}'s remain undecided as to whether or not they are polytopal, and we refer to them as *doubted polytopes*. Since every neighborly 4-polytope is simplicial ([**14**, Theorem 4.2.1]) we obtain:

THEOREM 5. *The number of neighborly* 4-*polytopes with* 10 *vertices is at least* 333 *and at most* 432.

To summarize the present section, the 3573 simply obtainable N^{10}'s fall into four classes:

Class I: 333 polytopes (denoted N_i^{10} with $1 \leqq i \leqq 333$ in the final catalogue).

Class II: 99 doubted polytopes (denoted N_i^{10} with $334 \leqq i \leqq 432$).

Class III: 3107 non-polytopal simply obtainable spheres (denoted N_i^{10} with $433 \leqq i \leqq 3539$).

Class IV: 34 non-orientable simply obtainable 3-manifolds which are homeomorphic to N_{51}^9 (denoted N_i^{10} with $3540 \leqq i \leqq 3573$).

5. Classification of the non simply obtainable N^{10}'s. There remain 104 non simply obtainable N^{10}'s to be classified. First, we checked them for orientability and we found that 55 cases are orientable and the remaining 49 cases are non-orientable. Next, we calculated the fundamental group of each of the 104 cases. This was done by means of Corollary 6.3.9 of [**17**]. One of those cases, denoted by $N_{3574}{}^{10}$ (see Table 1), was shown to have the trivial group as its fundamental group. Each of the remaining 103 cases was shown to have Z, the free group on one variable, as its fundamental group. A priori, $N_{3574}{}^{10}$ is either a sphere or, if not a sphere, it is a counterexample to the famous Poincaré Conjecture. This case, however, was discussed in detail in [**7**], and was shown there to be a sphere. Thus, the 104 non simply obtainable N^{10}'s fall into 3 classes:

Class V: A unique non simply obtainable sphere $N_{3574}{}^{10}$.

Class VI: 54 orientable non simply obtainable N^{10}'s having Z as their fundamental group (they are denoted $N_i{}^{10}$ with $3575 \leqq i \leqq 3628$).

Class VII: 49 non-orientable non simply obtainable N^{10}'s having Z as their fundamental group (denoted $N_i{}^{10}$ with $3629 \leqq i \leqq 3677$).

The question now arises, whether or not the 54 cases of Class VI are homeomorphic to each other, and whether or not the 49 cases of Class VII are homeomorphic to each other and perhaps also to the cases of Class IV.

In order to answer those questions we will define a certain transformation τ on 3-manifolds, such that if M is a 3-manifold on which τ is applicable, then $\tau(M)$ is again a 3-manifold and is homeomorphic to M. We then proceed to define a graph G on the 3677 N^{10}'s such that the vertices of G are the 3677 N^{10}'s, and two vertices N_1, N_2 in G are joined by an edge if and only if $N_2 = \tau(N_1)$. The study of the connected components of G will yield an affirmative answer to all the above questions.

The following well known theorem, in piecewise linear topology, will be repeatedly used in the present section (our attention to it was drawn by Professor Marshall Cohen):

AUXILIARY THEOREM. *If D and E are polyhedra, A and B are piecewise linear n-balls, $A \cap D = \partial A$, $B \cap E = \partial B$ and $h : D \rightarrow E$ is a piecewise linear homeomorphism with $h(\partial A) = \partial B$, then there is a piecewise linear homeomorphism $H : D \cup A \rightarrow E \cup B$ such that $H|D = h$.*

Definition 6. Let M be a 3-manifold which contains as a subcomplex the simplicial 3-complex K composed of the five 3-simplices

$$xyab,\ xybc,\ xyca,\ xdef,\ ydef$$

and their faces. We assume the five vertices x, y, a, b, c to be distinct, and similarly for the five vertices x, y, d, e, f, but we allow the triangles abc and def to share at most two common vertices. If the triangle abc is not in M, we say that *the transformation τ is applicable on M at K*, and we define

$$\tau_K(M) = N = (M \backslash K) \cup L,$$

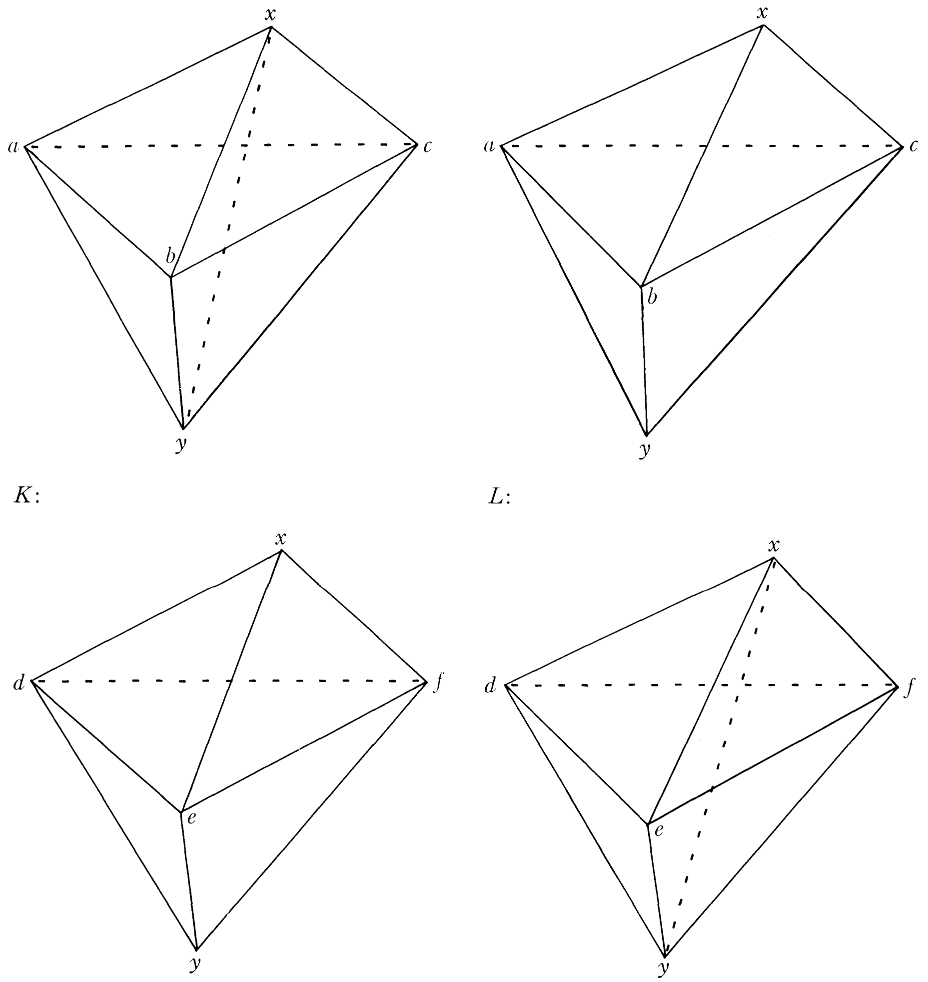

FIGURE 2

where L is the simplicial 3-complex composed of the five 3-simplices

$$xyde,\ xyef,\ xyfd,\ xabc,\ yabc$$

and their faces (see Figure 2).

THEOREM 7. *In the notation of Definition* 6, $N = \tau_K(M)$ *is a* 3-*manifold homeomorphic to* M.

Proof. We distinguish three cases:

Case 1: The triangles abc and def share two common vertices: $a = d$, $b = e$, $c \neq f$.

Case 2: The triangles abc and def share one common vertex: $a = d$, $b \neq e$, $c \neq f$.

Case 3: The triangles abc and def share no common vertex.

Note that in Case 1 both $|K|$ and $|L|$ are 3-balls, and they can be constructed geometrically in R^3 as shown in Figure 3: each of them has the appearance of a bipyramide (octahedron) over the basis $axby$, but the four vertices a, x, b, y are not in the same plane. In K the vertex x is pushed away from the plane aff$\{a, b, y\}$ towards the vertex c, while in L the vertex x is pushed away from the plane aff$\{a, b, y\}$ in the direction of the vertex f. Note also that in Case 1 the triangle abc is not in M even without the explicit assumption in Definition 6, since the link link(ab, M) of the edge ab in M consists of the circuit xy, yf, fx which does not contain the vertex c. (It is easy to see that the link of an edge in a 3-manifold is a 1-sphere, i.e., a circuit.)

In order to show that N is a 3-manifold, we treat only Case 1. The treatment of the other two cases is similar and is left to the reader.

We have to show that for every vertex v in N, link(v, N) is a 2-sphere. This is clear for every vertex v in N other than a, b, c, f, x and y. since then obviously link(v, N) = link(v, M). As for the remaining possibilities it is sufficient, because of the symmetry (see Figure 3), to deal with the two cases $v = a$ and $v = c$. In both cases, it is easily seen (Figure 4) that link(v, K) and link(v, L) are discs with a common boundary. Since link(v, N) is obtained from link(v, M) by removing link(v, K) and replacing it with link(v, L), link(v, N) will be a 2-sphere if we show that no simplex in L whose relative interior lies in relint link(v, L) has its relative interior also in link$(v, M\backslash K)$.

In case $v = a$, the simplices in L whose relative interiors are in relint-link(a, L) are the triangles xby, xfy, xbc, ybc and the edges xy, xb, yb, bc. The interior of the triangle xby is in link(a, K) and therefore not in link$(a, M\backslash K)$; the triangle xfy is not in M—and therefore its relative interior is not in link$(a, M\backslash K)$— since the link link(xy, M) of the edge xy in M consists of the circuit ab, bc, ca which does not contain f; relint xbc is not in link$(a, M\backslash K)$ since otherwise the edge xb would belong to three different triangles, namely xbc, xby and xbf, which is impossible; similarly also relint ybc is not in link$(a, M\backslash K)$. Each of the edges xy, xb, yb is in link(a, K) and therefore its relative interior is not in link$(a, M\backslash K)$; finally the edge bc is not in link(a, M) since otherwise M would contain the triangle abc, which is impossible by Definition 6, or also by a preceeding remark.

In the case $v = c$, the simplices in L whose relative interiors lay in relint link(a, L) are the triangles xab, yab and the edge ab. If the relative interior of any of these would be in link$(c, M\backslash K)$, it would imply the existence of the triangle abc in M, which is impossible.

Thus N is a 3-manifold. In order to show that $|N|$ is homeomorphic to $|M|$ we use the Auxiliary Theorem as follows:

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In Case 1, take $D = E = |M|\backslash\text{int}\,|K| = |N|\backslash\text{int}\,|L|$, $A = |K|$, $B = |L|$ and $h = 1$ in the Auxiliary Theorem, and the piecewise linear homeomorphism $H : |M| \to |N|$ follows.

In Cases 2, 3, define K_1 to be the subcomplex of K composed of the 3-simplices *xyab*, *xybc*, *xyac* and their faces, define K_2 to be the subcomplex of K composed of the 3-simplices *xdef*, *ydef* and their faces, define L_1 to be the subcomplex of L composed of *xabc*, *yabc* and their faces, and define L_2 to be the subcomplex of L composed of *xyde*, *xydf*, *xyef* and their faces. Now use the Auxiliary Theorem in two stages: first take $D = E = |M|\backslash\text{int}\,|K| = |N|\backslash\text{int}\,|L|$, $A = |K_1|$, $B = |L_1|$ and $h = 1$ to obtain that $M' = (|M|\backslash\text{int}\,|K|) \cup |K_1|$ is homeomorphic to $N' = (|N|\backslash\text{int}\,|L|) \cup |L_1|$, and next take $D = M'$, $E = N'$, $A = |K_2|$, $B = |L_2|$ and $h = 1$ to obtain that $|M|$ is homeomorphic to $|N|$. This completes the proof of Theorem 7.

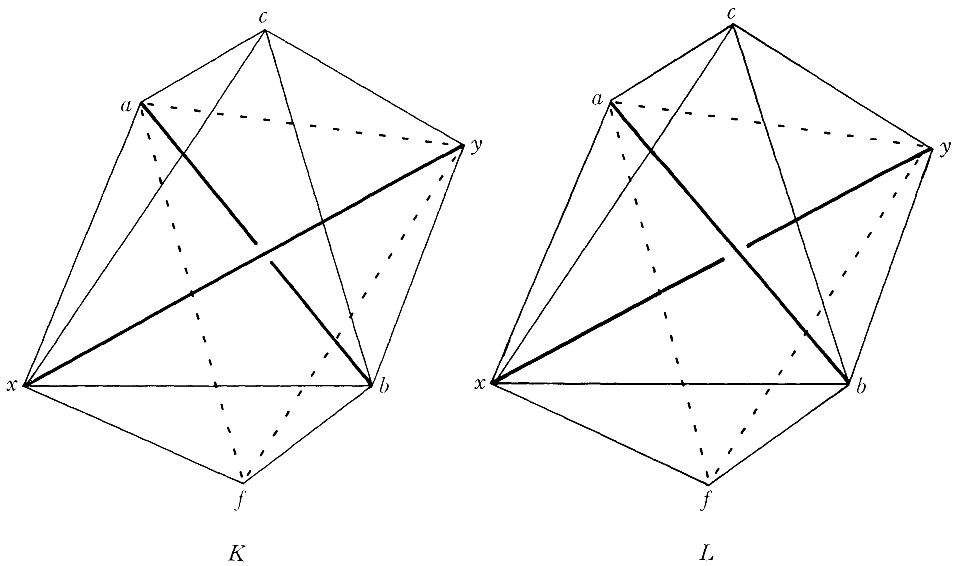

FIGURE 3

Note that, in the notation of Definition 6, the triangle *def* is not in N. Therefore the transformation τ is applicable on $N = \tau_K(M)$ at its subcomplex L, and it is easily seen that $\tau_L(N) = \tau_L(\tau_K(M)) = M$. In this sense we say that the transformation τ is *reversible*.

Note also that $\text{skel}_1 M = \text{skel}_1 N$, and therefore N is a neighborly 3-manifold if and only if M is a neighborly 3-manifold.

A standard device for looking for a subcomplex K in a 3-manifold M such that τ is applicable on M at K as is follows: Choose a triangle $def \in M$. *def* belongs to two 3-simplices of M, *xdef*, *ydef*, say. If the vertices x, y are not

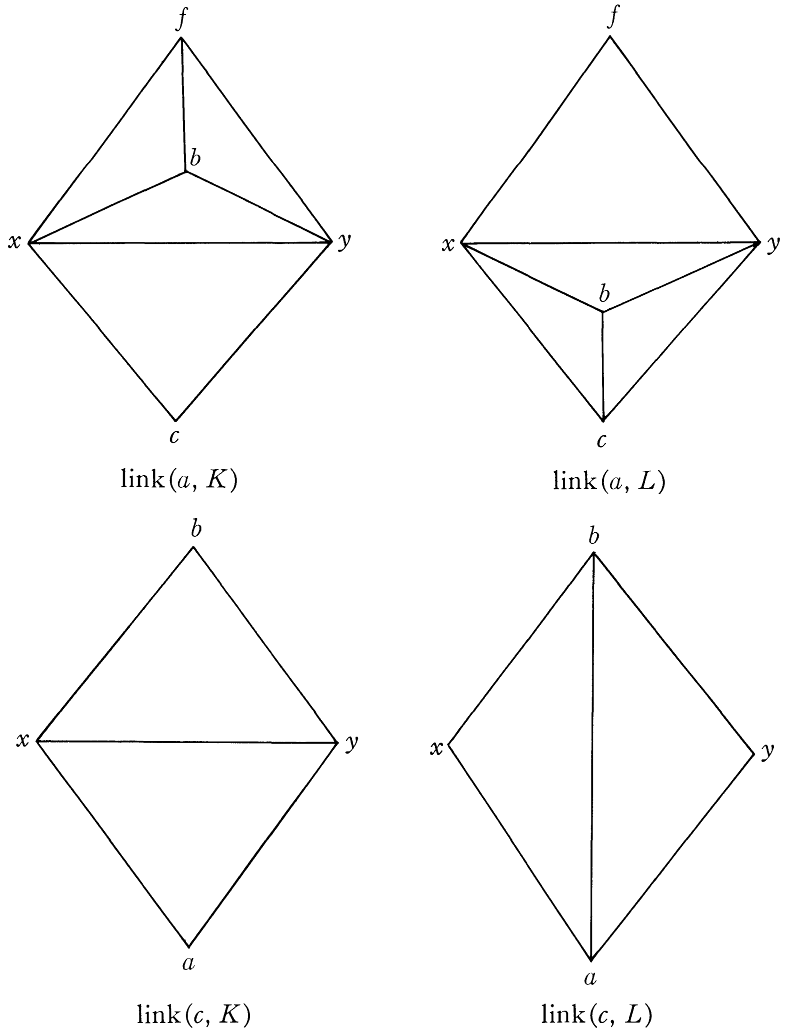

FIGURE 4

joined by an edge in M (which is not the case if M is neighborly) or they are joined by an edge of valence $\neq 3$, then we conclude that the triangle def does not yield any such subcomplex K. If, however, val $xy = 3$, we proceed as follows: Let the three 3-simplices containing xy be $xyab$, $xybc$, $xyac$. If the triangle abc is not in M, we have the desired subcomplex K (whose 3-simplices are $xdef$, $ydef$, $xyab$, $xybc$ and $xyac$); otherwise, we conclude that the starting triangle def does not yield any such subcomplex K.

The transformation τ is not applicable at all on the 3-manifold $N_{51}{}^9$ of [**5**] since, in the above notation, for every starting triangle $def \in N_{51}{}^9$ either the edge xy is not of valence 3, or the resulting triangle abc is in $N_{51}{}^9$.

Now define on our 3677 N^{10}'s a graph G as follows: The vertices of G are the 3677 N^{10}'s, and two distinct vertices M, N in G are joined by an edge if and only if $\tau_K(M) = N$ for some suitable subcomplex K of M. Since the choice of the particular subcomplex K of M is immaterial, we write simply $\tau(M) = N$. Note that by a preceeding remark τ is reversible, and therefore G is an undirected linear graph.

Since not all the 3677 N^{10}'s are homeomorphic to each other, Theorem 7 implies that the graph G is not connected. But the same Theorem 7 also implies that all the N^{10}'s which belong to the same connected component of G are homeomorphic to each other.

Thus we computed the graph G, and we found that G splits into precisely four connected components. One component contains all the 83 N^{10}'s of Classes IV and VII. A second component contains all the 54 N^{10}'s of Class VI. A third component contains 3539 of the 3540 N^{10}'s of Classes I, II, III and V, the exception is N_{425}^{10} (see Table 1) which belongs to Class II and forms the fourth component of the graph G.

Thus the proof of Theorem 1 is complete.

Note that N_{3574}^{10}, which is the unique 3-manifold of Class V, belongs to the same component of the graph G which contains also the N^{10}'s of Class I. This proves once again that N_{3574}^{10} is indeed a 3-sphere.

6. Remarks. 1) Table 2 summarizes the number of occurrences of the 2-spheres S_i ($1 \leqq i \leqq 50$) as links of vertices in our 3677 N^{10}'s. Note that S_{22}, which is a bipyramid over a 7-gon (see Figure 1), is not a link of any vertex in any of the 3677 N^{10}'s. This strengthens Conjecture 4.1 of [**5**]. Table 3 summarizes the number of N^{10}'s of Classes I, II (polytopes and doubted polytopes) directly obtainable from the polytopes N_i^9 ($1 \leqq i \leqq 23$) of [**4**]. Note that the cyclic polytope N_1^9 produces relatively few N^{10}'s of Classes I, II.

2) In the final catalogue, the manifolds within each of the seven classes are ordered according to increasing determinants of the edge-valence matrices. Altogether the 3677 N^{10}'s have 3669 distinct determinants. There are six N^{10}'s, four of which are polytopes, which share the determinant 0, and three pairs of N^{10}'s such that the members of each pair share the same determinant. Among those pairs is also the pair N_{3217}^{10}, N_{3218}^{10} (see Table 1) which share not only the same determinant but even the same edge-valence matrix. Thus Conjecture 1 of [**4**] and Conjecture 4.2 of [**5**] are false. Conjecture 2 of [**4**] is strengthened by the fact that the cyclic 4-polytope C(10, 4) with 10 vertices is N_4^{10}, which has the determinant 0.

3) The 3677 N^{10}'s have 3669 distinct sets of links of the vertices (and thus the set of links of the vertices, as well as the determinant, is a convenient device for discriminating between 3-manifolds), and the 432 N^{10}'s of Classes I, II have 431 different sets of links (the two equal sets of links belong to two

TABLE 1

Description of some neighborly 3-manifolds with 10 vertices.

N_i^{10}	3-Simplices				S_i	N_i^9	edge-valence matrix
4	1230	1560	3450	4789	1–41	1	*833444448
	1235	1670	3459	4890	2–41	1	8*84344443
	1240	1780	3567	5678	3–41	1	38*4834444
	1249	1890	3578	5689	4–41	1	344*444388
det = 0	1256	2340	3589	6789	5–41	1	4384*83444
	1267	2349	4560		6–41	1	44348*8344
	1278	2356	4569		7–41	1	444438*834
	1289	2367	4670		8–41	1	4443438*84
	1350	2378	4679		9–41	1	44484438*3
Class I	1490	2389	4780		0–41	1	834844443*
425	1230	1670	2458	4670	1–43	19	*464343468
	1239	1679	2578	4790	2–43	19	4*83464643
	1260	1790	2678	4890	3–43	19	68*4643434
	1269	1890	2689	5789	4–43	19	434*834646
	1345	2345	3560	6789	5–43	19	3468*46434
det = 7057326528	1348	2348	3567		6–43	19	46434*8346
	1350	2357	4560		7–43	19	343468*464
	1389	2360	4567		8–43	19	4646434*83
	1450	2367	4579		9–43	19	64343468*4
Class II	1480	2389	4589		0–43	19	834646434*
3574	1230	1570	2589	4560	1–11	0	*654345636
	1234	1680	2679	4589	2–11	0	6*36563454
	1248	1689	3478	4590	3–11	0	53*6436465
	1260	1780	3489	5670	4–11	0	466*533645
	1269	2340	3490	6780	5–11	0	3545*46366
det = 11670765568	1289	2456	3579		6–11	0	46334*6556
	1347	2458	3590		7–11	0	536366*544
	1350	2460	3678		8–11	0	6446355*63
	1357	2567	3679		9–11	0	35646546*3
Class V	1478	2579	3689		0–11	0	645566433*
3611	1230	1680	2458	4567	1–48	0	*435555348
	1235	1689	2589	4589	2–48	0	4*85534553
	1240	1790	2679	5670	3–48	0	38*5543554
	1245	1890	2789	6780	4–48	0	555*843435
	1350	2340	3450	7890	5–48	0	5558*34345
det = 11017073088	1457	2348	3459		6–48	0	53443*8555
	1460	2359	3489		7–48	0	543348*555
	1467	2367	3678		8–48	0	3554355*84
	1570	2369	3689		9–48	0	45534558*3
Class VI	1679	2378	4560		0–48	0	834555543*
3629	1230	1690	2458	4567	1–43	0	*434646348
	1235	1780	2489	4589	2–43	0	4*86443643
	1240	1789	2678	4670	3–43	0	38*4634464
	1245	1890	2689	6780	4–43	0	464*834346
	1350	2340	3450	6890	5–43	0	6468*43434
det = 0	1457	2349	3459		6–43	0	44334*8466

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TABLE 1—(*Continued*)

N_i^{10}	3-Simplices				S_i	N_i^9	edge-valence matrix
	1470	2358	3589		7–43	0	634438*644
	1560	2367	3679		8–43	0	3643446*84
	1567	2369	3789		9–43	0	44643648*3
Class VII	1679	2378	4560		0–43	0	834646443*
3631	1230	1690	2459	4567	1–43	0	*434646348
	1235	1780	2489	4589	2–43	0	4*86434463
	1240	1789	2679	4670	3–43	0	38*4643644
	1245	1890	2789	6780	4–43	0	464*834436
	1350	2340	3450	6890	5–43	0	6468*43344
det = 4729088448	1457	2348	3458		6–43	0	43434*8466
	1470	2359	3589		7–43	0	643438*644
	1560	2367	3678		8–43	0	3464346*84
	1567	2369	3689		9–43	0	46434648*3
Class VII	1679	2378	4560		0–43	0	834646443*
3217	1260	1570	2457	4567	1–2	0	*335663556
	1269	1580	2470	4678	2–5	0	3*64454376
	1290	1670	2579	4689	3–5	0	36*6544734
	1345	1890	2689	5790	4–14	0	546*556533
det = 9310329280	1348	2340	2790	5890	5–20	0	6455*36454
	1358	2345	3470		6–20	0	65453*6544
	1456	2359	3478		7–37	48	344666*337
	1469	2360	3589		8–39	34	5375453*73
	1489	2368	3670		9–39	39	57335437*5
Class III	1567	2389	3678		0–47	37	664344735*
3218	1250	1570	2467	4567	1–2	0	*335663556
	1259	1670	2470	4578	2–6	0	3*64454376
	1290	1680	2679	4589	3–6	0	36*6544734
	1346	1890	2689	6790	4–14	0	546*556533
	1348	2340	2790	6890	5–20	0	6455*36454*
det = 9310329280	1368	2346	3470		6–20	0	65453*6544
	1456	2350	3478		7–37	37	344666*337
	1459	2359	3570		8–39	34	5375453*73
	1489	2368	3578		9–39	39	57335437*5
Class III	1567	2389	3589		0–47	48	664344735*
416	1250	1479	2469	3789	1–29	22	*335633775
	1259	1568	2490	4569	2–29	22	3*53763357
	1290	1580	2567	4670	3–29	22	35*3337657
	1380	1789	2569	4678	4–29	22	533*376375
	1389	2357	2580	5678	5–37	11	6733*64643
det = 6674012800	1390	2358	3470		6–37	11	36376*6434
	1456	2360	3479		7–37	11	337646*643
	1459	2367	3490		8–37	11	7363646*34
	1468	2380	3578		9–46	20	75574343*4
Class II	1478	2460	3670		0–46	20	577534344*

The column N_i^{10} states the i for which N_i^{10} is described. det. is the determinant of the edge valence matrix. The column S_i states the i for which S_i is the link of the given vertex. The column N_i^9 states that the case is simply obtainable at the given vertex from N_i^9 with the stated i, where $i = 0$ means that the case is not directly obtainable at the given vertex. N_i^9 $(1 \leqq i \leqq 51)$ is that of [**4**] and [**5**]. * represents the number 14.

TABLE 2

i	All Classes	Classes I, II	Class I
1	253	0	0
2	351	0	0
3	79	0	0
4	425	0	0
5	480	0	0
6	799	0	0
7	130	0	0
8	245	0	0
9	894	0	0
10	244	0	0
11	412	0	0
12	266	0	0
13	349	0	0
14	567	0	0
15	694	0	0
16	203	0	0
17	814	0	0
18	1060	0	0
19	336	0	0
20	399	0	0
21	239	0	0
22	0	0	0
23	65	0	0
24	77	0	0
25	84	0	0
26	27	0	0
27	340	54	21
28	323	59	42
29	565	68	25
30	850	213	144
31	1794	449	345
32	772	138	106
33	1545	275	196
34	1382	151	99
35	762	79	62
36	617	43	38
37	1171	138	105
38	1501	202	150
39	1263	140	84
40	1277	100	55
41	920	557	557
42	2511	572	457
43	1216	190	169
44	1006	125	94
45	2165	268	198
46	862	95	65
47	1422	179	158
48	1041	75	53
49	1380	108	72
50	593	42	35

The number of occurrences of each 2-sphere S_i $(1 \leqq i \leqq 50)$ as link(v, N^{10}).

TABLE 3

i	Classes I, II	Class I
1	23	22
2	92	81
3	111	100
4	178	149
5	164	143
6	135	112
7	32	7
8	71	64
9	72	60
10	89	67
11	129	81
12	93	85
13	127	105
14	67	48
15	98	75
16	120	82
17	31	7
18	130	87
19	26	7
20	28	4
21	113	96
22	90	21
23	114	78

The number of N^{10}'s of Classes I, II (polytopes and doubted polytopes) directly obtainable from the polytope N_i^9 $(1 \leqq i \leqq 23)$.

polytopes, N_{65}^{10} and N_{134}^{10}). No two N^{10}'s have both the same determinant and the same set of links of the vertices.

Precisely 500 N^{10}'s use only S_i's with $27 \leqq i \leqq 50$ (i..e, stacked 2-spheres) as links of their vertices. Among those are of course all the 432 spheres of Classes I, II, but also 56 spheres of Class III and 12 N^{10}'s (among which are N_{3611}^{10}, N_{3629}^{10}, and N_{3631}^{10} of Table 1) of Classes VI, VII. Thus those 500 N^{10}'s split into 488 spheres and 12 non-spheres. It is interesting to note that each of the 488 spheres is simply obtainable at *each* of its vertices, while the 12 non-spheres are not simply obtainable at all. Thus problem 5 of [**4**] is answered in the negative (see also [**5**, page 135]).

There are precisely six N^{10}'s (they all appear in Table 1), one of which is the cyclic 4-polytope $C(10, 4)$ (N_4^{10} in our catalogue), which share with $C(10, 4)$ the property that all of their vertices have the same link. One of these six N^{10}'s is the sphere N_{425}^{10} which has already shown a peculiar behaviour with respect to the transformation τ of Section 5. Now, N_{425}^{10} is a doubted polytope. If it is a polytope, then it provides an affirmative answer to Problem 4 of [**4**].

The set of links of the vertices of a 3-manifold M reflects some local properties of M. In general, there is no reason to expect that those local properties will characterize M, or even the homotopy type of $|M|$. However, if M is neighborly, then for every vertex x in M the link of x in M involves all the vertices of M. Thus in this case there is perhaps some reason to expect that the set of links of the vertices will characterize the 3-manifold. Indeed, Shemmer proved in [**19**] that if K is a neighborly polytopal 3-sphere such that for every vertex $x \in K$, link(x, K) is isomorphic to link(y, C), where C is the boundary complex of a cyclic 4-polytope and y is a vertex in C, then K is isomorphic to C (link(y, C) is the analogue of S_{41} of Figure 1). Therefore, it is interesting to note that among the above six N^{10}'s are also N_{3629}^{10} and N_{3631}^{10}, both of Class VII, and they both share with N_{425}^{10} the property that the link of *each* of their vertices is S_{43}.

4) The programming in the present work was done by Mr. Bar-Yuda, who was not involved in [**5**], and therefore the present programming was to some extent different from the programming used in [**5**]. Moreover, because of the comparatively small size of the work in [**5**] we could use there a straightforward method for checking isomorphisms between manifolds in order to avoid duplicates, while here, because of the much larger number of cases, we had to develop and use much more sophisticated techniques for checking isomorphisms. Therefore, both for checking our program and checking the results of [**5**], we have let the present program produce once again all the neighborly 3-manifolds with 9 vertices, and we obtained precisely the same results. In this connection it is important to note that the number of hours it took for the present program to yield the 3677 N^{10}'s was greater than the number of seconds it took for the same program to yield the 51 neighborly 3-manifolds with 9 vertices. Due to this time consuming factor, it appears that our method cannot be further used to find *all* the neighborly 3-manifolds with n vertices for $n > 10$, or even to find *all* the non-neighborly 3-manifolds with 10 vertices (see [**3**, Section 4, Remark 1] and the next remark).

5) The transformation τ of Section 5 can also be used for *constructing* 3-manifolds. If M is a 3-manifold, one can apply τ on M in all the possible ways, i.e., at all the permissible subcomplexes K of M, thus obtaining a "second generation" of 3-manifolds $\tau(M)$ each of them being of the same topological type as M and sharing with M the same 1-skeleton. Next τ can be applied on each manifold of the second generation and thus a third generation is obtained, etc. Of course, each new manifold obtained this way should be compared to all the previous ones for isomorphism, in order to avoid duplicates. The process is finite, and yields the entire connected component which contains M, in the suitable τ-graph defined on all the 3-manifolds in analogy to the graph G of Section 5.

It follows from Section 5 that the transformation τ, applied in this manner on N_1^{10}, yields all the neighborly 3-spheres with 10 vertices except for N_{425}^{10}, while applying τ in this manner on $N_{425}{}^{10}$ yields no new manifolds at all. We applied this method and applied τ to a case of Class IV and also to a case of Class VI, and obtained the expected results. Thus, in a sense, we had an independent check of our catalogue. We applied τ in this manner on $N_1{}^9$ of [**5**], and we obtained all the 50 neighborly 3-spheres with 9 vertices. Moreover, we applied τ on the boundary complex of $C(4, 11)$, the cyclic 4-polytope with 11 vertices, which is of course a neighborly 3-sphere with 11 vertices. In order to save the time needed for checking isomorphisms in order to save duplicates, we calculated the determinant of the edge-valence matrix of each 3-sphere obtained, and considered two spheres to be "isomorphic" if they share the same determinant. During the process, we kept track of the ratio between the 3-spheres on which τ had already been applied and the total amount of 3-spheres constructed. This ratio was approximately 1 : 7 from the beginning, and did not change significantly as we reached an amount of 25,000 3-spheres, where we stopped the process. Thus we have good reason to beleive that the number of neighborly 3-spheres—not speaking of neighborly 3-manifolds—with 11 vertices is of at least six digits. This shows once again that one has to give up any hope of finding all the neighborly 3-manifolds with n vertices, where n is greater than 10, in the present generation of computers. This remark is further developed in [**8**].

6) While finding the 50 2-spheres with 9 vertices (the 50 S_i's), we let the program run over the 2-spheres with 10 vertices as well. It yielded altogether 233 distinct 2-spheres with 10 vertices (which agrees with [**12**] and with [**20**, p. 86]), and also yielded that precisely 93 of these 233 2-spheres are stacked (thus the number missing in Table 1 of [**11**] is 93).

7) It seems worthwhile to note that each N^{10} of Class IV, i.e., each N^{10} simply obtainable from $N_{51}{}^9$, is simply obtainable in at most two of its vertices. It is also interesting to note that the phenomenon, described in Section 4, of a neighborly 3-manifold M with a vertex x such that link(x, M) is stacked and nevertheless M is not simply obtainable at x, did not occur in any of our 3540 spheres. However, it did occur in every N^{10} which is not a sphere, with the exception of the very last case $N_{3677}{}^{10}$, which has no vertex with a stacked link.

8) Let N be a neighborly 3-manifold with v vertices. It is easily seen that every edge in N belongs to at most $v - 2$ 3-simplices in N. Following [**19**], we call an edge which belongs to precisely $v - 2$ 3-simplices (i.e., an edge of valence $v - 2$) a *universal edge* in N. The importance of this concept stems from the fact, used independently both in [**4**] and in [**19**], that if K is a neighborly 4-polytope with v vertices and C is a subcomplex of bd K composed of all but one of the facets of K which contain a certain universal edge of K and their faces, then there exists a point x which is beyond the facets of K which belong to C and beneath the other facets of K, and therefore conv $(K \cup \{x\})$ is a neighborly 4-polytope with $v + 1$ vertices.

Since all the known neighborly 4-polytopes have universal edges, a natural question is whether or not every neighborly 4-polytope has a universal edge. Among our N^{10}'s of Classes I and II there is precisely one case without a universal edge. It is $N_{416}{}^{10}$, which is in Class II (see Table 1). Thus it is of particular interest to find out whether or not $N_{416}{}^{10}$ is a polytope.

Another concept which is related to the concept of universal edge is the *universal vertex*, also defined and studied in [**19**]. Let S be a stacked 2-sphere with v vertices. A vertex p in S is a *universal vertex* in S if it is of valence $v - 1$ in S, i.e., it is joined by edges to all the other vertices of S. The relation between the last two concepts is given by the fact that if K is a neighborly polytopal 3-sphere and pq is a universal edge in K, then p is a universal vertex in the stacked 2-sphere link(q, K), and vice versa. Shemmer also proves that every stacked 2-sphere which contains a universal vertex is isomorphic to the link of some vertex in some neighborly polytopal 3-sphere (compare [**4**, Problem 1]), and uses this result to obtain a lower bound for the number of neighborly 4-polytopes with $v \geqq 5$ vertices.

9) In [**8**] it will be shown that each of the manifolds of Class VI is essentially a triangulation of $S^2 \times S^1$.

10) We also calculated the automorphism groups of the 3677 N^{10}'s (programmed by M. Aharoni under the supervision of M. A. Perles), and the results are as follows. In 3400 cases (among which are $N_{3217}{}^{10}$ and $N_{3218}{}^{10}$ of Table 1) the group consists of the identity only. In 255 cases it consists of two elements. In 14 cases it consists of four elements: in seven cases (among which is $N_{416}{}^{10}$ of Table 1) it is a cyclic group and in the other seven cases it is Klein's group. In one case it is a 5-element (cyclic) group. In one case it is an 8-element (dihedral) group. In two cases ($N_{3574}{}^{10}$ and $N_{3611}{}^{10}$ of Table 1) it is a 10-element (cyclic) group. In the remaining four cases ($N_4{}^{10}$, $N_{425}{}^{10}$, $N_{3629}{}^{10}$ and $N_{3631}{}^{10}$ of Table 1) it is a group with 20 elements. The groups of $N_4{}^{10}$, $N_{3629}{}^{10}$ and $N_{3631}{}^{10}$ are dihedral. The group of $N_{425}{}^{10}$ (which, in view of Remarks 3 and 5 in this section, is a most interesting case) is generated by the permutations $x = (13579)(24680)$, $y = (1296)(3870)(45)$ with the relations $x^5 = y^4 = e$ (= identity), $yx = x^3y$.

11) We conclude with an open question: Can every piecewise linear topological 3-manifold be so triangulated to yield a *neighborly* combinatorial 3-manifold?

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