A NOTE ON DOUBLE TRANSFORMATIONS OF CERTAIN HYPERGEOMETRIC FUNCTIONS

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(Received 2nd April 1964; revised MS. received 20th July 1964)

1. Introduction

Rainville (6) has discussed the following Euler transformation of the hypergeometric function ${}_pF_q$:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q \end{matrix} cx^k(1-x)^s\right]dx$$

$$= B(\alpha, \beta)_{p+k+s}F_{q+k+s}\left[\begin{matrix} a_1, a_2, \ldots, a_p, \Delta(k, \alpha), \Delta(s, \beta) \\ b_1, b_2, \ldots, b_q, \Delta(k+s, \alpha+\beta) \end{matrix}; c\delta\right], \quad (1.1)$$

where $\delta = \dfrac{k^k s^s}{(k+s)^{k+s}}$, $R(\alpha)>0$, $R(\beta)>0$, k and s are positive integers and $\Delta(k, \alpha)$ stands for the set of k parameters $\dfrac{\alpha}{k}, \dfrac{\alpha+1}{k}, \ldots, \dfrac{\alpha+k-1}{k}$. In a recent paper Abdul-Halim and Al-Salam (1) have given a double transformation of the hypergeometric function ${}_pF_q$ as follows:

$$[B(\alpha, \beta)B(\nu+1, \alpha+\beta+\mu)]^{-1} \int_0^1 \int_0^1 (uv)^\nu(1-uv)^\mu(1-u)^{\alpha-1}v^\alpha(1-v)^{\beta-1}$$

$$\cdot {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q \end{matrix} tv^s(1-u)^s(1-v)^k\right]dudv$$

$$= {}_{p+2k+2s}F_{q+2k+2s}\left[\begin{matrix} a_1, a_2, \ldots, a_p, \Delta(s, \alpha), \Delta(k, \beta), \Delta(s+k, \alpha+\beta+\mu) \\ b_1, b_2, \ldots, b_q, \Delta(s+k, \alpha+\beta), \Delta(s+k, \alpha+\beta+\mu+\nu+1) \end{matrix}; t\right], \quad (1.2)$$

where $R(\nu)>-1$, $R(\alpha)>0$, $R(\beta)>0$ and $R(\alpha+\beta+\mu)>0$.

Both the above transformations are effective tools for augmenting parameters in the ${}_pF_q$ function. We present here another double transformation of hypergeometric functions which will augment parameters in the ${}_pF_q$ function.

2. Some general formulae

We have the integral (3)

$$\int_0^\infty \int_0^\infty \phi(x+y)x^{\alpha-1}y^{\beta-1}dxdy = B(\alpha, \beta) \int_0^\infty \phi(z)z^{\alpha+\beta-1}dz,$$

where $R(\alpha)>0$, $R(\beta)>0$.

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Using the method of term by term integration, we can show that, if $R(\alpha)>0$, $R(\beta)>0$, and if k and s are non-negative integers, then inside the region of convergence of the resulting series

$$\int_0^\infty \int_0^\infty \phi(x+y)x^{\alpha-1}y^{\beta-1}{}_pF_q\left[\begin{matrix}a_1, a_2, \ldots, a_p\\ b_1, b_2, \ldots, b_q\end{matrix}; tx^sy^k\right]dxdy$$

$$= B(\alpha, \beta)\int_0^\infty \phi(z)z^{\alpha+\beta-1}$$

$$\cdot {}_{p+s+k}F_{q+s+k}\left[\begin{matrix}a_1, a_2, \ldots, a_p, \Delta(s, \alpha), \Delta(k, \beta)\\ b_1, b_2, \ldots, b_q, \Delta(s+k, \alpha+\beta)\end{matrix}; t\delta z^{s+k}\right]dz, \tag{2.1}$$

where $\delta = \dfrac{k^ks^s}{(k+s)^{k+s}}$. In particular, if we let $\phi(z) = e^{-z}z^\mu$, we can evaluate the above integral

$$\int_0^\infty \int_0^\infty e^{-(x+y)}(x+y)^\mu x^{\alpha-1}y^{\beta-1}{}_pF_q\left[\begin{matrix}a_1, a_2, \ldots, a_p\\ b_1, b_2, \ldots, b_q\end{matrix}; tx^sy^k\right]dxdy$$

$$= B(\alpha, \beta)\Gamma(\alpha+\beta+\mu)$$

$$\cdot {}_{p+2k+2s}F_{q+k+s}\left[\begin{matrix}a_1, \ldots, a_p, \Delta(s, \alpha), \Delta(k, \beta), \Delta(s+k, \alpha+\beta+\mu)\\ b_1, \ldots, b_q, \Delta(s+k, \alpha+\beta)\end{matrix}; ts^sk^k\right], \tag{2.2}$$

where $R(\alpha+\beta+\mu)>0$.

For brevity we shall use the operator notation

$$\Omega_{(\alpha, \beta, \mu)}\{\ \} = [B(\alpha, \beta)\Gamma(\alpha+\beta+\mu)]^{-1}\int_0^\infty \int_0^\infty e^{-(x+y)}(x+y)^\mu x^{\alpha-1}y^{\beta-1}\{\ \}dxdy, \tag{2.3}$$

in which $R(\alpha)>0$, $R(\beta)>0$ and $R(\alpha+\beta+\mu)>0$.

We mention here some results which are immediate consequences of the relation (2.3).

$$\Omega_{(\alpha, \beta, \mu)}\{1\} = 1, \tag{2.4}$$

$$\Omega_{(\alpha, \beta, \mu)}\{x^\lambda y^\delta(x+y)^\gamma\} = \frac{B(\alpha+\lambda, \beta+\delta)\Gamma(\alpha+\beta+\mu+\lambda+\delta+\gamma)}{B(\alpha, \beta)\Gamma(\alpha+\beta+\mu)}, \tag{2.5}$$

$$\Omega_{(\alpha, \beta, \mu)}\{e^{xt}\} = {}_2F_1[\alpha, \alpha+\beta+\mu;\ \alpha+\beta;\ t], \tag{2.6}$$

$$\Omega_{(\alpha, \beta, 0)}\{(1-xt)^n\} = {}_2F_0[-n, \alpha;\ -;\ t], \tag{2.7}$$

$$\Omega_{(\alpha, \beta, \mu)}\{{}_1F_1[a;\ b;\ tx]\} = {}_3F_2\left[\begin{matrix}a, \alpha, \alpha+\beta+\mu\\ b, \alpha+\beta\end{matrix}; t\right]. \tag{2.8}$$

3. Some classical polynomials

From (2.8) it is obvious that

$$\Omega_{(\alpha, 1-\alpha, n)}\{{}_1F_1[-n;\ b;\ tx]\} = {}_3F_2\left[\begin{matrix}-n, \alpha, 1+n\\ b, 1\end{matrix}; t\right],$$

https://doi.org/10.1017/S0013091500008804 Published online by Cambridge University Press

where $0<\alpha<1$. Therefore we have

$$\Omega_{(\alpha,\,1-\alpha,\,n)}\{L_n^{b-1}(tx)\} = \frac{(b)_n}{n!} H_n(\alpha, b, t), \tag{3.1}$$

where $L_n^{\alpha}(x)$ and $H_n(\alpha, b, t)$ are Laguerre and Rice's polynomials respectively.

Taking $b = \alpha$ and replacing t by $(1-t)/2$ in (3.1), we have

$$\Omega_{(\alpha,\,1-\alpha,\,n)}\left\{L_n^{\alpha-1}\left(\frac{1-t}{2}\,.\,x\right)\right\} = \frac{(\alpha)_n}{n!} P_n(t), \tag{3.2}$$

where $P_n(t)$ are Legendre's polynomials.

Again taking $a = -n$, $b = 1+\alpha$, $\mu = 0$, and replacing α by $(1+\alpha+\beta+n)$ in (2.8), we obtain

$$\Omega_{(1+\alpha+\beta+n,\,\delta,\,0)}\left\{L_n^{\alpha}\left(\frac{1-t}{2}\,.\,x\right)\right\} = P_n^{(\alpha,\beta)}(t), \tag{3.3}$$

where $P_n^{(\alpha,\beta)}(t)$ are Jacobi's polynomials.

We also obtain from (2.8)

$$\Omega_{(1+\alpha-\beta,\,\beta,\,n+\beta)}\left\{L_n^{\alpha-\beta}\left(\frac{1-t}{2}\,.\,x\right)\right\} = \frac{(1+\alpha-\beta)_n}{(1+\alpha)_n} P_n^{(\alpha,\beta)}(t). \tag{3.4}$$

For $\alpha = \beta$, (3.3) and (3.4) yield results corresponding to ultraspherical polynomials.

Again from (2.8), we have

$$\Omega_{(\alpha,\,1-\alpha-\mu,\,\nu+\mu)}\left\{{}_1F_1\left[-\nu;\ \alpha;\ \frac{1-t}{2}\,.\,x\right]\right\} = {}_2F_1\left[-\nu,\ 1+\nu;\ 1-\mu;\ \frac{1-t}{2}\right],$$

where $R(1-\alpha-\mu)>0$. Since (4 p. 143) associated Legendre's functions

$$P_\nu^{\mu}(x) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\mu/2} {}_2F_1\left[-\nu,\ 1+\nu;\ 1-\mu;\ \frac{1-x}{2}\right],$$

and Laguerre functions (4, p. 268)

$$L_\nu^{\alpha}(x) = \frac{1}{\Gamma(1+\nu)}\,\phi(-\nu;\ 1+\alpha;\ x),$$

therefore, we have

$$\Omega_{(\alpha,\,1-\alpha-\mu,\,\nu+\mu)}\left\{L_\nu^{\alpha-1}\left(\frac{1-t}{2}\,.\,x\right)\right\} = \frac{\Gamma(1-\mu)}{\Gamma(1+\nu)}\left(\frac{1-t}{1+t}\right)^{\mu/2} P_\nu^{\mu}(t). \tag{3.5}$$

Again since (4 p. 148)

$$P_\nu^{m}(x) = \frac{(-2)^{-m}\Gamma(\nu+m+1)}{m!\,\Gamma(\nu-m+1)}(1-x^2)^{m/2}{}_2F_1\left[\begin{matrix}1+m+\nu,\ m-\nu\\ 1+m\end{matrix};\ \frac{1-x}{2}\right],$$

therefore, using (2.6), we obtain

$$\Omega_{(m-\nu,\,1+\nu,\,\nu)}\{e^{x(1-t)/2}\} = \frac{(-2)^m\Gamma(\nu-m+1)m!}{\Gamma(\nu+m+1)}(1-t^2)^{-m/2}P_\nu^{m}(t), \tag{3.6}$$

where $0<R(m-\nu)<1$.

Further, we have Hermite polynomials

$$H_n(x) = (2x)^n {}_2F_0\left[-n/2,\ -n/2+\tfrac{1}{2};\ -;\ -\frac{1}{x^2}\right],$$

and (4 p. 128 (24))

$$P_\nu^\mu(z) = \frac{2^\mu z^{(\nu+\mu)}(z^2-1)^{-\mu/2}}{\Gamma(1-\mu)}\ {}_2F_1\left[\begin{matrix}-\frac{\nu}{2}-\frac{\mu}{2}, \frac{1}{2}-\frac{\mu}{2}-\frac{\nu}{2};\ 1-\frac{1}{z^2}\\ 1-\mu\end{matrix}\right],$$

where $Rz>0$. Taking $\mu = 0$ in the latter and replacing $(1-1/z^2)$ by $-x/t^2$, we have

$$P_n\left(\frac{t}{\sqrt{x+t^2}}\right) = \left(\frac{t}{\sqrt{x+t^2}}\right)^n {}_2F_1\left[-\frac{n}{2}, \frac{1}{2}-\frac{n}{2};\ 1;\ -\frac{x}{t^2}\right].$$

Now we can easily obtain the relation

$$\Omega_{(1,\beta,0)}\left\{(x+t^2)^{n/2}P_n\left(\frac{t}{\sqrt{x+t^2}}\right)\right\} = \frac{H_n(t)}{2^n}. \tag{3.7}$$

Bateman polynomials (4 p. 193) are defined by

$$z^{-\mu}J_n^{\mu,\nu}(z) = \frac{\Gamma\left(1+n+\nu+\frac{\mu}{2}\right)}{n!\Gamma(\mu+1)\Gamma\left(1+\nu+\frac{\mu}{2}\right)}\ {}_1F_2\left[-n;\ \mu+1,\ 1+\nu+\frac{\mu}{2};\ z^2\right].$$

More particularly

$$J_n^{0,\alpha}(z) = \frac{(1+\alpha)_n}{n!}\ {}_1F_2[-n;\ 1,\ 1+\alpha;\ z^2].$$

Since we have

$$\Omega_{(1,\alpha,0)}\{{}_1F_2[-n;\ 1,1+\alpha;\ xt]\} = {}_1F_1[-n;\ 1+\alpha;\ t],$$

therefore from the above relation, we obtain

$$\Omega_{(1,\alpha,0)}\{J_n^{0,\alpha}(\sqrt{xt})\} = L_n^\alpha(t). \tag{3.8}$$

The generalised Bessel polynomials (6 p. 293) are defined as follows:

$$Y_n(a,\ b,\ x) = {}_2F_0[-n,\ a-1+n;\ -;\ -x/b].$$

Using (2.7), we easily obtain

$$\Omega_{(a-1+n,\beta,0)}\left\{\left(1+\frac{xt}{b}\right)^n\right\} = Y_n(a,\ b,\ t). \tag{3.9}$$

4. Generating functions and expansions

The operator $\Omega_{(\alpha,\beta,\mu)}\{\ \}$ is sometimes useful in deriving the generating functions and expansions of one function from the known generating function and expansion of another function. We mention here a few such cases.

In view of the relation (5, p. 267 (22)), the generating function for $J_n^{0,\alpha}(\sqrt{xt})$ may be written as

$$(1-u)^{-1-\alpha}{}_0F_1\left[-;\ 1;\ -\frac{xtu}{1-u}\right]=\sum_{n=0}^{\infty} J_n^{0,\alpha}(\sqrt{xt})u^n. \tag{4.1}$$

Operating on both sides by $\Omega_{(1,\alpha,0)}\{\ \}$ and using (3.8), we obtain the generating function for Laguerre polynomials

$$(1-u)^{-1-\alpha}e^{-\frac{tu}{1-u}}=\sum_{n=0}^{\infty} L_n^{\alpha}(t)u^n. \tag{4.2}$$

Again replacing β by $\beta-n$ in the relation (3.3), we have

$$\Omega_{(1+\alpha+\beta,\delta,0)}\left\{L_n^{\alpha}\left(\frac{1-t}{2}.x\right)\right\}=P_n^{(\alpha,\beta-n)}(t),$$

which may also be written as

$$\Omega_{(1+\alpha+\beta,\delta,0)}\left\{\sum_{n=0}^{\infty} L_n^{\alpha}\left(\frac{1-t}{2}.x\right)u^n\right\}=\sum_{n=0}^{\infty} P_n^{(\alpha,\beta-n)}(t)u^n.$$

Using (4.2) and finally (2.6), we obtain the generating function for Jacobi polynomials

$$2^{1+\alpha+\beta}(1-u)^{\beta}(2-u-ut)^{-(1+\alpha+\beta)}=\sum_{n=0}^{\infty} P_n^{(\alpha,\beta-n)}(t)u^n. \tag{4.3}$$

Starting with (3.3), we may also obtain

$$2^{1+\alpha+\beta}(1+u)^{\alpha}(2+u-ut)^{-(1+\alpha+\beta)}=\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta)}(t)u^n. \tag{4.4}$$

Again, (3.9) may be adjusted as follows:

$$\Omega_{(a-1,\beta,0)}\left\{\sum_{n=0}^{\infty}\frac{u^n}{n!}\left(1+\frac{xt}{b}\right)^n\right\}=\sum_{n=0}^{\infty} Y_n(a-n,b,t)\frac{u^n}{n!}.$$

Thus, after performing the operation, we obtain the generating function for Bessel polynomials

$$e^u\left(1-\frac{ut}{b}\right)^{1-a}=\sum_{n=0}^{\infty} Y_n(a-n,b,t)\frac{u^n}{n!}. \tag{4.5}$$

In view of the relation (6, p. 207 (2)), we have

$$\left(\frac{1-t}{2}.x\right)^n=\sum_{k=0}^{n}\frac{(-1)^k n!(1+\alpha)_n}{(n-k)!(1+\alpha)_k}L_k^{\alpha}\left(\frac{1-t}{2}.x\right). \tag{4.6}$$

Operating on both sides of (4.6) by $\Omega_{(1+\alpha+\beta,\delta,0)}\{\ \}$, and using (3.3), we obtain the expansion for Jacobi polynomials

$$\left(\frac{1-t}{2}\right)^n=\sum_{k=0}^{n}\frac{(-1)^k n!(1+\alpha)_n}{(n-k)!(1+\alpha)_k}P_k^{(\alpha,\beta-k)}(t). \tag{4.7}$$

5. Some formulae involving Appell's functions

Appell's functions of two variables (4, p. 224) are defined as follows:

$$F_1(a;\ b, b';\ c;\ x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_{m+n} m! n!}, \tag{5.1}$$

$$F_2(a;\ b, b';\ c, c';\ x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_m (c')_n m! n!}, \tag{5.2}$$

$$F_3(a, a';\ b, b';\ c;\ x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n x^m y^n}{(c)_{m+n} m! n!}, \tag{5.3}$$

$$F_4(a;\ b;\ c, c';\ x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n} x^m y^n}{(c)_m (c')_n m! n!}. \tag{5.4}$$

Now it follows from the identity

$$e^{(xu+yv)} = \sum_{m,n=0}^{\infty} \frac{(xu)^m (yv)^n}{m! n!},$$

and the formula (2.5) that

$$\Omega_{(\alpha,\beta,\mu)}\{e^{xu+yv}\} = F_1(\alpha+\beta+\mu;\ \alpha, \beta;\ \alpha+\beta;\ u, v). \tag{5.5}$$

The confluent form of (5.1) is

$$\phi_1(a, b, c, x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_n x^m y^n}{(c)_{m+n} m! n!}. \tag{5.6}$$

This form is readily obtained from the identity

$$\Gamma\alpha I_{(\alpha-1)}(2\sqrt{ux})(ux)^{\frac{1-\alpha}{2}} e^{vy} = \sum_{m,n=0}^{\infty} \frac{(ux)^m (vy)^n}{(\alpha)_m m! n!},$$

where $I_\nu(z)$ is the modified Bessel function, and the formula (2.5), giving

$$\Omega_{(\alpha,\beta,\mu)}\{\Gamma\alpha(xu)^{\frac{1-\alpha}{2}} I_{\alpha-1}(2\sqrt{xu})e^{yv}\} = \phi_1(\alpha+\beta+\mu, \beta, \alpha+\beta, u, v). \tag{5.7}$$

We also obtain Appell's functions from the confluent series (4, p. 225) as follows:

$$\Omega_{(\beta,\beta',0)}\{\psi_2(\alpha, \gamma, \gamma', ux, vy)\} = F_2[\alpha;\ \beta, \beta';\ \gamma, \gamma';\ u, v], \tag{5.8}$$

$$\Omega_{(\alpha,\alpha',0)}\{\phi_2(\beta, \beta', \gamma, ux, vy)\} = F_3[\alpha, \alpha';\ \beta, \beta';\ \gamma;\ u, v]. \tag{5.9}$$

Further we have (4, p. 264) that

$$e^{x/2} x^{-\frac{1}{2}-\mu} M_{k,\mu}(x) = \phi(a;\ c;\ x);\ k = c/2-a,\ \mu = c/2-\tfrac{1}{2},$$

where $M_{k,\mu}(x)$ is Whittaker's M-function. Now consider the identity

$$e^{(ux+vy)/2}(ux)^{-c/2}(vy)^{-(1+a+b-c)/2} M_{c/2-a,\, c/2-\frac{1}{2}}(ux)$$
$$\cdot M_{(1+b-c-a)/2,\, (a+b-c)/2}(vy) = \phi(a;\ c;\ xu)\phi(a;\ 1+a+b-c;\ yv).$$

Operating on both sides by $\Omega_{(b,b,0)}\{\ \}$ and observing that (6, p. 269)

$$F_4[a;\ b;\ c, (1+a+b-c);\ x(1-y), y(1-x)]$$
$$= {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix};\ x\right] {}_2F_1\left[\begin{matrix} a, b \\ 1+a+b-c \end{matrix};\ y\right],$$

we immediately obtain

$$\Omega_{(b,b,0)}\{e^{(ux+vy)/2}(ux)^{-c/2}(vy)^{-(1+a+b-c)/2}$$
$$\cdot M_{c/2-a,\,c/2-\frac{1}{2}}(ux)M_{(1+b+c-a)/2,\,(a+b-c)/2}(vy)\}$$
$$= F_4[a;\ b;\ c,\ (1+a+b-c);\ u(1-v),\ v(1-u)]. \qquad (5.10)$$

6. An identity

The operator $\Omega_{(\alpha,\beta,\mu)}\{\ \}$ may also be used for establishing certain identities. We illustrate the method by considering the expansion for Laguerre polynomials. We have the relation (2, p. 142) that

$$L_n^{\lambda}(x+y) = \sum_{\gamma=0}^{n} \frac{(-1)^{\gamma}y^{\gamma}}{\gamma!} L_{n-\gamma}^{\lambda+\gamma}(x),$$

which may also be written as

$$\frac{(1+\lambda)_n}{n!}\sum_{\gamma=0}^{n}\frac{(-n)_{\gamma}(x+y)^{\gamma}}{\gamma!(1+\lambda)_{\gamma}} = \sum_{\gamma=0}^{n}\sum_{k=0}^{n-\gamma}\frac{(-1)^{\gamma}(1+\lambda+\gamma)_{n-\gamma}(-n+\gamma)_k x^k y^{\gamma}}{\gamma!k!(n-\gamma)!(1+\lambda+\gamma)_k}.$$

Operating on both sides by $\Omega_{(\alpha,\beta,\mu)}\{\ \}$, we obtain

$$\frac{(1+\lambda-\alpha-\beta-\mu)_n}{n!} = \sum_{\gamma=0}^{n}\frac{(-1)^{\gamma}(1+\lambda+\gamma)_{n-\gamma}(\beta)_{\gamma}(\alpha+\beta+\mu)_{\gamma}}{\gamma!(n-\gamma)!(\alpha+\beta)_{\gamma}}$$
$$\cdot {}_3F_2\left[\begin{matrix}-n+\gamma,\ \alpha,\ \alpha+\beta+\mu+\gamma \\ 1+\lambda+\gamma,\ \alpha+\beta+\gamma\end{matrix};\ 1\right]. \qquad (6.1)$$

Taking $\lambda = \alpha+\mu-n$, the right hand side becomes of the Saalschütz form and finally reduces to $(-1)^n\dfrac{(\beta)_n}{n!}$.

I wish to record my sincere thanks to Dr K. N. Srivastava for his kind help and guidance during the preparation of this note.

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