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RESEARCH ARTICLE

Deformations of Calabi–Yau varieties with *k*-liminal singularities

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Abstract
The goal of this paper is to describe certain nonlinear topological obstructions for the existence of first-order smoothings of mildly singular Calabi–Yau varieties of dimension at least 4. For nodal Calabi–Yau threefolds, a necessary and sufficient linear topological condition for the existence of a first-order smoothing was first given in [Fri86]. Subsequently, Rollenske–Thomas [RT09] generalized this picture to nodal Calabi–Yau varieties of odd dimension by finding a necessary nonlinear topological condition for the existence of a first-order smoothing. In a complementary direction, in [FL22a], the linear necessary and sufficient conditions of [Fri86] were extended to Calabi–Yau varieties in every dimension with 1-liminal singularities (which are exactly the ordinary double points in dimension 3 but not in higher dimensions). In this paper, we give a common formulation of all of these previous results by establishing analogues of the nonlinear topological conditions of [RT09] for Calabi–Yau varieties with weighted homogeneous k-liminal hypersurface singularities, a broad class of singularities that includes ordinary double points in odd dimensions.

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1. Introduction

The deformation theory of generalized Fano and Calabi–Yau threefolds with ordinary double points (or nodes), or more generally isolated canonical hypersurface singularities, has been extensively studied [Fri86], [Nam94], [NS95], [Nam02], [Ste97], [Gro97]. This paper is part of a series [FL22a, FL22b, FL22c, FL23] that aims to revisit and sharpen these results and explore generalizations to higher dimensions. A motivating question throughout has been the problem of understanding the local structure of compactified moduli spaces of Calabi–Yau varieties. Let Y be a generalized Calabi–Yau variety in a suitable sense (Definition 1.1). Two natural questions arise: (1) Is the first-order deformation space of Y unobstructed (i.e., is the moduli space smooth at the point corresponding to Y)? (2) If the singularities of Y are of some prescribed type, is there a smoothing of Y (i.e. a proper flat morphism $\mathcal{Y} \to \Delta$ whose fiber over 0 is isomorphic to Y and whose general fiber is smooth)?

To put Question (1) in context, the deformations of a Calabi–Yau manifold Y are unobstructed by the Bogomolov-Tian-Todorov theorem. This result was generalized to the case where Y is allowed to have ordinary double points by Kawamata, Ran and Tian [Kaw92], [Ran92], [Tia92]. In [FL22a, FL22c], this was further extended to a much wider class of singularities, 1*-Du Bois singularities* (possibly non-isolated starting in dimension 4). Turning to Question (2), a natural class of singularities to consider is isolated Gorenstein canonical (or equivalently rational) singularities. If the singularities are also local complete intersections, there are no local obstructions to smoothability. For isolated hypersurface singularities, there is a natural local condition on first-order deformations (i.e., deformations over $\operatorname{Spec} \mathbb{C}[\varepsilon]$), which we call a *strong first-order smoothing* (Definition 1.4). This condition guarantees that any deformation of Y over Δ whose associated first-order deformation over $\operatorname{Spec} \mathbb{C}[\varepsilon]$ is a strong first-order smoothing is a smoothing of Y. If there is a first-order deformation of Y that is a strong first-order smoothing at every singular point, and if, in addition, the deformation space of Y is unobstructed, then Question (2) has a positive answer.

Already in dimension 3, there is a somewhat paradoxical aspect to Question (2): the 'more rational' the singularities of Y, the harder it is to decide if Y is smoothable to first order. In dimension 3, this corresponds to the fact that there is a certain (linear) topological constraint in order for the ordinary double points of Y to be smoothable [Fri86], whereas no such constraint exists for more complicated rational hypersurface singularities [NS95], [FL22a]. In higher dimensions, this phenomenon is even more striking: if Y has rational hypersurface singularities that are 'not too rational' (not 1-Du Bois), then Y is smoothable at least to first order [FL22a], but these methods do not apply if the singularities are 'too rational' (1-rational).

A framework for understanding these results is the theory of *higher Du Bois* and *higher rational singularities*. Mustaţă, Popa and Saito along with their collaborators and the authors have introduced the notion of k*-Du Bois* and k*-rational* singularities for a complex algebraic variety X (for $0 \le k \le \dim X$), extending the usual notions of Du Bois and rational singularities, respectively (which correspond to the case $k = 0$) [MOPW23], [JKSY22], [FL22c]. If X has local complete intersection (lci) singularities, then k-rational $\implies$ k-Du Bois $\implies$ $(k-1)$-rational [CDM22], [FL22c], [FL24]. Thus, as k increases, the singularities become milder: A local complete intersection singularity that is k-Du Bois with $k > \frac{1}{2}(\dim X - 1)$ is smooth, and it is an ordinary double point if $k = \frac{1}{2}(\dim X - 1)$. Varieties with k-rational and k-Du Bois singularities satisfy various vanishing and non-vanishing results (e.g., [Ste85], [Ste97], [MP20], [FL24]), which, in turn, are closely related to the deformation theory of Calabi–Yau varieties in case $k = 1$ [FL22a]. In particular, the deformation theory of Y is especially well-behaved when the singularities are 1-Du Bois but not 1-rational. In this case, Question (1) has a positive answer and, for Question (2), there is a necessary and sufficient condition for the existence of a strong first-order smoothing in case the singular points of Y are isolated hypersurface singularities.

As noted above, the methods of [FL22a] unfortunately say nothing about the answer to Question (2) if the singularities are 1-rational. On the positive side, for odd-dimensional Calabi–Yau varieties Y (of dimension at least 5) with only ordinary double points, Rollenske and Thomas found a *nonlinear*

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obstruction to the existence of a first-order smoothing of Y [RT09], which we state more precisely below (Theorem 1.3). To find an appropriate generalization of this result, we make the following definition [FL24, Definition 6.10], Definition 2.4: An isolated hypersurface singularity is *k-liminal* if it is k-Du Bois, but not k-rational. In dimension 3, the only 1-liminal singularities are ordinary double points. More generally in odd dimension $2k+1$, the only k-liminal singularities are ordinary double points. However, ordinary double points in even dimensions are not k-liminal for any value of k. By Lemma 2.7 below, for every $n \ge 3$, there exist k-liminal singularities of dimension $n \iff 0 \le k \le \left[\frac{n-1}{2}\right]$. In particular, for every $n \ge 3$, there exist k-liminal singularities of dimension n for some $k \ge 1$. Thus, k-liminal singularities are important boundary/transition cases and are a far-reaching generalization of ordinary double points in odd dimensions.

Since the ordinary double points are exactly the k-liminal lci singularities in dimension $2k+1$, the Rollenske–Thomas theorem can then be rephrased as follows: If Y is a Calabi–Yau variety of dimension $n = 2k+1$ with only k-liminal lci singularities, there is a topological obstruction to the existence of a strong first-order smoothing of Y (i.e., a necessary condition for the existence of a strong first-order smoothing) that is (roughly) k-linear. (In dimension 3, the obstruction is a linear condition, and it is also sufficient [Fri86].) The main result of this paper is a generalization of the Rollenske–Thomas theorem to the case where Y is a Calabi–Yau variety with isolated hypersurface weighted homogeneous k-liminal singularities.

To explain our results in more detail, we begin with the following definition:

Definition 1.1. A *canonical Calabi–Yau variety* Y is a compact analytic variety Y with at worst canonical Gorenstein (or equivalently rational Gorenstein) singularities, such that $\omega_Y \cong \mathcal{O}_Y$, and such that either Y is a scheme or Y has only isolated singularities and the $\partial\bar{\partial}$-lemma holds for some resolution of Y.

For a compact analytic variety Y with at worst ordinary double point singularities, recall that a *first-order deformation of* Y is a flat proper morphism $f\colon \mathcal{Y} \to \operatorname{Spec} \mathbb{C}[\epsilon]$, together with an isomorphism from the fiber over 0 to Y, and these are classified by $\mathbb{T}^1_Y = \operatorname{Ext}^1(\Omega^1_Y, \mathcal{O}_Y)$. Given a class $\theta \in \mathbb{T}^1_Y$, its image in $H^0(Y; T^1_Y) = \bigoplus_{x \in Y_{\text{sing}}} T^1_{Y,x}$ measures the first-order change to the singularities of Y, and θ is a *first-order smoothing of* Y if the image of θ in $T^1_{Y,x} \cong \mathbb{C}$ is nonzero for every $x \in Y_{\text{sing}}$. Then by [Fri86, §4] (also [Fri91, Prop. 8.7]), we have the following:

Theorem 1.2. *Suppose that Y is a canonical Calabi–Yau variety of dimension 3 whose only singularities are ordinary double points. Let $\pi\colon Y' \to Y$ be a small resolution of the singularities of Y, so that $\pi^{-1}(x) = C_x \cong \mathbb{P}^1$ for every $x \in Y_{\text{sing}}$, and let $[C_x]$ be the fundamental class of C_x in $H^2(Y'; \Omega^2_{Y'})$. Then a first-order smoothing of Y exists $\iff$ there exist $a_x \in \mathbb{C}$, $a_x \neq 0$ for every x, such that $\sum_{x \in Y_{\text{sing}}} a_x[C_x] = 0$ in $H^2(Y'; \Omega^2_{Y'})$.*

Next, we describe the partial extension of Theorem 1.2 to all odd dimensions $n = 2k+1 \ge 3$ due to Rollenske–Thomas. For $n > 3$, there is no small resolution of an ordinary double point. Instead, consider the standard blowup of a node. The exceptional divisor is an even dimensional quadric, whose primitive cohomology is generated by the difference $[A] - [B]$, where A and B are two complementary linear spaces of dimension k such that $A \cdot B = 1$. For Y a projective variety of dimension $2k+1$ whose only singular points are nodes and $\pi : \widehat{Y} \to Y$ a standard resolution as above, for each $x \in Y_{\text{sing}}$, there is thus a class $[A_x] - [B_x] \in H^{k+1}(\widehat{Y}; \Omega^{k+1}_{\widehat{Y}})$. The following is equivalent to the necessity part of Theorem 1.2 in dimension 3 and generalizes it to all odd dimensional nodal canonical Calabi–Yau varieties [RT09]:

Theorem 1.3. *Suppose that Y is a canonical Calabi–Yau variety of odd dimension $n = 2k+1$ whose only singularities are ordinary double points, and let $\widehat{Y} \to Y$ be a standard resolution as above. Then there exist identifications $T^1_{Y,x} \cong \mathbb{C}$ such that the following holds: If θ is a first-order smoothing of*

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Y with image in $T^1_{Y,x}$ equal to $\lambda_x \in \mathbb{C}$ via the above isomorphisms $T^1_{Y,x} \cong \mathbb{C}$, then, with notation as above,

$$\sum_{x\in Y_{\mathrm{sing}}} \lambda_x^k([A_x]-[B_x]) = 0 \tag{1.1}$$

in $H^{k+1}(\widehat{Y};\Omega^{k+1}_{\widehat{Y}})$.

We can interpret Theorem 1.3 in the following way. First, if there exists a first-order smoothing of Y, then the classes $[A_x]-[B_x]$ are not linearly independent in $H^{k+1}(\widehat{Y};\Omega^{k+1}_{\widehat{Y}})$, and in fact satisfy a linear relation whose coefficients are all nonzero. Second, the image of $\mathbb{T}^1_Y$ in $H^0(Y;T^1_Y)$, which is a vector subspace of $H^0(Y;T^1_Y)$, is contained in the subvariety of $H^0(Y;T^1_Y)$ defined by the nonlinear Equation 1.1, which is roughly speaking an intersection of affine varieties of Fermat type.

The goal of this paper is to generalize Theorem 1.3. To state the result, let Y be as before a compact analytic variety with isolated singularities. If $x \in Y_{\mathrm{sing}}$ is a singular point, let $\pi\colon \widehat{Y} \to Y$ be some log resolution of Y, and let $E_x = \pi^{-1}(x)$ be the exceptional divisor over x. In the case of ordinary double points, $\dim T^1_{Y,x} = 1$ for a singular point and there are distinguished classes $[A_x]-[B_x] \in H^{k+1}(\widehat{Y};\Omega^{k+1}_{\widehat{Y}})$ that are defined locally around the singular points. In general, $\dim T^1_{Y,x} \neq 1$, so we must define the types of smoothings to which our methods will apply:

Definition 1.4. Let (X,x) be the germ of an isolated hypersurface singularity, so that $T^1_{X,x} = \mathcal{O}_{X,x}/J$ is a cyclic $\mathcal{O}_{X,x}$-module. Thus, $\dim T^1_{X,x}/\mathfrak{m}_xT^1_{X,x} = 1$. Then an element $\theta_x \in T^1_{X,x}$ is a *strong first-order smoothing* if $\theta_x \notin \mathfrak{m}_xT^1_{X,x}$. In case x is an ordinary double point, $\theta_x \in T^1_{X,x}$ is a strong first-order smoothing $\iff$ $\theta_x \neq 0$. For a compact Y with only isolated hypersurface singularities, a first-order deformation $\theta \in \mathbb{T}^1_Y$ is a *strong first-order smoothing* if the image θ_x of θ in $T^1_{X,x}$ is a strong first-order smoothing for every $x \in Y_{\mathrm{sing}}$. A standard argument (e.g., [FL22a, Lemma 1.9]) shows that if $f\colon \mathcal{Y} \to \Delta$ is a deformation of Y over the disk, then its Kodaira-Spencer class θ is a strong first-order smoothing $\iff$ $\mathcal{Y}$ is smooth, and in particular, the nearby fibers $Y_t = f^{-1}(t)$, $0 < |t| \ll 1$, are smooth.

Remark 1.5. For $k \geq 1$, a k-liminal singularity is in particular 1-Du Bois. Hence, by [FL22c, Corollary 1.5], a canonical Calabi–Yau variety Y with only isolated k-liminal hypersurface singularities has unobstructed deformations. In particular, if there exists a strong first-order smoothing of Y, then Y is smoothable.

To deal with the correct generalization of the class $[A_x]-[B_x]$, recall that for each $x \in Y_{\mathrm{sing}}$ (assumed throughout to be an isolated hypersurface singularity), we have the corresponding link L_x at x. There is a natural mixed Hodge structure on $H^\bullet(L)$ (see, for example, [PS08, §6.2]). Moreover, for all k, there is a natural map

$$\varphi\colon\ \mathrm{Gr}^{n-k}_F H^n(L_x) \to H^{k+1}(\widehat{Y};\Omega^{n-k}_{\widehat{Y}}) \tag{1.2}$$

given as the composition

$$\begin{aligned}\mathrm{Gr}^{n-k}_F H^n(L_x) = H^k(E_x;\Omega^{n-k}_{\widehat{Y}}(\log E_x)|E_x) \to \mathrm{Gr}^{n-k}_F H^{n+1}_E(\widehat{Y}) = H^k(E_x;\Omega^{n-k}_{\widehat{Y}}(\log E_x)/\Omega^{n-k}_{\widehat{Y}})\\ \xrightarrow{\partial} H^{k+1}(\widehat{Y};\Omega^{n-k}_{\widehat{Y}}).\end{aligned}$$

In case there is a Hodge decomposition for $\widehat{Y}$ (for example, if $\widehat{Y}$ is Kähler or more generally satisfies the $\partial\bar{\partial}$-lemma), the above maps are consistent in the obvious sense with the topological maps

$$H^n(L_x) \to H^{n+1}_{E_x}(\widehat{Y}) \to H^{n+1}(\widehat{Y}),$$

where via Poincaré duality, the map $H^n(L_x) \to H^{n+1}(\widehat{Y})$ is the same as the natural map $H_{n-1}(L_x) \to H_{n-1}(\widehat{Y})$. In the special case where x is an ordinary double point and $n = 2k+1$, $\dim H^n(L_x) = 1$, so that

$H^n(L_x) = \mathbb{C}\varepsilon_x$ for some $\varepsilon_x \in H^n(L_x)$, and, for an appropriate choice of ε_x, $\varphi(\varepsilon_x) = [A_x] - [B_x] \in H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}) = H^{k+1}(\widehat{Y}; \Omega^{k+1}_{\widehat{Y}})$.

The link of a k-liminal singularity is formally analogous to that of an ordinary double point in odd dimensions, by the following result, essentially due to Dimca-Saito [DS12, §4.11] (cf. also [FL24, Corollary 6.14]):

Theorem 1.6. *If (X, x) is the germ of an isolated k-liminal hypersurface singularity and L is the corresponding link, then* $\dim \mathrm{Gr}^{n-k}_F H^n(L) = 1$.

For 1-liminal singularities, we showed [FL22a, Lemma 5.6, Corollary 5.12] that there is a necessary and sufficient linear condition for there to exist a strong first-order smoothing of Y, and hence an actual smoothing by Remark 1.5. This statement (see Theorem 2.11 below for a precise version) can be viewed as a natural generalization of Theorem 1.2. The main results of this paper, Theorem 4.4 and Corollary 4.5, are then further generalizations that apply to all weighted homogeneous k-liminal singularities. However, as in Theorem 1.3, we are only able to obtain *necessary* conditions for $k \geq 2$:

Theorem 1.7. *Let Y be a canonical Calabi–Yau variety of dimension n with isolated k-liminal weighted homogeneous hypersurface singularities and $k \geq 1$. For each singular point $x \in Y$, let L_x be the link at x, and write $\mathrm{Gr}^{n-k}_F H^n(L_x) = H^k(E_x; \Omega^{n-k}_{\widehat{Y}}(\log E_x)|E_x) = \mathbb{C} \cdot \varepsilon_x$ for some choice of a generator ε_x. Let $\varphi\colon \mathrm{Gr}^{n-k}_F H^n(L) = \bigoplus_{x \in Y_{\mathrm{sing}}} \mathrm{Gr}^{n-k}_F H^n(L) \to H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}})$ be the natural map.*

Finally, for each $x \in Y_{\mathrm{sing}}$, fix an identification $T^1_{Y,x}/\mathfrak{m}_x T^1_{Y,x} \cong \mathbb{C}$. Then, for each $x \in Y_{\mathrm{sing}}$, there exist $c_x \in \mathbb{C}^*$ with the following property: If $\theta \in \mathbb{T}^1_Y$ induces $\lambda_x \in \mathbb{C}$, then

$$\sum_{x \in Y_{\mathrm{sing}}} c_x \lambda_x^k \varphi(\varepsilon_x) = 0 \in H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}).$$

In particular, if a strong first-order smoothing of Y exists, then the classes $\varphi(\varepsilon_x)$ are not linearly independent.

In some sense, the proof of Theorem 1.7 follows the main outlines of [RT09]. A key aspect of our arguments is that by restricting to weighted homogeneous singularities, we can work as if there exists a log resolution with a single (smooth) exceptional divisor E as in loc. cit. More precisely, for Y with such singularities, there is the weighted blowup (i.e., an orbifold resolution of singularities $Y^\# \to Y$ whose exceptional divisors E are smooth divisors in the sense of orbifolds). There are stacks naturally associated to $Y^\#$ and E, a picture that is worked out in detail in [FL22a, §3] (whose methods we use systematically). Thus, we can proceed as if $Y^\#$ and E were smooth and use the familiar numerology of hypersurfaces in weighted projective space. It would be interesting to generalize the proof of Theorem 1.7 to the case where the singularities are not necessarily weighted homogeneous.

The outline of this paper is as follows. In §2.1, we collect some necessary preliminaries about isolated singularities. k-liminal singularities are defined in §2.2, and the stack point of view is recalled in §2.3. Section 3 deals with the geometry of k-liminal weighted homogeneous singularities and establishes the existence of a nonzero homogeneous pairing between two one-dimensional vector spaces. In §4.1, this construction is globalized to establish Theorem 1.7 (Theorem 4.4 and Corollary 4.5). There is also a brief discussion in §4.2 of the interplay between the Hodge theory of Y or of $\widehat{Y}$ and of a smoothing Y_t of Y.

2. Preliminaries

2.1. *Some general Hodge theory*

Let X be a contractible Stein neighborhood of the isolated singularity x of dimension $n \geq 3$, and let $\pi\colon \widehat{X} \to X$ be a good (log) resolution (i.e., π is a resolution of singularities, and $E = \pi^{-1}(x)$ (with its reduced structure) is a divisor with simple normal crossings). For every coherent sheaf $\mathcal{F}$ on $\widehat{X}$, $H^i(\widehat{X}; \mathcal{F}) \cong H^0(X; R^i\pi_*\mathcal{F})$. Let $U = X - \{x\} = \widehat{X} - E$. In the global setting, Y will denote a projective

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variety of dimension n with isolated singularities, $Z = Y_{\mathrm{sing}}$ the singular locus of Y, and $\pi\colon \widehat{Y} \to Y$ a good (log) resolution at each singular point. We will also use E to denote the exceptional divisor in this context (i.e., $E = \pi^{-1}(Z)$), again viewed as a reduced divisor, and $V = \widehat{Y} - E = Y - Z$. Instead of assuming that Y is projective, it is more generally enough to assume that Y has a resolution satisfying the $\partial\bar{\partial}$-lemma.

Lemma 2.1. *With Y and $\pi\colon \widehat{Y} \to Y$ as above, and for all p, q, the groups $H^q(\widehat{X}; \Omega^p_{\widehat{X}}(\log E))$, $H^q(\widehat{X}; \Omega^p_{\widehat{X}}(\log E)(-E))$, $H^q(\widehat{Y}; \Omega^p_{\widehat{Y}}(\log E))$ and $H^q(\widehat{Y}; \Omega^p_{\widehat{Y}}(\log E)(-E))$ are all independent of the choice of resolution.*

Proof. The independence of $H^q(\widehat{Y}; \Omega^p_{\widehat{Y}}(\log E))$ is a result of Deligne [Del71, 3.2.5(ii)]. The independence of $H^q(\widehat{Y}; \Omega^p_{\widehat{Y}}(\log E)(-E))$ then follows because $H^q(\widehat{Y}; \Omega^p_{\widehat{Y}}(\log E)(-E))$ is Serre dual to $H^{n-q}(\widehat{Y}; \Omega^{n-p}_{\widehat{Y}}(\log E))$. The local results for $\widehat{X}$ can then be reduced to this case (cf. [FL22a, Remark 3.15]). □

Remark 2.2. In case Y is projective, we can understand the birational invariance as follows: Let $\underline{\Omega}^\bullet_{Y,Z}$ be the relative filtered de Rham complex as defined by Du Bois [DB81]. By [DB81, Théorème 2.4], $\underline{\Omega}^\bullet_{Y,Z}$ is an invariant of Y as an object in the filtered derived category, and the corresponding Hodge spectral sequence degenerates at E_1 in case Y is projective. By [PS08, Example 7.25], $\underline{\Omega}^p_{Y,Z} \cong R\pi_*\Omega^p_{\widehat{Y}}(\log E)(-E)$. Applying the Leray spectral sequence for hypercohomology gives

$$\mathrm{Gr}^p_F H^{p+q}(Y, Z) = \mathbb{H}^q(Y; \underline{\Omega}^p_{Y,Z}) = H^q(\widehat{Y}; \Omega^p_{\widehat{Y}}(\log E)(-E)).$$

Hence, $H^q(\widehat{Y}; \Omega^p_{\widehat{Y}}(\log E)(-E)) = \mathrm{Gr}^p_F H^{p+q}(Y, Z)$ does not depend on the choice of a resolution.

Note that from the exact sequence

$$\cdots \to H^{i-1}(Z) \to H^i(Y, Z) \to H^i(Y) \to H^i(Z) \to \cdots,$$

$H^i(Y, Z) \cong H^i(Y)$ except for $i = 0, 1$ since $\dim Z = 0$. Moreover, the hypercohomology of the exact sequence

$$0 \to \Omega^\bullet_{\widehat{Y}}(\log E)(-E) \to \Omega^\bullet_{\widehat{Y}} \to \Omega^\bullet_E/\tau^\bullet_E \to 0$$

gives the Mayer–Vietoris sequence, an exact sequence of mixed Hodge structures:

$$\cdots \to H^{i-1}(E) \to H^i(Y, Z) \to H^i(\widehat{Y}) \to H^i(E) \to \cdots.$$

Finally, the duality between $\mathbb{H}^\bullet(\widehat{Y}; \Omega^\bullet_{\widehat{Y}}(\log E)(-E))$ and $\mathbb{H}^\bullet(\widehat{Y}; \Omega^\bullet_{\widehat{Y}}(\log E))$ corresponds to Poincaré duality (cf. [PS08, §5.5, B.21, B.24])

$$H^i(Y, Z) \cong H^i_c(Y - Z) \cong (H^{2n-i}(Y - Z))^\vee(-n) = (H^{2n-i}(\widehat{Y} - E))^\vee(-n).$$

Lemma 2.3. *With Y and $\pi\colon \widehat{Y} \to Y$ as above, the map*

$$\mathrm{Gr}^{n-k}_F H^{n+1}(Y) = H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}(\log E)(-E)) \to H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}})$$

is injective for all $k \geq 0$.

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Proof. We have the exact sequence

$$H^k(\widehat{Y};\Omega^{n-k}_{\widehat{Y}}) \to H^k(E;\Omega^{n-k}_E/\tau^{n-k}_E) \to H^{k+1}(\widehat{Y};\Omega^{n-k}_{\widehat{Y}}(\log E)(-E)) \to H^{k+1}(\widehat{Y};\Omega^{n-k}_{\widehat{Y}}).$$

By semipurity in the local setting [Ste83, (1.12)], the map $H^n_E(\widehat{X}) \to H^n(E)$ is an isomorphism. Since it factors by excision as $H^n_E(\widehat{X}) \cong H^n_E(\widehat{Y}) \to H^n(\widehat{Y}) \to H^n(E)$, the map $H^n(\widehat{Y}) \to H^n(E)$ is therefore surjective, and hence, by strictness of morphisms, so is the map

$$\mathrm{Gr}^{n-k}_F H^n(\widehat{Y}) = H^k(\widehat{Y};\Omega^{n-k}_{\widehat{Y}}) \to \mathrm{Gr}^{n-k}_F H^n(E) = H^k(E;\Omega^{n-k}_E/\tau^{n-k}_E).$$

Thus, the map $H^{k+1}(\widehat{Y};\Omega^{n-k}_{\widehat{Y}}(\log E)(-E)) \to H^{k+1}(\widehat{Y};\Omega^{n-k}_{\widehat{Y}})$ is injective. □

2.2. k-Du Bois, k-rational, and k-liminal singularities

The k-Du Bois and k-rational singularities, natural extensions of Du Bois and rational singularities, respectively (the case $k=0$), were recently introduced by [MOPW23], [JKSY22], [KL20], [FL22c] and [MP22]. The relevance of these classes of singularities (especially for $k=1$) to the deformation theory of singular Calabi–Yau and Fano varieties is discussed in [FL22a], which additionally singles out the k-*liminal singularities* (for $k=1$) as particularly relevant to the deformation theory of such varieties. The k-liminal singularities should be understood as the frontier case between $(k-1)$-rational and k-rational. For the convenience of the reader, we summarize the relevant facts for these classes of singularities.

Definition 2.4. Let (X,x) be the germ of an isolated local complete intersection (lci) singularity of dimension $n \ge 3$, and let $\pi\colon \widehat{X} \to X$ be a good resolution with exceptional divisor E. Then X is k-*Du Bois* if $R^i\pi_*\Omega^p_{\widehat{X}}(\log E)(-E) = 0$ for $i > 0$ and $p \le k$, and is k-*rational* if $R^i\pi_*\Omega^p_{\widehat{X}}(\log E) = 0$ for $i > 0$ and $p \le k$. By [FL22c], [MP22], if (X,x) is k-rational, then it is k-Du Bois and by [FL24], [CDM22], if (X,x) is k-Du Bois, then it is $(k-1)$-rational.

Finally, (X,x) is k-*liminal* if it is k-Du Bois but not k-rational. In this case, if X is a hypersurface singularity, then $\dim \mathrm{Gr}^{n-k}_F H^n(L) = 1$, by Theorem 1.6.

The following collects some basic facts about k-liminal singularities:

Lemma 2.5. *Let X be the germ of an isolated hypersurface singularity.*

(i) *If* $\dim X = 3$ *and X is not smooth, then X is not* 1*-rational, and X is* 1*-liminal* $\iff$ *X is* 1*-Du Bois* $\iff$ *X is an ordinary double point.*

(ii) *More generally, if X is a k-Du Bois singularity and* $k > \frac{1}{2}(n-1)$*, then X is smooth. If* $\dim X = 2k+1$ *and X is not smooth, then X is k-Du Bois* $\iff$ *X is k-liminal* $\iff$ *X is an ordinary double point.*

(iii) *Suppose that X is weighted homogeneous. Viewing X as locally analytically isomorphic to the subvariety* $\{f=0\}$ *of* $(\mathbb{C}^{n+1},0)$*, where* $\mathbb{C}^*$ *acts on* $\mathbb{C}^{n+1}$ *with weights* $a_1,\dots,a_{n+1} \ge 1$*, and f is weighted homogeneous of degree d, define* $w_i = a_i/d$*. Then*

(a) *X is k-Du Bois* $\iff \sum_{i=1}^{n+1} w_i \ge k+1$.

(b) *X is k-rational* $\iff \sum_{i=1}^{n+1} w_i > k+1$.

(c) *X is k-liminal* $\iff \sum_{i=1}^{n+1} w_i = k+1$.

Proof. (i) This is a result of Namikawa-Steenbrink [NS95, Theorem 2.2] (cf. also [FL24, Corollary 6.12]).

(ii) This is [DM23, Corollary 6.3] (cf. also [FL24, Corollary 4.4]).

(iii) This is a result of Saito [Sai16, (2.5.1)] (see also [FL24, Corollary 6.8]). □

Remark 2.6. (i) By definition, a 0-liminal singularity is 0-Du Bois (i.e. Du Bois in the terminology of [Ste83]), but not rational. Thus, these singularities fall outside the scope of this paper. If X is an isolated normal Gorenstein surface singularity that is Du Bois but not rational, then by [Ste83, 3.8], X is either a simple elliptic or a cusp singularity. Such singularities are known to be deeply connected to

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degenerations of $K3$ surfaces. In [FL23], we explore the analogous picture for Calabi–Yau varieties in higher dimensions in case Y has hypersurface singularities.

(ii) Assume that X is a weighted homogeneous hypersurface singularity. If X is the cone over a smooth hypersurface E of degree d in $\mathbb{P}^n$, then, by Lemma 2.5(iii), the k-liminal condition is $n+1 = d(k+1)$, and in particular, $n+1$ is divisible by d and by $k+1$. Thus, these examples are somewhat sparse. By Theorem 1.6, the Hodge structure on $H^{n-1}(E)$ is (up to a Tate twist) of Calabi–Yau type. Primarily for this reason, such hypersurfaces are exceptions to Donagi's proof for generic Torelli ([Don83]; cf. Voisin [Voi22] for recent work along these lines).

Despite Remark 2.6(ii) above, there are many examples of isolated weighted homogeneous k-liminal singularities:

Lemma 2.7. *For all k with $1 \le k \le \left[\dfrac{n-1}{2}\right]$, there exists an isolated weighted homogeneous k-liminal singularity given by a diagonal hypersurface $f(z) = z_1^{e_1} + \cdots + z_{n+1}^{e_{n+1}}$.*

Proof. Given k such that $1 \le k \le \left[\dfrac{n-1}{2}\right]$, let $f(z) = z_1^{e_1} + \cdots + z_{n+1}^{e_{n+1}}$. First suppose that $n = 2a+1$ is odd, so $\left[\dfrac{n-1}{2}\right] = a$. Then choose 2ℓ of the e_i equal to 2 and the remaining $n+1-2\ell = 2(a+1-\ell)$ equal to $\dfrac{n+1-2\ell}{2} = a+1-\ell$. Here, $0 \le \ell \le a-1$ because the value $\ell = a$ would give some $e_i = 1$. Then

$$\sum_i w_i = \sum_i \frac{1}{e_i} = \frac{1}{2}(2\ell) + (n+1-2\ell)\left(\frac{2}{n+1-2\ell}\right) = \ell + 2,$$

and hence, $k = \sum_i w_i - 1 = \ell + 1$ can take on all possible values from 1 to a.

Similarly, if $n = 2a$ is even, so that $\left[\dfrac{n-1}{2}\right] = a-1$, and $1 \le \ell \le a-2$, choose $2\ell - 1$ of the e_i to be 2, 2 of the e_i to be 4, and the remaining $n+1-(2\ell+1) = 2a-2\ell$ to be $\dfrac{n+1-(2\ell+1)}{2} = a-\ell$. Then

$$\sum_i w_i = \sum_i \frac{1}{e_i} = \frac{1}{2}(2\ell-1) + \frac{1}{2} + (n+1-(2\ell+1))\left(\frac{2}{n+1-(2\ell+1)}\right) = \ell + 2,$$

and hence, $k = \sum_i w_i - 1 = \ell + 1$ can take on all possible values from 2 to $a-1$. For the remaining possibility $k = 1$, take $n-1 = 2a-1$ of the e_i equal to a and the remaining two equal to $2a$ to get $\sum_i w_i = 2$ and hence, $k = 1$. □

The following then generalizes [RT09, 2.6]:

Lemma 2.8. *If the singularities of X are isolated 1-Du Bois lci singularities, then $H^0(X; T^1_X) \cong H^1(\widehat{X}; \Omega^{n-1}_{\widehat{X}}(\log E))$. In the global case, $\mathbb{T}^1_Y \cong H^1(\widehat{Y}; \Omega^{n-1}_{\widehat{Y}}(\log E))$, compatibly with the map $\mathbb{T}^1_Y \to H^0(Y; T^1_Y)$ and restriction – that is, the following diagram commutes:*

$$\begin{array}{ccc} H^1(\widehat{Y}; \Omega^{n-1}_{\widehat{Y}}(\log E)) & \longrightarrow & H^0(Y; R^1\pi_*\Omega^{n-1}_{\widehat{Y}}(\log E)) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{T}^1_Y & \longrightarrow & H^0(Y; T^1_Y). \end{array}$$

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Proof. First, by a result of Schlessinger (see, for example, [FL22a, Lemma 1.16]), $H^0(X;T^1_X) \cong H^1(U;T^0_X|U)$. Clearly, $H^1(U;T^0_X|U) = H^1(U;\Omega^{n-1}_{\widehat{X}}(\log E)|U)$. The local cohomology sequence gives

$$H^1_E(\widehat{X};\Omega^{n-1}_{\widehat{X}}(\log E)) \to H^1(\widehat{X};\Omega^{n-1}_{\widehat{X}}(\log E)) \to H^0(X;T^1_X) \to H^2_E(\widehat{X};\Omega^{n-1}_{\widehat{X}}(\log E)).$$

Since 1-Du Bois lci singularities are rational, $H^1_E(\widehat{X};\Omega^{n-1}_{\widehat{X}}(\log E)) = 0$ by [FL22a, 1.8], and the 1-Du Bois assumption implies that $H^2_E(\widehat{X};\Omega^{n-1}_{\widehat{X}}(\log E)) = 0$ (cf. [FL22a, 2.8]). Hence, $H^0(X;T^1_X) \cong H^1(\widehat{X};\Omega^{n-1}_{\widehat{X}}(\log E))$. The global case is similar, using $\mathbb{T}^1_Y \cong H^1(V;\Omega^{n-1}_{\widehat{Y}}(\log E)|V)$, and the compatibility is clear. □

There is a similar result for 1-rational singularities:

Lemma 2.9. *If the singularities of X are isolated 1-rational lci singularities, then $H^0(X;T^1_X) \cong H^1(\widehat{X};\Omega^{n-1}_{\widehat{X}}(\log E)(-E))$. Globally, $\mathbb{T}^1_Y \cong H^1(\widehat{Y};\Omega^{n-1}_{\widehat{Y}}(\log E)(-E))$, and there is a commutative diagram*

$$\begin{array}{ccc} H^1(\widehat{Y};\Omega^{n-1}_{\widehat{Y}}(\log E)(-E)) & \longrightarrow & H^0(Y;R^1\pi_*\Omega^{n-1}_{\widehat{Y}}(\log E)(-E)) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{T}^1_Y & \longrightarrow & H^0(Y;T^1_Y). \end{array}$$

Proof. Since isolated 1-rational singularities are 1-Du Bois, it suffices by Lemma 2.8 to show that the map $H^1(\widehat{X};\Omega^{n-1}_{\widehat{X}}(\log E)(-E)) \to H^1(\widehat{X};\Omega^{n-1}_{\widehat{X}}(\log E))$ is an isomorphism. We have the long exact sequence

$$H^0(E;\Omega^{n-1}_{\widehat{X}}(\log E)|E) \to H^1(\widehat{X};\Omega^{n-1}_{\widehat{X}}(\log E)(-E)) \to H^1(\widehat{X};\Omega^{n-1}_{\widehat{X}}(\log E)) \\ \to H^1(E;\Omega^{n-1}_{\widehat{X}}(\log E)|E).$$

Moreover, $H^1(E;\Omega^{n-1}_{\widehat{X}}(\log E)|E) = \mathrm{Gr}^{n-1}_F H^n(L)$, which has dimension $\ell^{n-1,1} = \ell^{1,n-2} = 0$ by the 1-rational condition [FL24, Theorem 5.3(iv)]. Likewise, $\dim H^0(E;\Omega^{n-1}_{\widehat{X}}(\log E)|E) = \ell^{n-1,0}$. Since X is a rational singularity, $\ell^{n-1,0} = 0$ by a result of Steenbrink [Ste97, Lemma 2]. Hence,

$$H^1(\widehat{X};\Omega^{n-1}_{\widehat{X}}(\log E)(-E)) \cong H^1(\widehat{X};\Omega^{n-1}_{\widehat{X}}(\log E)).$$

The global case and the compatibility are again clear. □

Remark 2.10. In the global case, where we do not make the assumption that $\omega_Y \cong \mathcal{O}_Y$, the above lemmas remain true provided that we replace $H^1(\widehat{Y};\Omega^{n-1}_{\widehat{Y}}(\log E))$ resp. $H^1(\widehat{Y};\Omega^{n-1}_{\widehat{Y}}(\log E)(-E))$ by $H^1(\widehat{Y};\Omega^{n-1}_{\widehat{Y}}(\log E)\otimes\pi^*\omega_Y^{-1})$ resp. $H^1(\widehat{Y};\Omega^{n-1}_{\widehat{Y}}(\log E)(-E)\otimes\pi^*\omega_Y^{-1})$.

To illustrate how these results may be used in practice, we give a quick proof of a slight variant of [FL22a, Corollary 5.8]:

Theorem 2.11. *Suppose that Y is a canonical Calabi–Yau variety of dimension $n \geq 3$ with isolated 1-liminal hypersurface singularities. Then a strong first-order smoothing of Y exists $\iff$ for every $x \in Z$, there exists $a_x \in \mathbb{C}$, $a_x \neq 0$, such that $\sum_x a_x\varphi(\varepsilon_x) = 0$ in $H^2(\widehat{Y};\Omega^{n-1}_{\widehat{Y}})$, where $\varepsilon_x \in \mathrm{Gr}^{n-1}_F H^n(L_x)$ is a generator and φ is the composition*

$$H^1(E;\Omega^{n-1}_{\widehat{Y}}(\log E)|E) \xrightarrow{\partial} H^2(\widehat{Y};\Omega^{n-1}_{\widehat{Y}}(\log E)(-E)) \to H^2(\widehat{Y};\Omega^{n-1}_{\widehat{Y}}).$$

In particular, if Y satisfies the above condition, it is smoothable.

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Proof. By Lemma 2.8, there are isomorphisms

$$H^0(X; T^1_X) \cong H^1(U; T^0_X|U) \cong H^1(U; \Omega^{n-1}_{\widehat{X}}(\log E)|U).$$

Following the isomorphism $H^0(X; T^1_X) \cong H^1(U; \Omega^{n-1}_{\widehat{X}}(\log E)|U)$ with the restriction map

$$H^1(U; \Omega^{n-1}_{\widehat{X}}(\log E)|U) \to H^1(E; \Omega^{n-1}_{\widehat{Y}}(\log E)|E)$$

gives a homomorphism $H^0(Y; T^1_Y) \to H^1(E; \Omega^{n-1}_{\widehat{Y}}(\log E)|E)$, such that the following diagram is commutative:

$$\begin{array}{ccccc} \mathbb{T}^1_Y & \longrightarrow & H^0(Y; T^1_Y) & & \\ \cong\big\downarrow & & \big\downarrow & & \\ H^1(\widehat{Y}; \Omega^{n-1}_{\widehat{Y}}(\log E)) & \longrightarrow & H^1(E; \Omega^{n-1}_{\widehat{Y}}(\log E)|E) & \xrightarrow{\partial} & H^2(\widehat{Y}; \Omega^{n-1}_{\widehat{Y}}(\log E)(-E)). \end{array}$$

Here, if as usual $E_x = \pi^{-1}(x)$, $H^1(E_x; \Omega^{n-1}_{\widehat{Y}}(\log E)|E_x)$ has dimension one for every $x \in Z$ by the 1-liminal assumption. Let ε_x be a basis vector. By [FL22a, Lemma 2.6, Theorem 2.1(v)], the map $T^1_{Y,x} \to H^1(E_x; \Omega^{n-1}_{\widehat{Y}}(\log E)|E_x)$ is surjective, and its kernel is $\mathfrak{m}_x T^1_{Y,x}$. Thus, Y has a strong first-order smoothing $\iff$ for every $x \in Z$, there exists $a_x \in \mathbb{C}$, $a_x \neq 0$, such that $\sum_{x\in Z} a_x \partial(\varepsilon_x) = 0$ in $H^2(\widehat{Y}; \Omega^{n-1}_{\widehat{Y}}(\log E)(-E))$. By Lemma 2.3, the map $H^2(\widehat{Y}; \Omega^{n-1}_{\widehat{Y}}(\log E)(-E)) \to H^2(\widehat{Y}; \Omega^{n-1}_{\widehat{Y}})$ is injective. It follows that $\sum_x a_x \partial(\varepsilon_x) = 0$ in $H^2(\widehat{Y}; \Omega^{n-1}_{\widehat{Y}}(\log E)(-E)) \iff \sum_x a_x \varphi(\varepsilon_x) = 0$ in $H^2(\widehat{Y}; \Omega^{n-1}_{\widehat{Y}})$. Thus, a strong first-order smoothing exists $\iff \sum_x a_x \varphi(\varepsilon_x) = 0$. The final statement then follows from [FL22c, Corollary 1.5]. □

Remark 2.12. There is a similar result in the 1-liminal Fano case: Assume that Y has only isolated 1-liminal hypersurface singularities and that ω_Y^{-1} is ample. In this case, the above construction produces an obstruction to a strong first-order smoothing – namely, $\sum_{x\in Z} c_x \lambda_x^k \partial(\varepsilon_x) \in H^2(\widehat{Y}; \Omega^{n-1}_{\widehat{Y}}(\log E)(-E) \otimes \pi^*\omega_Y^{-1})$. The group $H^2(\widehat{Y}; \Omega^{n-1}_{\widehat{Y}}(\log E)(-E) \otimes \pi^*\omega_Y^{-1})$ is Serre dual to $H^{n-2}(\widehat{Y}; \Omega^1_{\widehat{Y}}(\log E) \otimes \pi^*\omega_Y)$. In many cases, $H^{n-2}(\widehat{Y}; \Omega^1_{\widehat{Y}}(\log E)) \otimes \pi^*\omega_Y) = 0$. For example, if there exists a smooth Cartier divisor H on Y, thus not passing through the singular points of Y, such that $\omega_Y = \mathcal{O}_Y(-H)$, and in addition $H^{n-3}(H; \Omega^1_H) = 0$, then an argument with the Goresky-MacPherson-Lefschetz theorem in intersection cohomology [GM83] shows that

$$H^{n-2}(\widehat{Y}; \Omega^1_{\widehat{Y}}(\log E) \otimes \pi^*\omega_Y) = H^{n-2}(\widehat{Y}; \Omega^1_{\widehat{Y}}(\log E) \otimes \mathcal{O}_{\widehat{Y}}(-H)) = 0,$$

where we identify the divisor H on Y with its preimage π^*H on $\widehat{Y}$. The proof of Theorem 2.11 then shows that, under these assumptions, a strong first-order smoothing of Y always exists, and hence, Y is smoothable by [FL22a, Theorem 4.5]. A somewhat stronger statement is proved in [FL22a, Corollary 4.10].

Remark 2.13. In dimension three, a singular point can be k-liminal only for $k = 1$. Since this case is covered by Theorem 2.11, we are free to make the assumption that $n \geq 4$, as needed in what follows.

2.3. Weighted homogeneous singularities and quotient stacks

For the remainder of this section, we are concerned with generalizing the above picture, and in particular, Lemma 2.9, in the context of stacks: Assume that the isolated singularity X is locally analytically isomorphic to a weighted cone in $\mathbb{C}^{n+1}$ over a weighted hypersurface $E \subseteq WP^n$. Thus, we may as well assume that X is the weighted cone as in Lemma 2.5(iii), with an isolated singularity at 0.

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Definition 2.14. Let X be the weighted cone in $\mathbb{C}^{n+1}$ over a weighted hypersurface $E \subseteq WP^n$, where WP^n is a weighted projective space, and $X^\#$ the weighted blowup of X as in [FL22a, §3]. Let $\underline{E}$, $\underline{X}^\#$, $\underline{WP^n}$ be the corresponding quotient stacks. If X has an isolated singularity at 0, then $\underline{X}^\#$ and $\underline{E}$ are quotient stacks for an action of $\mathbb{C}^*$ on smooth schemes with finite stabilizers. Hence, $\underline{X}^\#$ is a smooth stack, $\underline{E}$ is a smooth divisor in $\underline{X}^\#$, and there is a morphism $\underline{X}^\# \to X$ that defines an isomorphism $\underline{X}^\# - \underline{E} \to X - \{0\}$.

Globally, let Y be a projective variety of dimension n with isolated weighted homogeneous hypersurface singularities. Let $\pi\colon Y^\# \to Y$ denote the weighted blowup of Y at the singularities, and let E be the exceptional divisor (i.e., $E = \pi^{-1}(Z)$ where $Z = Y_{\mathrm{sing}}$). We can also construct a stacky version of $Y^\#$ as follows: For each $x \in Z$, we have the corresponding exceptional divisor E_x. Let X denote the corresponding weighted cone in $\mathbb{C}^{n+1}$. There is a (Zariski) open neighborhood $U \subseteq Y$ of x and an étale morphism $U \to X$. We can then pull back the stack $\underline{X}^\#$ to a stack $\underline{U}^\#$ and glue $\underline{U}^\#$ and $Y - \{x\}$ along the Zariski open subset $U - \{x\}$. Doing this for each singular point defines the stack $\underline{Y}^\#$.

A similar construction works in the analytic category, where we view an analytic stack as a functor on the category of complex analytic spaces. This allows for the possibility that, in Definition 1.1, Y is a compact analytic, not necessarily algebraic space.

As in Definition 2.14, let $X^\#$ be the weighted blowup of X, with $\underline{X}^\#$ the associated stack, and let $\widehat{X}$ be an arbitrary log resolution. Given a projective Y with isolated weighted homogeneous hypersurface singularities, we define $\underline{Y}^\#$ as before and let $\pi\colon \widehat{Y} \to Y$ be a log resolution. To avoid confusion, we denote the exceptional divisor of $\pi\colon \widehat{X} \to X$ or $\pi\colon \widehat{Y} \to Y$ by $\widehat{E}$. We claim that, in the statement of Lemmas 2.8 and 2.9, we can replace ordinary cohomology with stack cohomology. First, we recall the following definition, due to Steenbrink [Ste77b, §1], [Ste77a, §2]:

Definition 2.15. Let W be an analytic space that is an orbifold 'viewed as an analytic space' (i.e., locally $W = \widetilde{W}/G$, where G is a small subgroup of $GL(n,\mathbb{C})$ in the sense of [Ste77a] and $\widetilde{W}$ is a G-invariant neighborhood of the origin on $\mathbb{C}^n$). Let W_0 be the open subset where W is (locally) a free quotient so that, by hypothesis, $W - W_0$ has codimension at least 2. Define Ω^p_W to be $i_*\Omega^p_{W_0}$, where $i\colon W_0 \to W$ is the inclusion. If $\pi\colon \widehat{W} \to W$ is a resolution of singularities, then $\Omega^p_W = \pi_*\Omega^p_{\widehat{W}}$. If (locally) $W = \widetilde{W}/G$ as above, then $\Omega^p_W = (\Omega^p_{\widetilde{W}})^G$. If D is an orbifold normal crossing divisor of W in the obvious sense, then $\Omega^p_W(\log D)$ is defined similarly.

By [Ste77b, (1.9), (1.12)], the complex $(\Omega^\bullet_W, d)$ is a resolution of the constant sheaf $\mathbb{C}$ and, if W is projective, the hypercohomology spectral sequence with $E_1^{p,q} = H^q(W;\Omega^p_W) \implies \mathbb{H}^{p+q}(W;\Omega^\bullet_W) \cong H^{p+q}(W;\mathbb{C})$ degenerates at E_1. Likewise, if D is an orbifold normal crossing divisor of W, then $\mathbb{H}^k(W;\Omega^\bullet_W(\log D)) \cong H^k(W - D;\mathbb{C})$, and the analogous spectral sequence also degenerates at E_1.

There is an extension of Lemma 2.1 to this situation:

Lemma 2.16. *In the notation of Definition 2.14, for all p, q, there are isomorphisms*

$$H^q(\underline{X}^\#;\Omega^p_{\underline{X}^\#}(\log \underline{E})) \cong H^q(X^\#;\Omega^p_{X^\#}(\log E)) \cong H^q(\widehat{X};\Omega^p_{\widehat{X}}(\log \widehat{E}));$$

$$H^q(\underline{Y}^\#;\Omega^p_{\underline{Y}^\#}(\log \underline{E})) \cong H^q(Y^\#;\Omega^p_{Y^\#}(\log E)) \cong H^q(\widehat{Y};\Omega^p_{\widehat{Y}}(\log \widehat{E})),$$

where $\Omega^p_{X^\#}(\log E)$ and $\Omega^p_{Y^\#}(\log E)$ are the sheaves defined in Definition 2.15 for the spaces $X^\#$ and $Y^\#$. Likewise, with similar definitions of $\Omega^p_{X^\#}(\log E)(-E)$ and $\Omega^p_{Y^\#}(\log E)(-E)$,

$$H^q(\underline{X}^\#;\Omega^p_{\underline{X}^\#}(\log \underline{E})(-E)) \cong H^q(X^\#;\Omega^p_{X^\#}(\log E)(-E)) \cong H^q(\widehat{X};\Omega^p_{\widehat{X}}(\log \widehat{E})(-\widehat{E}));$$

$$H^q(\underline{Y}^\#;\Omega^p_{\underline{Y}^\#}(\log \underline{E})(-E)) \cong H^q(Y^\#;\Omega^p_{Y^\#}(\log E)(-E)) \cong H^q(\widehat{Y};\Omega^p_{\widehat{Y}}(\log \widehat{E})(-\widehat{E})).$$

Proof. These statements follow from the arguments of [FL22a, Lemma 3.13, Lemma 3.14] and Lemma 2.1. □

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Thus, for example, in the situation of Lemma 2.9, we have the following:

Corollary 2.17. *If all of the singularities of Y are weighted homogeneous isolated* 1*-rational singularities, then there is a commutative diagram*

$$\begin{array}{ccc} H^1(\underline{Y}^\#;\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-\underline{E})) & \longrightarrow & H^0(Y;R^1\pi_*\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-\underline{E})) \\ \cong\downarrow & & \downarrow\cong \\ H^1(\widehat{Y};\Omega^{n-1}_{\widehat{Y}}(\log \widehat{E})(-\widehat{E})) & \longrightarrow & H^0(Y;R^1\pi_*\Omega^{n-1}_{\widehat{Y}}(\log \widehat{E})(-\widehat{E})) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{T}^1_Y & \longrightarrow & H^0(Y;T^1_Y). \end{array}$$

Here, $H^0(Y;R^1\pi_*\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-\underline{E}))$ *is a direct sum of terms isomorphic to the corresponding local terms* $H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-\underline{E}))$.

In the local setting, we note the following for future reference:

Lemma 2.18. *There is an exact sequence*

$$0 \to \Omega^k_{\underline{E}} \to \Omega^k_{\underline{X}^\#}(\log \underline{E})|\underline{E} \to \Omega^{k-1}_{\underline{E}} \to 0.$$

Proof. Poincaré residue induces a surjection $\Omega^1_{\widehat{X}}(\log \underline{E})|\underline{E} \to \mathcal{O}_{\underline{E}}$ whose kernel is easily checked to be $\Omega^1_{\underline{E}}$ as $\underline{E}$ is smooth. Taking the k^{th} exterior power gives the exact sequence. □

Remark 2.19. With E as in Definition 2.14, we can either think of E as a scheme or as a stack. We will denote by $H^i(E) = H^i(E;\mathbb{C})$ the usual singular cohomology. By the remarks at the end of Definition 2.15, there is a spectral sequence $E_1^{p,q} = H^q(E;\Omega^p_{\underline{E}}) \implies H^{p+q}(E;\mathbb{C})$, and it degenerates at E_1. Moreover, the corresponding filtration defines a (pure) Hodge structure on $H^i(E)$ [Ste77b]. The method of proof of [FL22a, Lemma 3.13] shows that $H^q(E;\Omega^p_{\underline{E}}) \cong H^q(\underline{E};\Omega^p_{\underline{E}})$. Thus, in particular,

$$\mathrm{Gr}^p_F H^{p+q}(E) \cong H^q(\underline{E};\Omega^p_{\underline{E}}).$$

As noted in the introduction, the cohomology of the link L of the isolated singularity X carries a mixed Hodge structure. (We will not try to give a stacky interpretation of L.) Arguments as in the case where E is smooth show that

$$\mathrm{Gr}^p_F H^{p+q}(L) \cong H^q(\underline{E};\Omega^p_{\underline{X}^\#}(\log \underline{E})|\underline{E}).$$

3. Local calculations

3.1. Numerology

In this section, we consider the local case. We keep the notation of the previous section: X is the affine weighted cone over a hypersurface E in a weighted projective space WP^n, with an isolated singularity at 0, and $X^\#$ is the weighted blowup, with $\underline{X}^\#$, $\underline{E}$ and $\underline{WP}^n$ the corresponding stacks. Let $a_1,\dots,a_{n+1}$ be the $\mathbb{C}^*$ weights, let d be the degree of E, and set $w_i = a_i/d$. Setting $N = \sum_i a_i - d$, as a line bundle on the stack $\underline{E}$,

$$K_{\underline{E}} = \mathcal{O}_{\underline{E}}(-N) = \mathcal{O}_{\underline{E}}(d - \sum_i a_i).$$

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Since $\sum_i w_i = N/d + 1$, the k-liminal condition is equivalent to

$$k = \sum_i w_i - 1 = N/d \iff N = dk.$$

Thus, $K_{\underline{E}} = \mathcal{O}_{\underline{E}}(-dk)$. As for $K_{\underline{X}^\#}$, we have $K_{\underline{X}^\#} = \mathcal{O}_{\underline{X}^\#}(r\underline{E})$ for some $r \in \mathbb{Z}$. By adjunction,

$$K_{\underline{E}} = \mathcal{O}_{\underline{E}}(-dk) = K_{\underline{X}^\#} \otimes \mathcal{O}_{\underline{X}^\#}(\underline{E})|\underline{E} = \mathcal{O}_{\underline{X}^\#}((r+1)\underline{E})|\underline{E} = \mathcal{O}_{\underline{E}}(-(r+1)\underline{E}).$$

Thus, $r + 1 = dk$, $r = dk - 1$, and

$$K_{\underline{X}^\#} = \mathcal{O}_{\underline{X}^\#}((dk-1)\underline{E}) = \mathcal{O}_{\underline{X}^\#}((N-1)\underline{E}).$$

To simplify the notation, set

$$a = d(k-1) = N - d = \sum_i a_i - 2d.$$

Thus, $a = 0 \iff k = 1$ (i.e., X is 1-liminal). Moreover,

$$K_{\underline{E}}(a) = K_{\underline{E}} \otimes \mathcal{O}_{\underline{E}}(a) = \mathcal{O}_{\underline{E}}(-d).$$

3.2. Some cohomology calculations

Assumption 3.1. From now on, we assume that X is a k-liminal weighted homogeneous isolated hypersurface singularity with $n = \dim X \geq 4$ and $k \geq 2$. In particular, X is 1-rational, so that Lemma 2.9 and Corollary 2.17 apply.

Lemma 3.2. *With notation as above, if $j \leq 2$ and $1 \leq i \leq a-1$, then*

$$H^j(\underline{E}; \Omega^{n-1}_{\underline{E}}(i)) = H^j(\underline{E}; \Omega^{n-2}_{\underline{E}}(i)) = 0.$$

For $i = a$, we have $H^j(\underline{E}; \Omega^{n-1}_{\underline{E}}(a)) = 0$ for $j \leq 2$ and $H^j(\underline{E}; \Omega^{n-2}_{\underline{E}}(a)) = 0$ for $j = 0, 2$, but $\dim H^1(\underline{E}; \Omega^{n-2}_{\underline{E}}(a)) = \dim H^1(\underline{E}; T_{\underline{E}}(-d)) = 1$.

Proof. First, $H^j(\underline{E}; \Omega^{n-1}_{\underline{E}}(i)) = H^j(\underline{E}; K_{\underline{E}}(i)) = H^j(\underline{E}; \mathcal{O}_{\underline{E}}(-kd+i))$. We have the exact sequence

$$0 \to \mathcal{O}_{\underline{WP^n}}(r) \to \mathcal{O}_{\underline{WP^n}}(r+d) \to \mathcal{O}_{\underline{E}}(r+d) \to 0.$$

Since $H^i(\underline{WP^n}; \mathcal{O}_{\underline{WP^n}}(r)) = 0$ for $i = 1, 2, 3$ and all r, $H^j(\underline{E}; K_{\underline{E}}(i)) = 0$ for $j = 1, 2$ and all i. For $j = 0$, since $0 \leq i \leq a - 1 = kd - d - 1$, $-kd + i \leq -d - 1 < 0$, and hence, $H^0(\underline{E}; K_{\underline{E}}(i)) = H^0(\underline{E}; \mathcal{O}_{\underline{E}}(-kd+i)) = 0$ in this range as well.

For $H^j(\underline{E}; \Omega^{n-2}_{\underline{E}}(i))$, note first that, as E has dimension $n - 1$,

$$\Omega^{n-2}_{\underline{E}}(i) \cong T_{\underline{E}} \otimes K_{\underline{E}}(i) = T_{\underline{E}}(-kd+i).$$

From the normal bundle sequence

$$0 \to T_{\underline{E}} \to T_{\underline{WP^n}}|\underline{E} \to \mathcal{O}_{\underline{E}}(d) \to 0,$$

we therefore obtain

$$0 \to T_{\underline{E}}(-kd+i) \to T_{\underline{WP^n}}(-kd+i)|\underline{E} \to \mathcal{O}_{\underline{E}}(-kd+d+i) \to 0.$$

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For $i \le a-1$, $-kd+d+i \le -1$. Then an argument as before shows that, for $j \le 2$,

$$H^j(\underline{E};\Omega^{n-2}_{\underline{E}}(i)) \cong H^j(\underline{E};T_{\underline{WP^n}}(-kd+i)|\underline{E}).$$

We have the Euler exact sequence

$$0 \to \mathcal{O}_{\underline{E}} \to \bigoplus_{i=1}^{n+1} \mathcal{O}_{\underline{E}}(a_i) \to T_{\underline{WP^n}}|\underline{E} \to 0.$$

Still assuming that $j \le 2$ and $i \le a-1$, it suffices to show that

$$H^{j+1}(\underline{E};\mathcal{O}_{\underline{E}}(-kd+i)) = H^j(\underline{E};\mathcal{O}_{\underline{E}}(-kd+i+a_i)) = 0$$

for $j \le 2$. This is certainly true if $n \ge 5$, again using $-kd+i+a_i \le a_i - d \le -1$ since X is not smooth, and hence, $a_i < d$. For $n = 4$, $H^3(\underline{E};\mathcal{O}_{\underline{E}}(-kd+i)) = H^3(\underline{E};K_{\underline{E}}(i))$ that is Serre dual to $H^0(\underline{E};\mathcal{O}_{\underline{E}}(-i))$, so we are done as before since $i \ge 1$.

To prove the second statement, note that $\Omega^{n-1}_{\underline{E}}(a) = K_{\underline{E}}(a) = \mathcal{O}_{\underline{E}}(-d)$ and $H^j(\underline{E};\mathcal{O}_{\underline{E}}(-d)) = 0$ for $j \le 2$ by the same reasons as before. Likewise, $\Omega^{n-2}_{\underline{E}}(a) \cong T_{\underline{E}} \otimes K_{\underline{E}}(a) = T_{\underline{E}}(-d)$. Via the Euler exact sequence

$$0 \to \mathcal{O}_{\underline{E}}(-d) \to \bigoplus_{i=1}^{n+1} \mathcal{O}_{\underline{E}}(a_i - d) \to T_{\underline{WP^n}}(-d)|\underline{E} \to 0,$$

we see that $H^j(\underline{E};T_{\underline{WP^n}}(-d)|\underline{E}) = 0$ for $j \le 2$. Moreover, the normal bundle sequence gives

$$0 \to T_{\underline{E}}(-d) \to T_{\underline{WP^n}}(-d)|\underline{E} \to \mathcal{O}_{\underline{E}} \to 0.$$

Thus, $H^0(\underline{E};T_{\underline{E}}(-d)) = H^2(\underline{E};T_{\underline{E}}(-d)) = 0$, but the coboundary map $H^0(\mathcal{O}_{\underline{E}}) \to H^1(\underline{E};T_{\underline{E}}(-d))$ is an isomorphism. □

Corollary 3.3. *Under Assumption 3.1,*

(i) $H^0(\underline{E};\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-i\underline{E})|\underline{E}) = 0$ *for* $1 \le i \le a$*;*
(ii) $H^1(\underline{E};\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-i\underline{E})|\underline{E}) = 0$ *for* $1 \le i < a$*;*
(iii) $\dim H^1(\underline{E};\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E}) = 1$.

Proof. By Lemma 2.18, there is an exact sequence

$$0 \to \Omega^{n-1}_{\underline{E}}(i) \to \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-i\underline{E})|\underline{E} \to \Omega^{n-2}_{\underline{E}}(i) \to 0.$$

By Lemma 3.2, if $1 \le i \le a$, then $H^0(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-i\underline{E})|\underline{E}) = 0$, and

$$H^1(\underline{E};\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-i\underline{E})|\underline{E}) \cong H^1(\underline{E};\Omega^{n-2}_{\underline{E}}(i)),$$

which is 0 for $1 \le i < a$ and has dimension 1 for $i = a$. □

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Theorem 3.4. *Under Assumption 3.1,*

$$H^0(X;T^1_X) \cong H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})).$$

Moreover, the natural map $H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})) \to H^1(\underline{E};\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E})$ *induces an isomorphism*

$$H^0(X;T^1_X)/\mathfrak{m}_x H^0(X;T^1_X) \cong H^1(\underline{E};\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E}).$$

Proof. For the first part, we have an exact sequence

$$0 \to \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-(i+1)\underline{E}) \to \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-i\underline{E}) \to \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-i\underline{E})|\underline{E} \to 0.$$

Thus, by Corollary 3.3, for $1 \le i < a$, we have an isomorphism

$$H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-(i+1)\underline{E}) \to H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-i\underline{E})),$$

and by induction, starting with the isomorphism $H^0(X;T^1_X) \cong H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-\underline{E}))$ of Lemma 2.9 and Corollary 2.17, we see that $H^0(X;T^1_X) \cong H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E}))$.

To see the final statement, we have an exact sequence

$$H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-(a+1)\underline{E})) \to H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})) \to H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E}),$$

and hence an injection

$$H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E}))\Big/ \operatorname{Im} H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-(a+1)\underline{E})) \to H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E}).$$

By Corollary 3.3(iii), $\dim H^1(\underline{E};\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E}) = 1$. Thus, if the map

$$H^0(X;T^1_X)/\mathfrak{m}_x H^0(X;T^1_X) \to H^1(\underline{E};\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E})$$

is nonzero, it is an isomorphism. However, to prove that this map is nonzero, it is necessary to consider the $\mathbb{C}^*$ picture as in [FL22a, §3]: The vector bundle $\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})$ on $\underline{X}^\#$ is of the form ρ^*W for some vector bundle W on $\underline{E}$, where $\rho\colon \underline{X}^\# \to \underline{E}$ is the natural morphism, and $\mathcal{O}_{\underline{X}^\#}(-\underline{E}) = \rho^*\mathcal{O}_{\underline{E}}(1)$. Then

$$H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})) = \bigoplus_{r\ge 0} H^1(\underline{E};W(r)) = \bigoplus_{r\ge -N} H^0(X;T^1_X)(r) = \bigoplus_{r\ge -d} H^0(X;T^1_X)(r).$$

Here, the final equality holds because $-d$ is the smallest weight occurring in $H^0(X;T^1_X)$ and $-N = -dk \le -d$. Note also that $\bigoplus_{r\ge -d+1} H^0(X;T^1_X)(r) = \mathfrak{m}_x H^0(X;T^1_X)$. Taking the tensor product with $\mathcal{O}_{\underline{X}^\#}(-i\underline{E})$ has the effect of shifting the weight spaces by i since

$$\rho^*W \otimes \mathcal{O}_{\underline{X}^\#}(-i\underline{E}) \cong \rho^*W \otimes \rho^*\mathcal{O}_{\underline{E}}(i) = \rho^*(W \otimes \mathcal{O}_{\underline{E}}(i)).$$

Thus,

$$H^1(\underline{X}^\#;\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-i\underline{E})) = \bigoplus_{r\ge 0} H^1(\underline{E};W(r+i)) = \bigoplus_{r\ge i} H^1(\underline{E};W(r)) = \bigoplus_{r\ge -N+i} H^0(X;T^1_X)(r).$$

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Here, $-d \geq -N + i \iff i \leq N - d = a$. This recovers the fact that $H^1(\underline{X}^\#; \Omega^{n-1}_{\widehat{\underline{X}}}(\log \underline{E})(-i\underline{E})) \cong H^0(X; T^1_X)$ for $i \leq a$, whereas

$$H^1(\underline{X}^\#; \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-(a+1)\underline{E})) = \bigoplus_{r \geq -d+1} H^0(X; T^1_X)(r) = \mathfrak{m}_x H^0(X; T^1_X),$$

as claimed. □

3.3. Definition of the nonlinear map

We now consider the analogue of [RT09, Lemma 4.10]. First, we have the subsheaf $T_{\underline{X}^\#}(-\log \underline{E}) \subseteq T_{\underline{X}^\#}$ (on the stack $\underline{X}^\#$), which is the kernel of the map $T_{\underline{X}^\#} \to N_{\underline{E}/\underline{X}^\#}$ or, equivalently, is dual to $\Omega^1_{\underline{X}^\#}(\log \underline{E})$. There is thus a commutative diagram

$$\begin{array}{ccc} T_{\underline{X}^\#}(-\log \underline{E}) & \longrightarrow & T_{\underline{X}^\#} \\ \cong\downarrow & & \downarrow\cong \\ \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-\underline{E}) \otimes K^{-1}_{\underline{X}^\#} & \longrightarrow & \Omega^{n-1}_{\underline{X}^\#} \otimes K^{-1}_{\underline{X}^\#}. \end{array}$$

There are compatible isomorphisms

$$\begin{aligned} T_{\underline{X}^\#}(d\underline{E}) &\cong \Omega^{n-1}_{\underline{X}^\#} \otimes K^{-1}_{\underline{X}^\#} \otimes \mathcal{O}_{\underline{X}^\#}(d\underline{E}) = \Omega^{n-1}_{\underline{X}^\#}((d - dk + 1)\underline{E}) = \Omega^{n-1}_{\underline{X}^\#}(-a\underline{E} + \underline{E}); \\ T_{\underline{X}^\#}(-\log \underline{E})(d\underline{E}) &\cong \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-\underline{E}) \otimes K^{-1}_{\underline{X}^\#} \otimes \mathcal{O}_{\underline{X}^\#}(d\underline{E}) \\ &= \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})((d - dk)\underline{E}) = \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E}). \end{aligned}$$

Taking k^{th} exterior powers, $\bigwedge^k T_{\underline{X}^\#}$ is dual to $\Omega^k_{\underline{X}^\#}$ and hence is isomorphic to $\Omega^{n-k}_{\underline{X}^\#} \otimes K^{-1}_{\underline{X}^\#}$, and $\bigwedge^k T_{\underline{X}^\#}(-\log \underline{E})$ is dual to $\Omega^k_{\underline{X}^\#}(\log \underline{E})$ and hence is isomorphic to $\Omega^{n-k}_{\underline{X}^\#}(\log \underline{E})(-\underline{E}) \otimes K^{-1}_{\underline{X}^\#}$. There are compatible isomorphisms

$$\bigwedge^k (T_{\underline{X}^\#}(d\underline{E})) \cong \Omega^{n-k}_{\underline{X}^\#} \otimes K^{-1}_{\underline{X}^\#} \otimes \mathcal{O}_{\underline{X}^\#}(dk\underline{E}) = \Omega^{n-k}_{\underline{X}^\#} \otimes \mathcal{O}_{\underline{X}^\#}(\underline{E});$$

$$\bigwedge^k (T_{\underline{X}^\#}(-\log \underline{E})(d\underline{E})) \cong \Omega^{n-k}_{\underline{X}^\#}(\log \underline{E})(-\underline{E}) \otimes K^{-1}_{\underline{X}^\#} \otimes \mathcal{O}_{\underline{X}^\#}(dk\underline{E}) = \Omega^{n-k}_{\underline{X}^\#}(\log \underline{E}).$$

So we have a commutative diagram

$$\begin{array}{ccc} \bigwedge^k (T_{\underline{X}^\#}(-\log \underline{E})(d\underline{E})) \cong \bigwedge^k \left(\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})\right) & \longrightarrow & \bigwedge^k \left(\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E}\right) \\ \cong\downarrow & & \downarrow\cong \\ \Omega^{n-k}_{\underline{X}^\#}(\log \underline{E}) & \longrightarrow & \Omega^{n-k}_{\underline{X}^\#}(\log \underline{E})|\underline{E}. \end{array}$$

There is also the induced map $T_{\underline{X}^\#}(-\log \underline{E}) \to T_{\underline{E}}$, and the following commutes:

$$\begin{array}{ccc} T_{\underline{X}^\#}(-\log \underline{E}) & \longrightarrow & T_{\underline{E}} \\ \cong\downarrow & & \downarrow\cong \\ \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-\underline{E}) \otimes K^{-1}_{\underline{X}^\#} & \xrightarrow{\text{Res}} & \Omega^{n-2}_{\underline{E}} \otimes K^{-1}_{\underline{E}}, \end{array}$$

using the adjunction isomorphism $\mathcal{O}_{\underline{X}^\#}(-\underline{E}) \otimes K^{-1}_{\underline{X}^\#}|\underline{E} = (K_{\underline{X}^\#} \otimes \mathcal{O}_{\underline{X}^\#}(\underline{E}))^{-1}|\underline{E} \cong K^{-1}_{\underline{E}}$.

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The exact sequence of Lemma 2.18 yields an exact sequence

$$0 \to \Omega^{n-1}_{\underline{E}}(a) \to \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E} \to \Omega^{n-2}_{\underline{E}}(a) \to 0.$$

By Lemma 3.2, there is an induced isomorphism $H^1(\underline{E}; \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E}) \to H^1(\underline{E}; \Omega^{n-2}_{\underline{E}}(a))$. Moreover,

$$\Omega^{n-2}_{\underline{E}}(a) \cong T_{\underline{E}} \otimes K_{\underline{E}}(a) = T_{\underline{E}}(-kd + a) = T_{\underline{E}}(-d).$$

Taking k^{th} exterior powers,

$$\bigwedge^k (T_{\underline{E}}(-d)) = \left(\bigwedge^k T_{\underline{E}}\right)(-kd) = \left(\bigwedge^k T_{\underline{E}}\right) \otimes K_{\underline{E}} \cong \Omega^{n-k-1}_{\underline{E}}.$$

A combination of wedge product and cup product induces symmetric homogeneous degree k maps

$$\nu_{\underline{X}^\#}\colon H^1(\underline{X}^\#; \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})) \to H^k(\underline{X}^\#; \Omega^{n-k}_{\underline{X}^\#}(\log \underline{E}));$$
$$\mu_{\underline{X}^\#} = \nu_{\underline{X}^\#}|\underline{E}\colon H^1(\underline{E}; \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E}) \to H^k(\underline{E}; \Omega^{n-k}_{\underline{X}^\#}(\log \underline{E})|\underline{E}),$$

and a commutative diagram (with nonlinear vertical maps)

$$\begin{array}{ccc} H^1(\underline{X}^\#; \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})) & \longrightarrow & H^1(\underline{E}; \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E}) \\ {\scriptstyle \nu_{\underline{X}^\#}}\downarrow & & \downarrow{\scriptstyle \mu_{\underline{X}^\#}} \\ H^k(\underline{X}^\#; \Omega^{n-k}_{\underline{X}^\#}(\log \underline{E})) & \longrightarrow & H^k(\underline{E}; \Omega^{n-k}_{\underline{X}^\#}(\log \underline{E})|\underline{E}). \end{array}$$

There are similarly compatible symmetric homogeneous degree k maps

$$\nu'_{\underline{X}^\#}\colon H^1(\underline{X}^\#; T_{\underline{X}^\#}(-\log \underline{E})(d\underline{E})) \to H^k(\underline{X}^\#; \bigwedge^k (T_{\underline{X}^\#}(-\log \underline{E})(d\underline{E}))) \cong H^k(\underline{X}^\#; \Omega^{n-k}_{\underline{X}^\#}(\log \underline{E}));$$
$$\mu'_{\underline{X}^\#}\colon H^1(\underline{E}; T_{\underline{E}}(-d)) \cong H^1(\underline{E}; \Omega^{n-2}_{\underline{E}}(a)) \to H^k(\underline{E}; \bigwedge^k (T_{\underline{E}}(-d))) \cong H^k(\underline{E}; \Omega^{n-k-1}_{\underline{E}}).$$

The following diagram with nonlinear vertical maps commutes:

$$\begin{array}{ccc} H^1(\underline{E}; \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E}) & \xrightarrow{\cong} & H^1(\underline{E}; T_{\underline{E}}(-d)) \cong H^1(\underline{E}; \Omega^{n-2}_{\underline{E}}(a)) \\ {\scriptstyle \mu_{\underline{X}^\#}}\downarrow & & \downarrow{\scriptstyle \mu'_{\underline{X}^\#}} \\ H^k(\underline{E}; \Omega^{n-k}_{\underline{X}^\#}(\log \underline{E})|\underline{E}) & \xrightarrow{\text{Res}} & H^k(\underline{E}; \wedge^k (T_{\underline{E}}(-d))) \cong H^k(\underline{E}; \Omega^{n-k-1}_{\underline{E}}). \end{array}$$

By Lemma 3.2, Corollary 3.3(iii) and Theorem 1.6,

$$\dim H^1(\underline{E}; \Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E}) = \dim H^1(\underline{E}; T_{\underline{E}}(-d)) = 1;$$
$$\dim H^k(\underline{E}; \Omega^{n-k}_{\underline{X}^\#}(\log \underline{E})|\underline{E}) = \dim \mathrm{Gr}^{n-k}_F H^n(L) = 1.$$

The map $H^k(\underline{X}^\#; \Omega^{n-k}_{\underline{X}^\#}(\log \underline{E})) \to H^k(\underline{E}; \Omega^{n-k-1}_{\underline{E}})$ factors through the (surjective) map

$$H^k(\underline{X}^\#; \Omega^{n-k}_{\underline{X}^\#}(\log \underline{E})) \to H^k(\underline{E}; \Omega^{n-k}_{\underline{E}}(\log \underline{E})|\underline{E}) = \mathrm{Gr}^{n-k}_F H^n(L),$$

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and the map

$$H^k(\underline{E};\Omega^{n-k}_{\underline{X}^\#}(\log \underline{E})|\underline{E}) = \mathrm{Gr}^{n-k}_F H^n(L) \\ \to H^k(\underline{E};\Omega^{n-k-1}_{\underline{E}}) = \mathrm{Gr}^{n-k-1}_F H^{n-1}(E) = \mathrm{Gr}^{n-k}_F H^{n-1}(E)(-1)$$

is an isomorphism in almost all cases. More precisely, let $H^{n-1}_0(E)$ be the primitive cohomology of E in dimension $n-1$, and let $H^{n-1-k}_0(\underline{E};\Omega^k_{\underline{E}}) = \mathrm{Gr}^k_F H^{n-1}_0(E)$ be the corresponding groups.

Lemma 3.5. *If X is not an ordinary double point or if n is even, then the map $\mathrm{Gr}^{n-k}_F H^n(L) \to H^k(\underline{E};\Omega^{n-k-1}_{\underline{E}})$ is an isomorphism, and hence,*

$$\dim H^k(\underline{E};\Omega^{n-k-1}_{\underline{E}}) = 1.$$

If X is an ordinary double point and $n = 2k+1$ is odd, then

$$\mathrm{Gr}^{n-k}_F H^n(L) \to H^k(\underline{E};\Omega^{n-k-1}_{\underline{E}}) = \mathrm{Gr}^{n-k-1}_F H^{n-1}(E)$$

is injective with image $H^k_0(\underline{E};\Omega^k_{\underline{E}}) = \mathbb{C}([A]-[B])$, and hence, $\dim H^k_0(\underline{E};\Omega^k_{\underline{E}}) = 1$.

Proof. As noted in Definition 2.4, if L is the link of the singularity, then $\dim \mathrm{Gr}^{n-k}_F H^n(L) = 1$. Then Lemma 2.18 gives the exact sequence

$$\mathrm{Gr}^{n-k}_F H^n(E) \to \mathrm{Gr}^{n-k}_F H^n(L) \to \mathrm{Gr}^{n-k-1}_F H^{n-1}(E) \to \mathrm{Gr}^{n-k}_F H^{n+1}(E).$$

Since E is an orbifold weighted hypersurface in WP^n, $\mathrm{Gr}^i_F H^j(E) = 0$ except for the cases $j = 2i$ or $i + j = n-1$. Thus, $\mathrm{Gr}^{n-k}_F H^n(E) = \mathrm{Gr}^{n-k-1}_F H^{n-1}(E) = 0$ unless $n = 2(n-k)$ (i.e., $k = \frac{1}{2}n$) or $n+1 = 2(n-k)$ (i.e., $k = \frac{1}{2}(n-1)$). The first case is excluded since we assumed that X is a singular point, and the second case only arises if $n = 2k+1$ and X is an ordinary double point (Lemma 2.5). This proves the first statement, and the second statement is the well-known computation of the primitive cohomology of an even-dimensional quadric. □

Proposition 3.6. *The map $\mu_{\underline{X}^\#}$ is not 0. Hence, there exist bases $v \in H^1(\underline{E};\Omega^{n-1}_{\underline{X}^\#}(\log \underline{E})(-a\underline{E})|\underline{E})$ and $\varepsilon \in H^k(\underline{E};\Omega^{n-k}_{\underline{X}^\#}(\log \underline{E})|\underline{E})$ of the two one-dimensional vector spaces and a nonzero $c \in \mathbb{C}$ such that, for all $\lambda \in \mathbb{C}$,*

$$\mu_{\underline{X}^\#}(\lambda v) = c\lambda^k \varepsilon.$$

Proof. It suffices to prove that the map $\mu'_{\underline{X}^\#}$ is nonzero. Taking the $(i+1)^{\mathrm{st}}$ exterior power of the normal bundle sequence

$$0 \to T_{\underline{E}}(-d) \to (T_{\underline{WP^n}}|\underline{E})(-d) \to \mathcal{O}_{\underline{E}} \to 0$$

gives exact sequences

$$0 \to \bigwedge^{i+1} T_{\underline{E}}(-(i+1)d) \to \bigwedge^{i+1}(T_{\underline{WP^n}}|\underline{E})(-(i+1)d) \to \bigwedge^{i} T_{\underline{E}}(-id) \to 0, \qquad (*)$$

and thus a sequence of connecting homomorphisms

$$\partial_i\colon H^i(\underline{E};\bigwedge^{i} T_{\underline{E}}(-id)) \to H^{i+1}(\underline{E};\bigwedge^{i+1} T_{\underline{E}}(-(i+1)d).$$

We claim the following:

Claim 3.7. *There exists a nonzero element* $\eta \in H^1(\underline{E}; T_{\underline{E}}(-d))$*, necessarily a generator, such that* $\mu'_{\underline{X}^\#}(\eta) = \pm\partial_{k-1} \circ \cdots \circ \partial_1(\eta)$.

Claim 3.8. *The connecting homomorphism* ∂_i *is an isomorphism for* $1 \leq i \leq k-2$ *and injective for* $i = k-1$.

Clearly, the two claims imply Proposition 3.6. □

Proof of Claim 3.7. The element $\eta = \partial_0(1) \in H^1(\underline{E}; T_{\underline{E}}(-d))$ is the extension class for the extension $0 \to T_{\underline{E}}(-d) \to (T_{\underline{W\,P^n}}|\underline{E})(-d) \to \mathcal{O}_{\underline{E}} \to 0$. By the last line of the proof of Lemma 3.2, the coboundary map ∂_0 is injective, and hence, $\eta \neq 0$. Then a calculation shows that, up to sign,

$$\wedge\eta \in H^1(\underline{E}; Hom(\bigwedge^{i} T_{\underline{E}}(-id), \bigwedge^{i+1} T_{\underline{E}}(-(i+1)d)))$$

is the corresponding extension class for the extension $(*)$. Since the connecting homomorphism is given by cup product with the extension class, we see that

$$\mu'_{\underline{X}^\#}(\eta) = \eta^k = \pm\partial_{k-1} \circ \cdots \circ \partial_1(\eta) \in H^k(\underline{E}; \bigwedge^{k} T_{\underline{E}}(-kd)). \qquad \square$$

Proof of Claim 3.8. It suffices to show that $H^i(\underline{E}; \bigwedge^{i+1}(T_{\underline{W\,P^n}}|\underline{E})(-(i+1)d)) = 0$ for $1 \leq i \leq k-1$. First, note that

$$\bigwedge^{i+1}(T_{\underline{W\,P^n}}|\underline{E})(-(i+1)d)) = \left(\Omega^{n-i-1}_{\underline{W\,P^n}}|\underline{E}\right)\left(\sum_k a_k - (i+1)d\right).$$

We have the exact sequence

$$0 \to \Omega^\ell_{\underline{W\,P^n}}(r-d) \to \Omega^\ell_{\underline{W\,P^n}}(r) \to (\Omega^\ell_{\underline{W\,P^n}}|\underline{E})(r) \to 0.$$

By Bott vanishing (or directly), $H^j(\underline{E}; (\Omega^\ell_{\underline{W\,P^n}}|\underline{E})(r)) = 0$ as long as $1 \leq j \leq n-2$ and $j \neq \ell$ or $\ell+1$. In our situation, $i \leq k-1 < k \leq \frac{1}{2}(n-1)$, and thus $i < n-i-1$. In particular, $i \neq n-i-1$ or $n-i$. Thus, $H^i(\underline{E}; \bigwedge^{i+1}(T_{\underline{W\,P^n}}|\underline{E})(-(i+1)d)) = 0$. □

Remark 3.9. The above calculations are connected with the computation of the Hodge filtration on E. For example, in case X is a cone over the smooth degree d hypersurface E in $\mathbb{P}^n$, then $H^0(\mathbb{P}^n; K_{\mathbb{P}^n} \otimes (n-k)d) = H^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(-n-1+d(k+1)) = H^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n})$ is identified via residues with $F^{n-k}H^n(L) = \mathrm{Gr}^{n-k}_F H^n(L)$.

4. The global setting

4.1. Deformation theory

We assume the following for the rest of this subsection:

Assumption 4.1. Y is a canonical Calabi–Yau variety of dimension $n \geq 4$, all of whose singularities are k-liminal isolated weighted homogeneous hypersurface singularities, with $k \geq 2$, as the case $k = 1$ has already been considered in Theorem 2.11. We freely use the notation of the previous sections, especially that of Definition 2.14. In particular, $Y^\#$ is the weighted blowup at each point x of $Z = Y_{\mathrm{sing}}$, with exceptional divisor E_x, and a_x is the integer defined in §2.1. We let $\underline{Y}^\#$ and $\underline{E} = \sum_{x \in Z} \underline{E}_x$ be the associated stacks. Let $\vec{a}\underline{E}$ denote the divisor $\sum_{x \in Z} a_x \underline{E}_x$.

The argument of Theorem 3.4 shows the following:

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Lemma 4.2. *There is a commutative diagram*

$$\begin{array}{ccc} \mathbb{T}^1_Y & \xrightarrow{\cong} & H^1(\underline{Y}^\#;\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-\vec{a}\underline{E})) \\ \downarrow & & \downarrow \\ H^0(Y;T^1_Y) & \xrightarrow{\cong} & H^0(Y;R^1\pi_*\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-\vec{a}\underline{E})). \end{array}$$

We also have the subsheaf $T_{\underline{Y}^\#}(-\log \underline{E}) \subseteq T_{\underline{Y}^\#}$. As in §2, globally there is an isomorphism

$$\bigwedge^k \left(\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-\vec{a}\underline{E})\right) \cong \Omega^{n-k}_{\underline{Y}^\#}(\log \underline{E}).$$

Then the global form of the discussion in §3 yields the following:

Theorem 4.3. *There is a commutative diagram*

$$\begin{array}{ccc} H^1(\underline{Y}^\#;\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-\vec{a}\underline{E})) & \longrightarrow & H^1(\underline{E};\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-\vec{a}\underline{E})|\underline{E}) \\ {\scriptstyle \nu_{\underline{Y}^\#}}\downarrow & & \downarrow{\scriptstyle \mu_{\underline{Y}^\#}} \\ H^k(\underline{Y}^\#;\Omega^{n-k}_{\underline{Y}^\#}(\log \underline{E})) & \longrightarrow & H^k(\underline{E};\Omega^{n-k}_{\underline{Y}^\#}(\log \underline{E})|\underline{E}). \end{array}$$

Here, $\mu_{\underline{Y}^\#}$ is the sum of the local maps $\mu_{\underline{X}^\#}$ at each component of E, and $\nu_{\underline{Y}^\#}$ is also a homogeneous map of degree k. Note that after we localize at a singular point x of Y,

$$\dim H^k(\underline{E}_x;\Omega^{n-k}_{\underline{X}^\#}(\log \underline{E}_x)|\underline{E}_x) = \dim \mathrm{Gr}^{n-k}_F H^n(L_x) = 1.$$

By Corollary 3.3(iii), $\dim H^1(\underline{E}_x;\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-a_x\underline{E})|\underline{E}_x) = 1$ as well, and so the $\mu_{\underline{Y}^\#}$ in the diagram, at each singular point x of Y, is a homogeneous degree k map between two one-dimensional vector spaces. For every $x \in Z$, fix an isomorphism $H^1(\underline{E}_x;\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-a_x\underline{E})|\underline{E}_x) \cong \mathbb{C}$ (i.e. a basis vector $v_x \in H^1(\underline{E}_x;\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-a_x\underline{E})|\underline{E}_x))$ and a basis vector $\varepsilon_x \in \mathrm{Gr}^{n-k}_F H^n(L_x)$. It follows by Proposition 3.6 that for every $x \in Z$, there exists a nonzero $c_x \in \mathbb{C}$, depending only on the above choices, such that, for every $\lambda = (\lambda_x) \in \mathbb{C}^Z \cong H^1(\underline{E};\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-\vec{a}\underline{E})|\underline{E})$,

$$\mu_{\underline{Y}^\#}(\lambda) = \sum_{x\in Z} c_x\lambda_x^k\varepsilon_x.$$

Consider the following diagram, where the vertical arrows are homogeneous of degree k and the bottom row is exact:

$$\begin{array}{ccccc} H^1(\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-\vec{a}\underline{E})) & \longrightarrow & H^1(\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-\vec{a}\underline{E})|\underline{E}) & & \\ {\scriptstyle \nu_{\underline{Y}^\#}}\downarrow & & \downarrow{\scriptstyle \mu_{\underline{Y}^\#}} & & \\ H^k(\Omega^{n-k}_{\underline{Y}^\#}(\log \underline{E})) & \longrightarrow & H^k(\Omega^{n-k}_{\underline{Y}^\#}(\log \underline{E})|\underline{E}) & \xrightarrow{\partial} & H^{k+1}(\Omega^{n-k}_{\underline{Y}^\#}(\log \underline{E})(-\underline{E})). \end{array}$$

The above diagram then implies the following: if a class

$$\alpha = (\alpha_x) \in H^1(\underline{E};\Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-\vec{a}\underline{E})|\underline{E})$$

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is the image of $\beta \in H^1(\underline{Y}^\#; \Omega^{n-1}_{\underline{Y}^\#}(\log \underline{E})(-\vec{a}\underline{E}))$, then $\mu_{\underline{Y}^\#}(\alpha)$ is the image of

$$\nu_{\underline{Y}^\#}(\beta) \in H^k(\underline{Y}^\#; \Omega^{n-k}_{\underline{Y}^\#}(\log \underline{E})),$$

and hence, $\partial(\mu_{\underline{Y}^\#}(\alpha)) = 0$ in $H^{k+1}(\Omega^{n-k}_{\underline{Y}^\#}(\log \underline{E})(-\underline{E}))$.

Returning to the world of spaces, as opposed to stacks, consider a log resolution $\pi\colon \widehat{Y} \to Y$ with exceptional divisor, which we continue to denote by E. Then via the isomorphism

$$H^{k+1}(\underline{Y}^\#; \Omega^{n-k}_{\underline{Y}^\#}(\log \underline{E})(-\underline{E})) \cong H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}(\log E)(-E))$$

of Lemma 2.16, the coboundary $\partial(\mu_{\underline{Y}^\#}(\alpha))$ defines an element of $H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}(\log E)(-E))$. Moreover, if $\partial(\mu_{\underline{X}^\#}(\alpha))$ is of the form $\sum_{x\in Z} c_x\lambda_x^k\partial(\varepsilon_x)$, then it has the same form when viewed as an element of $H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}(\log E)(-E))$, by the commutativity of the diagram

$$\begin{array}{ccc} H^k(\underline{E}; \Omega^{n-k}_{\underline{Y}^\#}(\log \underline{E})|\underline{E}) & \xrightarrow{\partial} & H^{k+1}(\underline{Y}^\#; \Omega^{n-k}_{\underline{Y}^\#}(\log \underline{E})(-\underline{E})) \\ \cong\downarrow & & \downarrow\cong \\ H^k(E; \Omega^{n-k}_{\widehat{Y}}(\log E)|E) & \xrightarrow{\partial} & H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}(\log E)(-E)). \end{array}$$

So finally, we obtain the following:

Theorem 4.4. *For every $x \in Z = Y_{\mathrm{sing}}$, fix isomorphisms*

$$H^0(T^1_{Y,x})/\mathfrak{m}_x H^0(T^1_{Y,x}) \cong H^1(E_x; \Omega^{n-1}_{\widehat{X}}(\log E_x)(-a_xE_x)|E_x) \cong \mathbb{C}.$$

Write $\mathrm{Gr}^{n-k}_F H^n(L_x) = H^k(E_x; \Omega^{n-k}_{\widehat{Y}}(\log E_x)|E_x) = \mathbb{C}\cdot\varepsilon_x$ for some fixed choice of a generator ε_x. For all $x \in Z$, there exist $c_x \in \mathbb{C}^*$ depending only on the above identifications, with the following property: Suppose that the class $(\bar{\theta}_x) \in \bigoplus_{x\in Y_{\mathrm{sing}}} H^0(T^1_{Y,x})/\mathfrak{m}_x H^0(T^1_{Y,x})$ is in the image of $\theta \in \mathbb{T}^1_Y$, and let $\lambda_x \in \mathbb{C}$ be the complex number corresponding to $\bar{\theta}_x$ via the above identification. If $\partial\colon H^k(E; \Omega^{n-k}_{\widehat{Y}}(\log E)|E) \to H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}(\log E)(-E))$ is the coboundary map, then

$$\sum_{x\in Z} c_x\lambda_x^k\partial(\varepsilon_x) = 0.$$

We can post-compose the coboundary map

$$\partial\colon \mathrm{Gr}^{n-k}_F H^n(L) = H^k(E; \Omega^{n-k}_{\widehat{Y}}(\log E)|E) \to H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}(\log E)(-E))$$

with the natural (injective) map

$$H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}(\log E)(-E)) \to H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}).$$

Let $\varphi\colon \mathrm{Gr}^{n-k}_F H^n(L) = H^k(E; \Omega^{n-k}_{\widehat{Y}}(\log E)|E) \to H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}})$ be the above composition. This is the same as the induced map on Gr^{n-k}_F of the natural map $H^n(L) \to H^{n+1}(\widehat{Y})$, which is the Poincaré dual of the map $H_{n-1}(L) \to H_{n-1}(\widehat{Y})$.

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Corollary 4.5. *With the notation and hypotheses of Theorem 4.4, and with* $\varphi\colon \mathrm{Gr}_F^{n-k} H^n(L) \to H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}})$ *the natural map as above, the following holds in* $H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}})$*:*

$$\sum_{x\in Z} c_x \lambda_x^k \varphi(\varepsilon_x) = 0.$$

In particular, if a strong first-order smoothing of Y exists, then for all $x \in Z$*, there exists* $\lambda_x \in \mathbb{C}^*$ *with* $\sum_{x\in Z} c_x\lambda_x^k\varphi(\varepsilon_x) = 0$.

Remark 4.6. (i) By Poincaré duality, the map $H^n(L) \to H^{n+1}(\widehat{Y})$ is the same as the map $H_{n-1}(L) \to H_{n-1}(\widehat{Y})$, which factors as $H_{n-1}(L) \to H_{n-1}(Y) \to H_{n-1}(\widehat{Y})$. By Remark 2.2, we can identify $\partial\colon \mathrm{Gr}_F^{n-k} H^n(L) \to H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}(\log E)(-E))$ with the corresponding map

$$\mathrm{Gr}_F^{n-k} H_{n-1}(L)(-n) \to \mathrm{Gr}_F^{n-k} H_{n-1}(Y)(-n).$$

This gives an equivalent statement to Theorem 4.4 that only involves Y, not the choice of a resolution.

(ii) By Lemma 2.3, the map $H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}(\log E)(-E)) \to H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}})$ is injective. Thus, $\sum_{x\in Z} c_x\lambda_x^k\varphi(\varepsilon_x) = 0 \iff \sum_{x\in Z} c_x\lambda_x^k\partial(\varepsilon_x) = 0$, so that Theorem 4.4 and Corollary 4.5 contain the same information.

Remark 4.7. It is certainly possible for $\partial(\varepsilon_x) = 0$. For example, suppose that $\dim Y = 2k+1$ and the singularities of Y are all k-liminal (i.e., ordinary double points). If Y is a Calabi–Yau hypersurface in $\mathbb{P}^{2k+2}$ with just a few singular points in general position, then they can be smoothed independently (i.e., for every $x \in Z$, there exists a $\theta \in \mathbb{T}^1_Y$ such that $\lambda_x \neq 0$, but $\lambda_{x'} = 0$ for all $x' \neq x$). Then $\partial(\varepsilon_x) = 0$ for every $x \in Z$.

Remark 4.8. As in Remark 2.12, we can also consider the Fano case, where Y has isolated k-liminal weighted homogeneous hypersurface singularities and ω_Y^{-1} is ample. In this case, the construction produces an obstruction to a strong first-order smoothing; namely, $\sum_{x\in Z} c_x\lambda_x^k\partial(\varepsilon_x) \in H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}(\log E)(-E) \otimes \pi^*\omega_Y^{-1})$. By Serre duality, $H^{k+1}(\widehat{Y}; \Omega^{n-k}_{\widehat{Y}}(\log E)(-E) \otimes \pi^*\omega_Y^{-1})$ is dual to $H^{n-k-1}(\widehat{Y}; \Omega^k_{\widehat{Y}}(\log E) \otimes \pi^*\omega_Y)$.

In many reasonable cases, however, $H^{n-k-1}(\widehat{Y}; \Omega^k_{\widehat{Y}}(\log E) \otimes \pi^*\omega_Y) = 0$. For example, if there exists a smooth Cartier divisor H on Y, thus not passing through the singular points, such that $\omega_Y = \mathcal{O}_Y(-H)$, and in addition $H^{n-k-2}(H; \Omega^k_H) = 0$, then the argument of Remark 2.12 shows that

$$H^{n-k-1}(\widehat{Y}; \Omega^k_{\widehat{Y}}(\log E) \otimes \pi^*\omega_Y) = H^{n-k-1}(\widehat{Y}; \Omega^k_{\widehat{Y}}(\log E) \otimes \mathcal{O}_{\widehat{Y}}(-H)) = 0,$$

where we identify the divisor H on Y with its preimage π^*H on $\widehat{Y}$. For example, these hypotheses are satisfied if Y is a hypersurface in $\mathbb{P}^{n+1}$ of degree $d \le n+1$. However, as soon as $n = 2k+1$ is odd and $n \ge 5$, there exist such hypersurfaces with only nodes as singularities (the k-liminal case with $k = \frac{1}{2}(n-1)$) such that the map $\mathbb{T}^1_Y \to H^0(Y; T^1_Y) = \bigoplus_{x\in Y_{\mathrm{sing}}} H^0(T^1_{Y,x})/\mathfrak{m}_x H^0(T^1_{Y,x})$ is not surjective (cf. for example, [FL22a, Remark 4.11(iv)]). Thus, the obstructions to the surjectivity of the map $\mathbb{T}^1_Y \to H^0(Y; T^1_Y)$ are not detected by the nonlinear obstruction $\sum_{x\in Z} c_x\lambda_x^k\partial(\varepsilon_x)$. Of course, a nodal hypersurface in $\mathbb{P}^{n+1}$ is smoothable, but the above examples show that, even in the Fano case, the nodes cannot always be smoothed independently.

4.2. Geometry of a smoothing

We make the following assumption throughout this subsection (except for Remark 4.14 at the end):

Assumption 4.9. Y denotes a projective variety, not necessarily satisfying ω_Y^{-1} ample or $\omega_Y \cong \mathcal{O}_Y$, with only isolated lci singular points (not necessarily weighted homogeneous). Denote by Z the singular

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locus of Y. Let $f\colon \mathcal{Y} \to \Delta$ be a projective smoothing of Y (i.e., $Y_0 = Y \cong f^{-1}(0)$ and the remaining fibers $Y_t = f^{-1}(t)$, $t \neq 0$, are smooth). For $x \in Z$, let L_x denote the link at x, and let M_x denote the Milnor fiber at x. Finally, let $M = \bigcup_{x\in Z} M_x$ and $L = \bigcup_{x\in Z} L_x$.

We have the Mayer–Vietoris sequence of mixed Hodge structures (where $H^i(Y_t)$ is given the limiting mixed Hodge structure):

$$\cdots \to H^{i-1}(M) \to H^i(Y, Z) \to H^i(Y_t) \to H^i(M) \to \cdots . \tag{4.1}$$

In particular, just under the assumption that Y has isolated lci singularities, $H^i(Y, Z) \to H^i(Y_t)$ is an isomorphism except for the cases $i = n, n+1$. There is a more precise result if we assume that the singularities are k-Du Bois:

Lemma 4.10. *Suppose that all singular points of Y are isolated lci k-Du Bois singularities. Then*

(i) $\mathrm{Gr}^p_F H^n(M_x) = 0$ *for* $p \le k$ *and* $\mathrm{Gr}^{n-p}_F H^n(M_x) = 0$ *for* $p \le k-1$.
(ii) *For all i, if* $p \le k$, *then* $\mathrm{Gr}^p_F H^i(Y_t) = \mathrm{Gr}^p_F H^i(Y)$, *and if* $p \le k-1$, *then* $\mathrm{Gr}^{n-p}_F H^i(Y_t) = \mathrm{Gr}^{n-p}_F H^i(Y)$.

Proof. The first statement follows from [FL24, §6] and the second from (i), (4.1), and strictness. □

Remark 4.11. Under the assumption of isolated lci k-Du Bois singularities as above (or more generally isolated lci $(k-1)$-rational singularities), the above implies that $\mathrm{Gr}^W_{2n-a} H^n(Y_t) = 0$ for all $a \le 2k-1$, and hence that for all $a \le 2k-1$, $\mathrm{Gr}^W_a H^n(Y_t) = 0$ as well. Thus, if T is the monodromy operator acting on $H^n(Y_t)$ and $N = \log T^m$ for a sufficiently divisible power of T, then $N^{n-2k+1} = 0$.

Under the assumption of k-liminal singularities, the proof of Lemma 4.10 and [FL24, Corollary 6.14] gives the following:

Lemma 4.12. *In the above notation, if all singular points of Y are isolated k-liminal hypersurface singularities, then*

$$\mathrm{Gr}^{n-k}_F H^n(M_x) \cong \mathrm{Gr}^{n-k}_F H^n(L_x) = \mathbb{C}\cdot \varepsilon_x$$

for some nonzero $\varepsilon_x \in \mathrm{Gr}^{n-k}_F H^n(L_x)$. *Moreover, there is an exact sequence*

$$0 \to \mathrm{Gr}^{n-k}_F H^n(Y) \to \mathrm{Gr}^{n-k}_F H^n(Y_t) \to \bigoplus_{x\in Z} \mathbb{C}\cdot\varepsilon_x \xrightarrow{\psi} \mathrm{Gr}^{n-k}_F H^{n+1}(Y) \to \mathrm{Gr}^{n-k}_F H^{n+1}(Y_t) \to 0.$$

We also have the natural map $\varphi\colon \mathrm{Gr}^{n-k}_F H^n(L) = \bigoplus_{x\in Z} \mathbb{C}\cdot\varepsilon_x \to \mathrm{Gr}^{n-k}_F H^{n+1}(\widehat{Y})$, and there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}^{n-k}_F H^n(M) & \xrightarrow{\cong} & \mathrm{Gr}^{n-k}_F H^n(L) \\ \psi\downarrow & & \downarrow\varphi \\ \mathrm{Gr}^{n-k}_F H^{n+1}(Y) & \longrightarrow & \mathrm{Gr}^{n-k}_F H^{n+1}(\widehat{Y}). \end{array}$$

By Lemma 2.3, $\mathrm{Gr}^{n-k}_F H^{n+1}(Y) \to \mathrm{Gr}^{n-k}_F H^{n+1}(\widehat{Y})$ is injective. Thus, the dimensions of the kernel and image of the map $\psi\colon \bigoplus_{x\in Z} \mathbb{C}\cdot\varepsilon_x \to \mathrm{Gr}^{n-k}_F H^{n+1}(Y)$ are the same as the dimensions of the kernel and image of the map $\varphi\colon \bigoplus_{x\in Z} \mathbb{C}\cdot\varepsilon_x \to \mathrm{Gr}^{n-k}_F H^{n+1}(\widehat{Y})$. Then we have the following generalization of [Fri91, Lemma 8.1(2)]:

Corollary 4.13. *Still assuming that all singular points of Y are isolated k-liminal hypersurface singularities, in the above notation, let* $s' = \dim \mathrm{Ker}\{\varphi\colon \bigoplus_{x\in Z} \mathbb{C}\cdot\varepsilon_x \to \mathrm{Gr}^{n-k}_F H^{n+1}(\widehat{Y})\}$, *and let* $s'' = \#(Z) - s' = \dim \mathrm{Im}\,\varphi$. *Then*

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(i) $h^{n-k,k}(Y_t) = h^{k,n-k}(Y_t) = \dim \mathrm{Gr}_F^{n-k} H^n(Y_t) = \dim \mathrm{Gr}_F^{n-k} H^n(Y) + s'$.
(ii) $\dim \mathrm{Gr}_F^k H^n(Y) = \dim \mathrm{Gr}_F^{n-k} H^n(Y) + s'$.
(iii) $h^{n-k,k+1}(Y_t) = \dim \mathrm{Gr}_F^{n-k} H^{n+1}(Y_t) = \dim \mathrm{Gr}_F^{n-k} H^{n+1}(Y) - s''$.

Proof. (i) and (iii) follow from the exact sequence in Lemma 4.12. As for (ii), by Lemma 4.10(ii),

$$\dim \mathrm{Gr}_F^k H^n(Y) = \dim \mathrm{Gr}_F^k H^n(Y_t) = \dim \mathrm{Gr}_F^{n-k} H^n(Y_t) = \dim \mathrm{Gr}_F^{n-k} H^n(Y) + s',$$

using (i). □

Remark 4.14. Let Y be a compact analytic threefold with all singular points 1-liminal, hence ordinary double points. Assume in addition that $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$. For a canonical Calabi–Yau threefold, since $\omega_Y \cong \mathcal{O}_Y$, this is a natural assumption to make: If $h^1(\mathcal{O}_Y) \neq 0$, Y is smooth by a result of Kawamata [Kaw85, Theorem 8.3], and $h^1(\mathcal{O}_Y) = 0 \iff h^2(\mathcal{O}_Y) = h^2(\omega_Y) = 0$ by Serre duality. Let Y' be a small resolution of Y, and let $[C_x] \in H^2(Y'; \Omega^2_{Y'}) = H^4(Y')$ be the fundamental class of the exceptional curve over the point $x \in Z$. Setting $\psi\colon \mathbb{C}^Z \to H^2(Y'; \Omega^2_{Y'})$ to be the natural map $(a_x) \mapsto \sum_{x\in Z} a_x[C_x]$, let $s' = \dim \operatorname{Ker} \psi$ and $s'' = \dim \operatorname{Im} \psi$. Then arguments similar to those above show that

$$b_4(Y) = b_4(Y') = b_2(Y') = b_2(Y) + s''.$$

Moreover, if Y is smoothable and Y_t denotes a general smoothing, then

$$b_2(Y_t) = b_2(Y) = b_2(Y') - s'';$$
$$b_3(Y_t) = b_3(Y') + 2s'.$$

In particular, if Y is a 1-liminal canonical Calabi–Yau threefold and $\psi = 0$, or equivalently $s'' = 0$ in the above notation (i.e., Y is $\mathbb{Q}$-factorial), then Y is smoothable by Theorem 1.2 and the Kawamata–Ran–Tian theorem, and the above formulas hold for Y_t.

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