TRIANGLES IN AN ORDINARY GRAPH

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1. Introduction. An ordinary graph is a finite linear graph which contains no loops or multiple edges, and in which all edges are undirected. In such a graph G, let N, L, and T denote respectively the number of nodes, edges, and triangles. One problem, suggested by P. Erdös **(1)**, is to determine the minimum number of triangles when the number of edges is specified, subject to suitable restrictions. For any ordinary graph with $N = 2u$, $L = u^2 + k$, and $k < u$, he conjectured that $T \geqslant ku$. The case $k = 1$ is an unpublished result due to Rademacher, and the cases $k = 2, 3$ (also unpublished) were established by Erdös, who also showed that the conjecture can fail when $k = u$. In this paper we seek a function $f(N, L)$ such that $T \geqslant f(N, L)$ for every G, and develop various inequalities for the minimum number of triangles, valid for the entire range of L. In particular we readily establish an inequality quadratic in L which for many graphs is the best possible. However, most of our efforts deal with inequalities linear in L. This approach was motivated by the observation that the conjecture of Erdös implies that $8T \geqslant N(4L - N^2)$ when $N^2 < 4L < N^2 + 2N$. One obvious modification is to write $RT \geqslant N(4L - N^2)$ and attempt to determine the positive number R (independent of L and N) as small as possible so that the resulting inequality is valid for all ordinary graphs. It is simple to show that this inequality holds for all graphs when $R = 12$ and that $R = 9$ is the minimal possible value of R. Of course this does not exclude smaller values of R, as in the Erdös conjecture, for certain subclasses of graphs in which the number of edges is suitably restricted.

We say that G is t-ary if the graph complementary to G has exactly t components, and are able to show that the inequality $9T \geqslant N(4L - N^2)$ holds for all graphs with N nodes if it holds for all "unary" graphs ($t = 1$). We also show that this inequality holds for all symmetric graphs, for all graphs satisfying $N^2 \leqslant 3L$, for all graphs with $N \leqslant 10$, and for certain other types of graphs.

2. Preliminary observations. Unless $4L > N^2$, we cannot be certain that a triangle will appear. If $N = 2u$, we may divide the nodes into two groups each with u nodes and insert edges only between nodes not in the same group; then $4L = N^2$ and no triangles occur. Similarly if $N = 2u + 1$ we divide the nodes into two groups with u and $u + 1$ nodes respectively, and insert edges as before, obtaining $4L = N^2 - 1$, and again no triangles occur. Accordingly, we shall usually assume that $4L > N^2$.

Received June 13, 1961.

https://doi.org/10.4153/CJM-1963-004-7 Published online by Cambridge University Press

Let x_i denote the number of edges terminating at the node P_i and call x_i the *order* of the node P_i. Then $2L = \sum x_i$, where the summation symbol, unless otherwise restricted, will indicate the range $i = 1, 2, \ldots, N$.

Let T_k be the number of sets of three nodes having k edges, $k = 0, 1, 2, 3$. Then $T_3 = T$ is the number of "full" triangles, and T_0 the number of "empty" triangles. Each edge e_j is an edge of T_{kj} "triangles" of type k, and

$$T_{1j} + T_{2j} + T_{3j} = N - 2,$$

for the edge itself joins two nodes, and with each of the other $N - 2$ nodes determines a triangle of type 1, 2, or 3. Summing the displayed expression for $j = 1, 2, \ldots, L$, we obtain

$$T_1 + 2T_2 + 3T = L(N - 2). \tag{1}$$

By considering the number of pairs of edges at each node we find

$$T_2 + 3T = \sum \binom{x_i}{2} = \sum \frac{x_i(x_i - 1)}{2},$$

which we write in the form

$$2T_2 + 6T = \sum x_i^2 - 2L. \tag{2}$$

Let $\sum'$ indicate summation over all pairs i and j, ranging independently from 1 to N, subject to the restriction $i > j$. Note that

$$\sum x_i^2 + 2\sum' x_i x_j = (\sum x_i)^2 = 4L^2,$$

$$(N - 1)\sum x_i^2 - 2\sum' x_i x_j = \sum' (x_i - x_j)^2,$$

$$N\sum x_i^2 = 4L^2 + \sum' (x_i - x_j)^2. \tag{3}$$

Eliminating T_2 from (1) and (2) we have

$$3T = T_1 + \sum x_i^2 - LN.$$

Then using (3) we obtain

$$3TN = NT_1 + L(4L - N^2) + \sum' (x_i - x_j)^2. \tag{4}$$

Since T_1 and $\sum'(x_i - x_j)^2$ are non-negative, we obtain the inequality (quadratic in L)

$$3TN \geqslant L(4L - N^2), \tag{5}$$

in which the equality sign holds if and only if $T_1 = 0$ and $\sum'(x_i - x_j)^2 = 0$. The latter condition of course implies that every node of G has the same order, that is, G is *regular*, and as shown in § 3, the condition $T_1 = 0$ implies that G is a symmetric graph. We also observe that $T > 0$ when $4L > N^2$, and from the examples given at the beginning of this section that no smaller value of L will guarantee that $T > 0$. Thus the critical condition is $4L > N^2$, as suspected. From (5) we observe that for every graph G the point (L, T) always lies above or on the parabola $3TN = L(4L - N^2)$, whose graph is shown in Figure 1.

Since the tangent line at point $A : (N^2/4, 0)$ has the equation $12T = N(4L - N^2)$, we have

THEOREM 1. *The relation* $12T \geqslant N(4L - N^2)$ *holds for all ordinary graphs.*

For the ideas expressed in (1) and (2) and the first proof of Theorem 1, we are indebted to J. H. McKay.

THEOREM 2. *The minimal value of R for which* $RT \geqslant N(4L - N^2)$ *can hold for all ordinary graphs is* $R = 9$.

Proof. The value $R = 9$ cannot be improved, for there are graphs for which $9T = N(4L - N^2)$. One such graph may be obtained by taking $N = 3U$, dividing the nodes into three sets each containing U nodes, and inserting edges between all nodes that are in different sets, but no other edges. Then $L = 3U^2$ and $T = U^3$. Furthermore, (see Figure 1) we note that the line

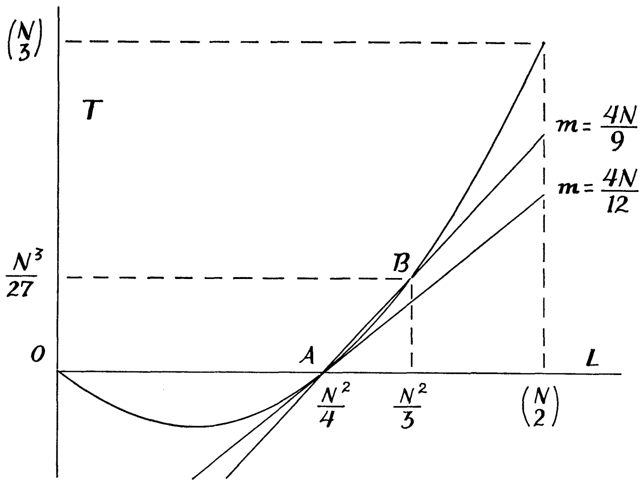

FIGURE 1

through points $A : (N^2/4, 0)$ and $B : (N^2/3, N^3/27)$ has the equation $9T = N(4L - N^2)$, and so the inequality $9T \geqslant N(4L - N^2)$ is valid for all ordinary graphs having $3L \geqslant N^2$ or $N^2 \geqslant 4L$. It remains to consider this inequality when $3L < N^2 < 4L$. For convenience, we define

$$J = 9T + N^3 - 4NL, \tag{6}$$

and conjecture that $J \geqslant 0$ for all ordinary graphs when $3L < N^2 < 4L$. The following sections are devoted mainly to a study of this conjecture.

https://doi.org/10.4153/CJM-1963-004-7 Published online by Cambridge University Press

3. Symmetric graphs. If in an ordinary graph G there is a path consisting of edges leading from node P to node Q, then P and Q are said to be connected. If this notion is extended to mean that each node is always considered to be connected to itself, then "being connected" is an equivalence relation dividing the nodes of G into disjoint subsets called "components" of G. In general not every pair of nodes in a component is connected by an edge, for it may be that every path joining the two nodes involves more than one edge. But if every pair of nodes in a component is connected by an edge, then the component is called "complete"; and a component consisting of a single node is also called complete.

By the "complement" G^* of a graph G is meant the graph obtained from G by using the same nodes, but removing all edges of G and inserting an edge in G^* wherever there was none in G (see **3**). We define the graph G to be "t-ary" if its complement G^* contains exactly t components. Thus if we say that G is "unary", then the complement G^* must have $t = 1$, that is, G^* is connected.

A graph G is called "symmetric" if all components of G^* are complete. Thus a symmetric t-ary graph G of N nodes can be constructed by dividing the nodes into t disjoint subsets containing $N_1, N_2, \ldots, N_t$ nodes with $N = N_1 + N_2 + \ldots + N_t$, where nodes of a subset are not joined by edges, but every possible edge is inserted between nodes of distinct subsets. In this description the completely disconnected subsets of G become the completely connected components of G^*. A symmetric graph may also be defined as the product of distinct empty graphs (see **4**).

THEOREM 3. *G is symmetric if and only if $T_1 = 0$.*

Proof. A "triangle" of type 1 consists of three nodes with exactly two edges missing. If a "triangle" in a symmetric graph has two edges missing, then the three nodes must belong to the same disconnected subset, hence the third edge is also missing. Thus a symmetric graph has $T_1 = 0$.

Conversely, if $T_1 = 0$, every "triangle" with two edges missing must have the third edge missing. The corresponding graph G must have the following characteristics. Consider any node P of G. Either P has the order $x = N - 1$ and the single node is itself to be considered a disconnected subset, or P has $x < N - 1$ and then we can show that P and every node not joined to P constitute a completely disconnected subset. For (a) if P_1 is not joined to P and if Q is joined to P, then Q must be joined to P_1, for otherwise $T_1 > 0$; hence the set of nodes not joined to P_1 can consist only of P and nodes not joined to P. And (b) if P_1 and P_2 are not joined to P, then P_1 and P_2 are not themselves joined, for otherwise $T_1 > 0$. Thus (a) and (b) show that P is a node of a completely disconnected subset of G corresponding to a complete component of G^* of exactly $N - x$ nodes. But P was any node of G. Hence $T_1 = 0$ implies that G is symmetric.

As an application of Theorem 3, we consider the problem of determining when a point (L, T) lies on the parabola of Figure 1, and note from the remarks following (5) that the corresponding G is regular and symmetric. If such a G is "t-ary" and each node has order x, then $t(N - x) = N$, so t is a divisor of N. The number of such regular symmetric graphs is therefore the total number of positive integer divisors of N. Since $t = 1, 2, 3$ gives $L = 0$, $N^2/4$, $N^2/3$ respectively, no such graph gives a point on the parabola when $3L < N^2 < 4L$.

4. An inequality for positive real numbers. It is possible to prove an inequality for the elementary symmetric functions

$$E_1 = \sum A_p, \quad E_2 = \sum A_pA_q, \quad E_3 = \sum A_pA_qA_r$$

of any t positive real numbers $A_1, A_2, \ldots, A_t$ similar to the conjectured inequality $J \geqslant 0$ for graphs.

THEOREM 4. *The inequality $J_t = 9E_3 + E_1^3 - 4E_1E_2 \geqslant 0$ holds for any set of t positive real numbers.*

Proof. The proof is by induction on t, starting from the obvious cases $t = 1$, where $J_1 = A_1^3 > 0$, and $t = 2$, where $J_2 = (A_1 + A_2)(A_1 - A_2)^2 \geqslant 0$. For the case $t + 1$ let A_0 be the additional number and assume (since the induction hypothesis will be $J_t \geqslant 0$ for *any* t positive numbers) that $0 < A_0 \leqslant A_1 \leqslant \ldots \leqslant A_t$. Then $tA_0 \leqslant E_1$ implies for $t \geqslant 2$ that $A_0 \leqslant E_1/2 < 4E_1/5$. Since

$$\begin{aligned} J_{t+1} &= 9(E_3 + A_0E_2) + (E_1 + A_0)^3 - 4(E_1 + A_0)(E_2 + A_0E_1) \\ &= J_t + A_0(E_1/2 - A_0)^2 + (4E_2 - E_1^2)5A_0/4, \end{aligned}$$

it is easy to see that if $4E_2 - E_1^2 > 0$, $J_t \geqslant 0$ implies $J_{t+1} > 0$. The other possibility of $4E_2 - E_1^2 \leqslant 0$ may be handled by use of the previous observation $A_0 < 4E_1/5$, for then

$$J_{t+1} \geqslant J_t - (E_1^2 - 4E_2)5A_0/4 \geqslant J_t - E_1(E_1^2 - 4E_2) = 9E_3 \geqslant 0.$$

By induction the proof is complete.

When $t > 2$ the first inequality of the last-displayed chain is strict because the omitted term $A_0(E_1/2 - A_0)^2$ is positive; thus $J_t > 0$ holds for $t \geqslant 4$. When $t = 2$ the first inequality of the chain is strict unless $A_0 = E_1/2$ and the second inequality is strict unless $E_1^2 - 4E_2 = (A_2 - A_1)^2 = 0$. Thus $J_3 > 0$, except when $A_0 = A_1 = A_2$.

THEOREM 5. *For every symmetric graph the inequality $J \geqslant 0$ is satisfied.*

Proof. For the symmetric graph described in § 3, we have the special relations:

$$N = \sum N_p, \quad L = \sum N_pN_q, \quad T = \sum N_pN_qN_r,$$

where N, L, T are the elementary symmetric functions of the t positive integers $N_1, N_2, \ldots, N_t$. Hence Theorem 4 may be applied to establish $J \geqslant 0$.

5. Reduction of the problem to unary graphs. If the graph G with N nodes is t-ary, let the connected components of G^* have $N_1, N_2, \ldots, N_t$ nodes. Let the vertices P_i and P_j belong to a component of G^* having N_r nodes where $N_r \geqslant 2$. Suppose that in G the nodes P_i and P_j are joined by an edge P_iP_j which is an edge of k_{ij} triangles of G. Let G_{ij} be the graph obtained from G by deleting the edge P_iP_j, and using the notation in (6) let J_{ij} be associated with G_{ij}.

LEMMA 1. *If $N_r \leqslant 5N/9$, then $J_{ij} \leqslant J$.*

Proof. Since $N_{ij} = N$, $L_{ij} = L - 1$, $T_{ij} = T - k_{ij}$, it follows that

$$J_{ij} = 9T_{ij} + N^3 - 4NL_{ij} = J - (9k_{ij} - 4N).$$

But from the concept of components the N_r nodes to which P_i and P_j belong are connected in G to all the other $N - N_r$ nodes. Consequently, if $N_r \leqslant 5N/9$, then $k_{ij} \geqslant N - N_r \geqslant 4N/9$; hence $J_{ij} \leqslant J$.

THEOREM 6. *If G has every $N_r \leqslant 5N/9$, then $J \geqslant 0$.*

Proof. By repeated application of Lemma 1 the edges in each subset G_r of G, corresponding to a component of G^* with N_r nodes when $N_r \geqslant 2$, can be deleted without increasing "J". After a sufficient number of deletions a symmetric graph G'' is reached with $J'' \leqslant J$. But from Theorem 5 we know $0 \leqslant J''$. Hence $J \geqslant 0$. In the above proofs an isolated node, corresponding to the case $N_r = 1$, causes no difficulty.

THEOREM 7. *If $J \geqslant 0$ holds for all G with N nodes which have $t = 1$ or $t = 2$, then $J \geqslant 0$ holds for all G with N nodes.*

Proof. We must consider G with $t \geqslant 3$. If every N_r of G satisfies $N_r \leqslant 5N/9$, then Theorem 6 applies. Since at most one N_r, say N_1, can satisfy $N_1 > 5N/9$, in the case which remains G has $t \geqslant 3$, $N_1 > 5N/9$, and $N_2 + N_3 + \ldots + N_t < 4N/9$. By Lemma 1 each of the subsets G_r of G corresponding to N_r, for $2 \leqslant r \leqslant t$, can be replaced by a completely disconnected subset without increasing "J". But $N_2 + N_3 < 4N/9$ shows that each edge of G joining a node of G_2 to a node of G_3 can be deleted (by the same argument as in the proof of Lemma 1), without increasing "J". By repetition of this argument all of G corresponding to the nodes counted by $N_2 + N_3 + \ldots + N_t$ can be combined into one completely disconnected subset without increasing "J". Thus the problem of establishing $J \geqslant 0$ for graphs with N nodes has been reduced to the two cases $t = 1$ and $t = 2$.

THEOREM 8. *If $J \geqslant 0$ holds for all unary G_1 with N_1 nodes, $N_1 \leqslant N$, then $J \geqslant 0$ holds for all G with N nodes.*

Proof. By Theorem 7 the only case requiring discussion occurs when G has $t = 2$, with $N_1 > 5N/9$ and $N_2 = N - N_1$, with G_2 completely disconnected itself and completely joined to G_1. Let the subgraph G_1 contain L_1 edges and T_{31} triangles. From the relations $N = N_1 + N_2$, $L = L_1 + N_1N_2$, $T = T_{31} + L_1N_2$, it follows that

$$J = N(N - 2N_1)^2 + 9T_{31} - (9N_1 - 5N)L_1. \tag{7}$$

We set $A = N(N - 2N_1)^2/(9N_1 - 5N)$ and note that $A > 0$ since $9N_1 > 5N$. Since $T_{31} \geqslant 0$, it follows directly from (7) that $J \geqslant 0$ if $A \geqslant L_1$. If $L_1 > A$ we make use of the hypothesis that $J \geqslant 0$ holds for G_1, so that $9T_{31} \geqslant N_1(4L_1 - N_1^2)$. Then if we study (7) it follows that $J \geqslant 0$ providing that $L_1 \geqslant B \equiv (3NN_1 - N^2 - N_1^2)/5$. However, $L_1 > A \geqslant B$, since the latter inequality reduces to $N_1(3N_1 - 2N)^2 \geqslant 0$. This completes the proof of Theorem 8.

THEOREM 9. *The relation $J \geqslant 0$ holds for every G of N nodes which is t-ary, $t \geqslant 2$, if the largest N_r, say N_1, satisfies $N_1 \leqslant 2N/3$.*

Proof. From Theorem 7 we may assume that G has $t = 2$ with G_2 completely disconnected and completely joined to G_1. Moreover, we may assume $9N_1 > 5N$ and $3L < N^2$. Since $L_1 = L - N_1N_2$, where $N_2 = N - N_1$, we find

$$(9N_1 - 5N)L_1 < (9N_1 - 5N)(N_1^2 - N_1N + N^2/3) \leqslant N[N - 2N_1]^2,$$

for the last inequality reduces to $(3N_1 - 2N)^3 \leqslant 0$, which is true from the hypothesis $N_1 \leqslant 2N/3$. From (7) we find $J \geqslant 0$.

Theorems 7 and 8 seem important because they reduce the study of $J \geqslant 0$ to the consideration of unary graphs. Unfortunately the general unary case is still undecided, so the positive result of Theorem 9 (which subsumes Theorem 6) for certain non-unary graphs has been included, since its proof is independent of the unary case and since these graphs, which may be symmetric but are not necessarily symmetric, considerably augment (without completely including) the class of symmetric graphs covered by Theorem 5.

6. Special results for unary graphs.

If $L > 0$, we say that G is "non-empty."

THEOREM 10. *If G is unary and non-empty, then $T_1 \geqslant N - 2$.*

Proof. The proof is by induction on N with $N \geqslant 3$.

When $N = 3$, if G is unary and non-empty, then G has exactly one edge, so $T_1 = 1 = 3 - 2$.

We assume that the statement of the theorem is correct for every non-empty unary graph with N' nodes, $3 \leqslant N' \leqslant N$, and we consider any non-empty unary graph G with $N + 1$ nodes. We fix attention on one node P of order x and the graph G' obtained from G by deleting P and all x edges joining P to nodes of G'. We then consider three cases:

A. G' is unary with $T_1' = 0$;

B. G' is unary with $T_1' > 0$;

C. G' is t-ary with $t \geqslant 2$ and $N = N_1 + N_2 + \ldots + N_t$.

Case A. Since $T_1' = 0$, it follows from Theorem 3 that G' is symmetric. But G' is unary, so $L' = 0$. Since G is unary and non-empty, it follows that $1 \leqslant x \leqslant N - 1$. Since $L' = 0$, we have $T_1 = x(N - x)$. If $x = 1$, $T_1 = N - 1 = (N + 1) - 2$. If $x > 1$, $T_1 = x(N - x) \geqslant (N - 1)$ because this latter inequality reduces to $N \geqslant x + 1$.

Case B. Since G' is unary with $T_1' > 0$, hence $L' > 0$, the induction hypothesis applies, and $T_1' \geqslant N - 2$. Since G is unary, $x = N$ cannot occur. If $x = 0$, then

$$T_1 = T_1' + L' \geqslant (N - 2) + 1 = (N + 1) - 2.$$

In the remaining cases, $1 \leqslant x \leqslant N - 1$, there is at least one node Q_1 of G' which is (in G) not joined to P and at least one node Q_2 of G' which is (in G) joined to P. Since G' is unary, there must exist in $(G')^*$ a path joining Q_1 to Q_2, say from $P_1 = Q_1$ to P_2 to $\ldots$ to $P_s = Q_2$. Then because of the properties of Q_1 and Q_2 there must be a minimal value j, $1 \leqslant j \leqslant s - 1$, such that (in G) P is not joined to P_j and P is joined to P_{j+1}. In $(G')^*$ the nodes P_j and P_{j+1} are joined, but in G' they are not joined, hence PP_jP_{j+1} is a "triangle" of G with exactly one edge. Thus again

$$T_1 \geqslant T_1' + 1 \geqslant (N - 2) + 1 = (N + 1) - 2.$$

Case C. We have that G' is t-ary with $t \geqslant 2$. Let G_i for $i = 1, 2, \ldots, t$ indicate the subgraph of G' corresponding to the N_i nodes that form one of the connected components of $(G')^*$. From the definition of the t-ary graph, every node of G_i is connected to every node of G_j, when $i \neq j$. Since G is unary, in each G_i there exists at least one node P_i which is not joined to P. We note that (a) when $i \neq j$, "triangle" PP_iP_j has exactly one edge.

The subgraph G_i^+ consisting of G_i, P, and the edges of G that join P to nodes of G_i is a unary graph with fewer than N nodes. Let T_{1i}^+ indicate the value of T_1 for G_i^+. If $T_{1i}^+ > 0$, then this implies $N_i \geqslant 2$, so the induction hypothesis applies to G_i^+. We note that

(b) $T_{1i}^+ \geqslant (N_i + 1) - 2 = N_i - 1$.

But if $T_{1i}^+ = 0$, then Theorem 3 shows that G_i^+ is symmetric. But G_i^+ is unary, so G_i^+ is empty, and there is no node of G_i joined to P. Since $t \geqslant 2$, there exists $j \neq i$, and then all the "triangles" with P and P_j as two nodes and any node of G_i as the third node have exactly one edge. We note that the number of these "triangles" is given by

(c) $N_i = 1 + (N_i - 1)$.

Combining the observations (a), (b), and (c) we have shown that

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$$T_1 \geqslant \binom{t}{2} + \sum_{i=1}^{t} (N_i - 1) \equiv C.$$

However, $C = N + (t^2 - 3t)/2 \geqslant (N + 1) - 2$ when $t \geqslant 2$.

Since cases A, B, and C exhaust all possibilities, the induction step from N to $N + 1$ is successful, completing the proof of Theorem 10.

THEOREM 11. *The inequality $J \geqslant 0$ holds for all G when $N \leqslant 10$.*

Proof. The conjecture $J \geqslant 0$ may be rewritten with the aid of (4) and (6) in the form

$$3NT_1 + 3D \geqslant V, \tag{8}$$

where $D = \sum'(x_i - x_j)^2$ and $V = (N^2 - 3L)(4L - N^2)$.

The maximum value of V, considered as a function of L, is $N^4/48$, obtained when $L = 7N^2/24$. Since $D \geqslant 0$, the relation (8) will hold if $T_1 \geqslant N^2/144$.

From Theorem 8 it will suffice to establish (8) for unary graphs. If $T_1 = 0$, then by Theorem 3 the graph must be symmetric, and we can either apply Theorem 5, or note that a symmetric graph cannot be unary unless it is empty, and then $V < 0$ so that (8) holds trivially. If $T_1 > 0$ and the graph is unary, then from Theorem 10 we have $T_1 \geqslant N - 2$. However, $N - 2 \geqslant N^3/144$ when $3 \leqslant N \leqslant 10$. The cases $N = 1$ and $N = 2$ are trivial, so the proof is complete.

The authors have also proved that $J \geqslant 0$ for all graphs with $N = 11$, 12, or 13, but the argument here becomes much more tedious, and we omit the details. It is evident from (8) that an estimate is needed for D. This can be obtained from the following lemma, whose proof we also omit.

LEMMA 2. *If $x_1, x_2, \ldots, x_N$ are non-negative integers, if $S = \sum x_i$, and if $S = QN + R$ with $0 \leqslant R < N$, then the minimal value m of $M = \sum x_i^2$, for a fixed value of S, is given by $m = (N - R)Q^2 + R(Q + 1)^2$.*

Using $S = \sum x_i = 2L$, and applying Lemma 2 to (3), we find $D \geqslant R(N-R)$, an estimate which can be used to establish $J \geqslant 0$ when $N = 11$. More refined estimates are used for the cases $N = 12$ and $N = 13$.

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