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A NORMAL FORM IN FREE FIELDS

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ABSTRACT. We give a normal form for the elements in the free field, following the lines of the minimization theory of noncommutative rational series.

1. **Introduction.** Free fields, first introduced by Amitsur [A], were described by the first author as universal field of fractions of the ring of noncommutative polynomials; they are universal objects in the category whose morphisms are specializations. A characteristic property is that each full polynomial matrix may be inverted in the free field. A normal form for the elements of the free field was given in [C3].

We propose here another normal form, inspired from automata theory. Each element of the free field is obtained by inverting some full linear matrix, and then multiplying on the left and right by a row and a column over the scalars. We call this a *representation*. It is known that each rational (or regular, or recognizable) language has a unique minimal deterministic automaton; this result was extended by Schützenberger to noncommutative rational formal series: he showed essentially that such a series has a minimal linear representation, unique up to the natural action of the scalar linear group (the form stated here is due to Fliess). This leads us to prove a similar result for elements of the free field: there is a natural action of the square of the linear group on representations of these elements (equivalently, by elementary row and column operations), and we show that minimal representations form a single orbit. It is this orbit that we may call "normal form".

We begin by studying closely the representations of formal series in $D_k\langle\langle X\rangle\rangle$, the ring of tensor ring series: the variables do not commute with the elements of the skew field D, but with the elements of the central subfield k. We extend to $D_k\langle\langle X\rangle\rangle$ the minimization theory of rational series in $k\langle\langle X\rangle\rangle$.

The first application is the minimization theory of elements of the free field $D_k\{X\}$ where D is infinite dimensional over its center k. The essential tool here is the specialization lemma of the first author, which allows us to work with formal series in $D_k\langle\langle X\rangle\rangle$, by "change of origin". The second application is the analogous theory for $k\{X\}$. Here, one works similarly, but some inertia properties of the embedding $k\{X\} \longrightarrow D_k\{X\}$ have to be established first; one cannot change the origin by translations over k, because some elements in $k\{X\}$ are not commutatively defined, *e.g.* $(xy - yx)^{-1}$.

The main results are the uniqueness theorems: Theorem 2.8, Theorem 4.1 and Theorem 4.3. The latter result has a striking analogy with a result of Roberts [R] (see also [C1]

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Section 5.8), who gives a normal form for matrices in $k\langle X\rangle$ and shows that, if minimal, they are equivalent in the sense given here; however, Theorem 4.3 does not seem to be a simple consequence of this result, although the methods of Roberts could perhaps be adapted. Worth mentioning in the present work are also Proposition 2.12 and Proposition 4.8: they state that minimal representations appear as block components of general representations.

In this paper, all fields are possibly skew.

2. **Rational series in $D_k\langle\langle X\rangle\rangle$.** Let D be a field containing the commutative field k in its center, and let X be a *finite* set of noncommuting indeterminates. The *tensor D-ring* on X over k is denoted by $D_k\langle X\rangle$; it is the direct sum

$$D_k\langle X\rangle = D \oplus D\otimes V\otimes D \oplus D\otimes V\otimes D\otimes V\otimes D \oplus\cdots$$

where V is the k-vector space $V = kX$, and where tensor products are taken over k. The product is induced by that of D.

An element of $D_k\langle X\rangle$ will be called a *polynomial*. On $D_k\langle X\rangle$, there is a degree-function and an order function obtained by associating degree (and order) one to each member of X; recall that the order (and the degree) of a monomial $d_0x_1d_1\cdots x_nd_n$ is n, and that the order of a polynomial P is the smallest order of the monomials appearing in it, for all possible representations of P. The completion of $D_k\langle X\rangle$ with respect to this order function is denoted by $D_k\langle\langle X\rangle\rangle$ and called the *power series D-ring on X over k*; a typical element of $D_k\langle\langle X\rangle\rangle$ has the form

$$f = \sum_{n\geq 0} f_n, \tag{1}$$

where each f_n is an homogeneous element of $D_k\langle X\rangle$, of degree n. The elements of $D_k\langle\langle X\rangle\rangle$ will also be called *series*. Note that f above is invertible in $D_k\langle\langle X\rangle\rangle$ if and only if its constant term $f_0\in D$ is nonzero.

A series is called *rational* if it belongs to the smallest subring of $D_k\langle\langle X\rangle\rangle$ which contains $D_k\langle X\rangle$ and which contains the inverses of all its invertible elements. A basic result, which may be proved as in the classical case of $k\langle\langle X\rangle\rangle$ ($= D_k\langle\langle X\rangle\rangle$ when $D=k$), is the following.

PROPOSITION 1. *A series f is rational if and only if there exist $n\geq 0$, series $f_1,\ldots,f_n$, linear polynomials without constant term p_{ij}, $1\leq i,j\leq n$, elements $\lambda_1,\ldots,\lambda_n,\rho_1,\ldots,\rho_n$ in D such that*

$$f = \lambda_1 f_1 + \cdots + \lambda_n f_n$$

and that for $i=1,\ldots,n$, one has

$$f_i = \sum_{j=1}^{n} p_{ij} f_j + \rho_i. \tag{2}$$

This result extends the characterization of rational series in $k\langle\langle X\rangle\rangle$ by Schützenberger [S2]; see also [E] Section VII.6, [SS] Section II.1 and [BR] Theorem I.6.1. A simple computation shows that, with the notations of the proposition, writing $M = (p_{ij})$, one has $f = \lambda(1-M)^{-1}\rho$ (observe that $1-M$ is invertible in ${}^nD_k\langle\langle X\rangle\rangle^n$). This justifies the following definition: a *representation* of a series $f \in D_k\langle\langle X\rangle\rangle$ is a triple $\pi = (\lambda, 1-M, \rho)$, where $\lambda \in D^n$, $\rho \in {}^nD$ and $M \in {}^nD_k\langle X\rangle^n$ is a linear matrix without constant term (*i.e.* each entry of M is an homogeneous element of $D_k\langle X\rangle$ of degree 1), such that

(3) $$f = \lambda(1-M)^{-1}\rho.$$

Note that in this case, if we define series $f_1, \dots, f_n$ by

(4) $$f_i = \big((1-M)^{-1}\rho\big)_i,$$

then the constant term of f_i is ρ_i, $f = \lambda_1 f_1 + \cdots + \lambda_n f_n$ and (2) holds, with $M = (p_{ij})$. Conversely (2) implies (3) and (4). Thus we have:

COROLLARY 2. *A series f is rational if and only if it admits a representation.*

The dimension of the representation above is n. A representation of f of smallest possible dimension is called *minimal*, and its dimension is by definition the *rank* of f.

It will be useful for the sequel to adopt the following terminology and notation: for a representation π as above, call the family $f_1, \dots, f_n$ the *left family* of π, and denote by $L(\pi)$ the left D-linear subspace of $D_k\langle\langle X\rangle\rangle$ which they span. We often identify the left family and the vector $(f_1, \dots, f_n)^T = (1-M)^{-1}\rho$. Similarly, writing $g_i = \big(\lambda(1-M)^{-1}\big)_i$, we call $g_1, \dots, g_n$ the *right family* of π and we denote by $R(\pi)$ the right D-linear space they span.

The aim of this section is to characterize minimal representations and to show how they are related to each other. Before that, we introduce a space $L(f)$, canonically attached to f, and closely related to minimal representations of f.

Let $(u_\lambda)_{\lambda\in\Lambda}$ be a basis of D over k. By (1), each f in $D_k\langle\langle X\rangle\rangle$ may be uniquely written

(5) $$f = \sum_{n\geq 0} \sum_{\substack{\lambda_0,\dots,\lambda_n\in\Lambda \\ x_1,\dots,x_n\in X}} \alpha(\lambda_0, \dots, \lambda_n; x_1, \dots, x_n) u_{\lambda_0} x_1 u_{\lambda_1} \cdots x_n u_{\lambda_n},$$

where α is in k, and where for fixed n, only finitely many $\alpha(\lambda_0, \dots, \lambda_n; x_1, \dots, x_n)$ are nonzero. Let $\lambda \in \Lambda$ and $x \in X$. We define a series, denoted by $(u_\lambda x)^{-1}f$: it is the right cofactor of $u_\lambda x$ in f; in symbols

$$(u_\lambda x)^{-1}f = \sum_{n\geq 1} \sum_{\substack{\lambda_1,\dots,\lambda_n\in\Lambda \\ x_2,\dots,x_n\in X}} \alpha(\lambda, \lambda_1, \dots, \lambda_n; x, x_2, \dots, x_n) u_{\lambda_1} x_2 \cdots x_n u_{\lambda_n}.$$

The mapping $f \mapsto (u_\lambda x)^{-1}f$ is well-defined and k-linear. For convenience, we call it a *left transduction*. We define $L(f)$ to be the smallest left D-subspace of $D_k\langle\langle X\rangle\rangle$ containing f and which is closed under all the left transductions. It is easy to show that the space $L(f)$ does not depend on the chosen basis. In the case of rational series, this will be a consequence of the sequel.

We can now give another characterization of rationality. It extends the Hankel characterization of rational series in $k\langle\langle X\rangle\rangle$ by Fliess [F], in a form given by Jacob [J]; see also [SS] Section II.3 and [BR] Proposition I.5.1.

PROPOSITION 3. *A series f is rational if and only if the space $L(f)$ is a finite dimensional D-vector space.*

We prove first three lemmas.

LEMMA 4. *Let $f \in D_k\langle\langle X\rangle\rangle$ such that $L(f)$ is a finite-dimensional left D-vector space. Then f has the following property:*

(6) *f is the sum of its constant term and of finitely many series of the form dxg with $d \in D$, $x \in X$, $g \in D_k\langle\langle X\rangle\rangle$.*

PROOF. By (5), we may write

$$f = f_0 + \sum_{\lambda,x} u_\lambda x f_{\lambda,x},$$

for some series $f_{\lambda,x}$ satisfying the property of local finiteness: for each n, only finitely many series $f_{\lambda,x}$ involve monomials of X-degree n. We have $(u_\lambda x)^{-1}f = f_{\lambda,x}$. Let $f_1, \ldots, f_r$ be a D-basis of $L(f)$. Then $f_{\lambda,x} = \sum_{i=1}^r d_{\lambda,x,i} f_i$, for some $d_{\lambda,x,i}$ in D. Thus

$$f = f_0 + \sum_{\lambda,x} u_\lambda x \sum_{i=1}^{r} d_{\lambda,x,i} f_i.$$

Let $f_i^{\leq n}$ denote the sum of the homogeneous components of f_i of degree $\leq n$. Since the f_i are linearly independent, so are the $f_i^{\leq N}$, for some N. Then, for each choice of $d_{\lambda,x,i}$, $i = 1, \ldots, r$, not all zero, the series $f_{\lambda,x} = \sum_{i=1}^r d_{\lambda,x,i} f_i$ involves some term of degree $\leq N$. We conclude by local finiteness that $d_{\lambda,x,i}$, $i = 1, \ldots, n$, vanish for almost all λ, x. ■

LEMMA 5. *Let π be a representation of f. Then $L(f) \subseteq L(\pi)$, and $L(\pi)$ is closed under left transductions.*

PROOF. Let $f_1, \ldots, f_n$ be the left family of π. Then $f \in L(\pi)$, because $f = \lambda_1 f_1 + \cdots + \lambda_n f_n$. We show that $L(\pi)$ is closed under left transduction, which will prove the lemma. For this, it is enough to show that for any $\lambda \in \Lambda$, $x \in X$, $d \in D$, $i \in 1, \ldots, n$, one has $(u_\lambda x)^{-1}(df_i) \in L(\pi)$. Now by (2), $df_i = \sum_{j=1}^n dp_{ij} f_j + d\rho_i$, and we can write $dp_{ij} = \sum u_\lambda x d_{\lambda,x,i,j}$ where the sum is over finitely many λ and x. Thus

$$df_i = \sum u_\lambda x \sum_j d_{\lambda,x,i,j} f_j + d\rho_i,$$

which implies $(u_\lambda x)^{-1}(df_i) = \sum_j d_{\lambda,x,i,j} f_j$. Thus, this series is in $L(\pi)$. ■

LEMMA 6. *Let $f_1, \ldots, f_n$ be series, let L be the left D-vector space they span, and suppose that L is closed under left transductions. Furthermore, define elements $d_{\lambda,x,i,j}$ of D by*

(7) $$(u_\lambda x)^{-1} f_i = \sum_j d_{\lambda,x,i,j} f_j,$$

define linear polynomials p_{ij} *by*

$$p_{ij} = \sum_{\lambda,x} u_\lambda x d_{\lambda,x,i,j}$$

and put $M = (p_{ij})_{1\leq i,j\leq n}$.

If ρ_i *denotes the constant term of* f_i, *and* $\rho = (\rho_1, \ldots, \rho_n)^T$, *then*

$$(f_1, \ldots, f_n)^T = (1-M)^{-1}\rho. \tag{8}$$

Moreover, if $f_1, \ldots, f_n$ *are linearly independent over* D, *then* M, *subject to (8), is unique.*

PROOF. Each series f_i satisfies the hypothesis of Lemma 4, hence we may write

$$f_i = \rho_i + \sum_{\lambda,x} u_\lambda x g_{\lambda,x,i}$$

for some series $g_{\lambda,x,i}$, where the summation is finite. Then $(u_\lambda x)^{-1} f_i = g_{\lambda,x,i}$, hence

$$\begin{aligned} f_i &= \rho_i + \sum_{\lambda,x} u_\lambda x \sum_j d_{\lambda,x,i,j} f_j \\ &= \rho_i + \sum_j p_{ij} f_j. \end{aligned}$$

The latter equality may be written

$$(f_1, \ldots, f_n)^T = \rho + (p_{ij})(f_1, \ldots, f_n)^T,$$

which implies (8). Note that if $f_1, \ldots, f_n$ are linearly independent over D, then the $d_{\lambda,x,i,j}$ are unique; hence so is M, because (8) implies (7). ∎

PROOF OF PROPOSITION 3. If f is rational, then $L(f)$ is finite-dimensional by Lemma 5. Conversely, if $L(f)$ is finite-dimensional, then by Lemma 6, f admits the representation $(\lambda, 1-M, \rho)$, where $f_1, \ldots, f_n$ span $L(f)$, and $\lambda_i \in D$ are defined by $f = \lambda_1 f_1 + \cdots + \lambda_n f_n$. ∎

Similarly to left transductions, we may define right transductions and the right D-vector space $R(f)$. The proofs above show that the following result holds.

COROLLARY 7. *The rank of* f *is equal to the common dimension of* $L(f)$ *and* $R(f)$.

Observe that for usual rational series $\sum a_n x^n$, the rank as defined here is the rank of the Hankel matrix $(a_{i+j})_{i,j\in N}$; see [F], [SS] Section II.3 or [BR] Theorem II.1.5 for this and its generalization to rational series in $k\langle\langle X\rangle\rangle$.

We say that two representations $\pi = (\lambda, 1-M, \rho)$ and $\pi' = (\lambda', 1-M', \rho')$ of the same dimension n are *similar* if for some square matrix $P \in \mathrm{GL}_n(D)$, one has $\lambda' = \lambda P$, $M' = P^{-1}MP$, $\rho' = P^{-1}\rho$. In this case, they represent the same series: $\lambda'(1-M')^{-1}\rho' = \lambda P(1-P^{-1}MP)^{-1}P^{-1}\rho = \lambda PP^{-1}(1-M)^{-1}PP^{-1}\rho = \lambda(1-M)^{-1}\rho$. Observe also that the two left families of π and π' are related by $(1-M')^{-1}\rho' = P^{-1}(1-M)^{-1}\rho$. In particular if one of them is left linearly independent, so is the other. Similarly for right families.

The next result extends [S1], in a form given by Fliess [F], for rational series in $k\langle\langle X\rangle\rangle$.

THEOREM 8. *Two minimal representations of a given series f are similar.*

We use the following lemma.

LEMMA 9. *Let π, π' be two representations of f which have the same left family, and suppose that this family is left D-linearly independent. Then $\pi = \pi'$.*

PROOF. Let $(1-M)^{-1}\rho = (f_1,\ldots,f_n)^T = (1-M')^{-1}\rho'$. Then $f = \lambda_1 f_1+\cdots+\lambda_n f_n = \lambda'_1 f_1 + \cdots + \lambda'_n f_n$, which shows that $\lambda = \lambda'$. We have $(f_1,\ldots,f_n)^T = M(f_1,\ldots,f_n)^T + \rho$, and similarly with M', ρ' instead of M, ρ; hence $\rho = \rho'$ is the vector of constant terms of the f_i. Moreover, the left D-space spanned by the f_i is closed under left transductions by Lemma 5. Lemma 6 then shows that M is unique, hence $M = M'$. ∎

PROOF OF THEOREM 8. Let $\pi = (\lambda, 1-M, \rho)$ and $\pi' = (\lambda', 1-M', \rho')$ be two minimal representations of f. Let $f_1,\ldots,f_n$ and $f'_i,\ldots,f'_n$ be their left families. Then by Lemma 5, Corollary 7 and minimality, both of these families form a basis of the space $L(f)$. Hence we may find $P \in \mathrm{GL}_n(D)$ such that $P(f'_i,\ldots,f'_n)^T = (f_1,\ldots,f_n)^T$ *i.e.* $P(1-M')^{-1}\rho' = (1-M)^{-1}\rho$. Let $\pi_1 = (\lambda_1, 1-M_1, \rho_1) = (\lambda P, 1-P^{-1}MP, P^{-1}\rho)$. Then this representation is similar to π, and its left family is $(1-M_1)^{-1}\rho_1 = P^{-1}(1-M)^{-1}PP^{-1}\rho = P^{-1}(1-M)^{-1}\rho = (1-M')^{-1}\rho'$. By Lemma 9, we conclude that $\pi_1 = \pi'$. ∎

The next result characterizes minimal representations. It extends [BR] Proposition II.2.1.

PROPOSITION 10. *Let π be a representation. Then π is minimal if and only if its left family is left D-linearly independent and its right family is right D-linearly independent.*

LEMMA 11. *Let $\pi = (\lambda, 1-M, \rho)$ be a representation of f with left family $f_1,\cdots,f_n$.*

(i) If $f_1,\ldots,f_n$ are left D-linearly independent and if the left D-space spanned by $f_1,\ldots,f_p$ is closed under left transduction and contains f, then π has the $p+(n-p)$ block form

$$\lambda = (\times, 0), \quad M = \begin{pmatrix} \times & 0 \\ \times & \times \end{pmatrix}$$

(ii) If $f_1 = \cdots = f_p = 0$ and $f_{p+1},\ldots,f_n$ are left D-linearly independent, then π has the $p+(n-p)$ block form

$$M = \begin{pmatrix} \times & 0 \\ \times & \times \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 \\ \times \end{pmatrix}.$$

PROOF. (i) By Lemma 5, the left D-space spanned by $f_1,\ldots,f_n$ is closed under left transductions. Hence, we can apply Lemma 6: equation (7) holds, and since the left D-space spanned by $f_1,\ldots,f_p$ admits this set as basis, we must have $d_{\lambda x,i,j} = 0$ for $1 \le i \le p$, $p+1 \le j \le n$. Hence $p_{ij} = 0$ for these i, j. Moreover $f = \sum_j \lambda_j f_j$ implies $\lambda_j = 0$ for $p+1 \le j \le n$. Thus λ and M have the indicated block form.

(ii) By Lemma 6 again, we must have $d_{\lambda x ij} = 0$ for $1 \le i \le p, p+1 \le j \le n$, since $f_1 = \cdots = f_p = 0$ and $f_{p+1},\ldots,f_n$ are linearly independent. Hence $p_{ij} = 0$ for these i, j.

Moreover, for $1 \leq i \leq p$, $0 = f_i = \rho_i + \sum_{j=1}^{n} p_{ij}f_j = \rho_i + \sum_{j=1}^{p} p_{ij}f_j + \sum_{j=p+1}^{n} p_{ij}f_j = \rho_i$. Hence M and ρ have the required block form.

PROOF OF PROPOSITION 10. The condition is necessary by Lemma 5 and Corollary 7, and by symmetry. Conversely, denote by $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ the left and right families. Suppose that $f_1, \ldots, f_n$ are left D-linearly independent and that π is not minimal. Then by Corollary 7 the left D-space spanned by the f_i is strictly bigger than $L(f)$. By replacing π by the similar representation $(\lambda P, 1-P^{-1}MP, P^{-1}\rho)$, we replace $(f_1, \ldots, f_n)^T$ by $P^{-1}(f_1, \ldots, f_n)^T$, and $(g_1, \ldots, g_n)$ by $(g_1, \ldots, g_n)P$. Hence we may suppose that $f_1, \ldots, f_p$ is a basis of $L(f)$, with $p < n$. Then Lemma 11(i) shows that π has the block form: $\lambda = (\times, 0)$, $M = \begin{pmatrix} \times & 0 \\ \times & \times \end{pmatrix}$. Hence the g_i are not independent. ∎

The next result shows how general representations are related to minimal ones.

PROPOSITION 12. *Let $\pi = (\lambda, 1-M, \rho)$ and $\bar{\pi} = (\bar{\lambda}, 1-\bar{M}, \bar{\rho})$ be two representations of the series f, the second being minimal. Then the first one is similar to a representation having the block form*

$$(\times, \bar{\lambda}, 0), \quad \begin{pmatrix} \times & 0 & 0 \\ \times & \bar{M} & 0 \\ \times & \times & \times \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \bar{\rho} \\ \times \end{pmatrix}.$$

PROOF. Suppose that the left family $(1-M)^{-1}\rho$ of π is not left D-independent. Then for some invertible matrix P over D, one has $P^{-1}(1-M)^{-1}\rho = (0, \ldots, 0, f_1', \ldots, f_p')^T$, where $f_1', \ldots, f_p'$ are D-linearly independent. Then π is similar to $\pi_1 = (\lambda P, 1 - P^{-1}MP, P^{-1}\rho)$. Observe that the left family of π_1 is $(0, \ldots, 0, f_1', \ldots f_p')^T$; hence Lemma 11 (ii) implies that π_1 has the block form

$$\lambda_1 = (\times, \lambda'), \quad M_1 = \begin{pmatrix} \times & 0 \\ \times & M' \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 \\ \rho' \end{pmatrix}.$$

Hence $(\lambda', 1 - M', \rho')$ is a representation of f, with left family $f_1', \ldots, f_p'$. Observe that if the right family $\lambda(1-M)^{-1}$ of π were right D-linearly independent, then so would the right family of π_1 be, hence also that of π', because

$$\lambda_1(1-M_1)^{-1} = (\times, \lambda') \begin{pmatrix} \times & 0 \\ \times & (1-M')^{-1} \end{pmatrix} = \big(\times, \lambda'(1-M')^{-1}\big).$$

Thus, in this case, the representation π' is minimal by Proposition 10.

If π' is not minimal, then its right family is not independent. Then we work symmetrically, and obtain a new representation with independent right family, and also independent left family, because that of π' is. Hence we conclude with Proposition 10. ∎

We have also obtained the following result.

COROLLARY 13. *Let π be a representation of f. Then $L(f) \subseteq L(\pi)$ (resp. $R(f) \subseteq R(\pi)$), with equality if π is minimal.*

3. Free fields.

3.1 *Representations.* A ring R is said to have a *universal field of fractions* K, if there is an embedding $R \longrightarrow K$ which is universal for specialization. In particular, such a universal field of fractions K exists whenever R is a semifir and K is then characterized by the property: every full matrix over R is invertible over K (see [C1] Corollary 7.5.11 or [C2] Theorem 4.C). Here a semifir is a ring in which every finitely generated left (or right) ideal is free of unique rank; a matrix M over R is full if it is square say $n \times n$ and for any factorization $M = AB$ where $A \in {}^nR^p$, $B \in {}^pR^n$, we have $p \geq n$.

In particular $D_k\langle X\rangle$ is a semifir and so has a universal field of fractions, written $D_k\{X\}$ and also called *free field*. Likewise $D_k\langle\langle X\rangle\rangle$ is a semifir and so has a universal field of fractions, which we denote by $D_k\{\{X\}\}$. Throughout this section we shall assume that k is the precise center of D.

We call *representation* of an element f of $D_k\{X\}$ a triple $\pi = (\lambda, M, \rho)$ where $\lambda \in D^n$, $\rho \in {}^nD$ and $M \in {}^nD_k\langle X\rangle^n$ is an *affine* matrix (*i.e.* each entry of M is a polynomial of degree ≤ 1) which is full and such that $f = \lambda M^{-1}\rho$.

Similarly to what has been done in the previous section, we associate to each representation π its *left family* $f_1, \ldots, f_n \in D_k\{X\}$, with $f_i = (M^{-1}\rho)_i$, and the left D-vector space $L(\pi)$ they span in $D_k\{X\}$; symmetrically, the *right family* of π and the right D-space $R(\pi)$.

We call n the *dimension* of π, and say that π is *minimal* if it has the least possible dimension among all the representations of the given element f.

The following result holds.

PROPOSITION 1. *Each element f of $D_k\{X\}$ has a representation.*

See [C1] Theorem 7. 1.2 or [C2] Theorem 4.2.1.

We call *translation* each automorphism of $D_k\{X\}$ sending each variable x onto $x + a_x$, for some a_x in D. Such an automorphism exists, for each choice of scalars $a_x (x \in X)$; see [C1] Theorem 7.5.14 or [C2] Theorem 4.3.3. Observe that a translation fixes D.

Let t be a translation. Then t extends to matrices, and representations, because t preserves the X-degree of polynomials, and invertible matrices. Note that t sends minimal representations of f onto minimal representations of $t(f)$.

We say that two representations of dimension n (λ, M, ρ) and (λ', M', ρ') are *equivalent* if for some invertible matrices $P, Q \in \mathrm{GL}_n(D)$, one has $\lambda' = \lambda P$, $M' = QMP$, $\rho' = Q\rho$. Then they represent the same element: $\lambda' M'^{-1}\rho' = \lambda PP^{-1}M^{-1}Q^{-1}Q\rho = \lambda M^{-1}\rho$. Observe that two representations are equivalent if and only if their images under some translation are. Note that, since we consider representations (λ, M, ρ) with no special requirement on the constant matrix of M (unlike in Section 2 where it was the identity matrix), we have to consider a wider equivalence relation of representations than in Section 2 (where it was similarity, which preserves the constant matrix). Note also that a representation is equivalent to a representation in the sense of Section 2 if and only if its constant matrix is invertible over D. When a representation (λ, M, ρ) is a representation in the sense of Section 2, *i.e.* when its constant matrix is the identity, then we say that the representation is an *S-representation*. The corresponding element of $D_k\{X\}$ will then be called *S-defined*.

All we have said applies also to the case $D = k$. In this case, we write $k\langle X\rangle$ instead of $D_k\langle X\rangle$ (it is then the free associative k-algebra on X), and $k\{X\}$ instead of $D_k\{X\}$. Note that $k\{X\}$ may be identified with the subfield of rational elements in the Mal'cev-Neumann series algebra of the free group on X, by a result of Lewin, see [L].

3.2 *Embeddings of free fields.* We suppose here and in the next section that k is infinite, and that D has infinite dimension over k. This will allow us to use the specialization lemma.

SPECIALIZATION LEMMA (SEE [C2] LEMMA 6.3.1). *Let $M(x)$ be a full matrix over $D_k\langle X\rangle$. Then for some choice of scalars a_x, $x \in X$, the matrix $M(a_x)$ is invertible over D.*

We shall use the specialization lemma under the following form: if $M \in {}^nD_k\langle X\rangle^n$ is full, then $t(M) \in {}^nD_k\langle X\rangle^n$ is invertible in ${}^nD_k\langle\langle X\rangle\rangle^n$ (*i.e.* the constant matrix of $t(M)$ is invertible in ${}^nD^n$), for some translation t.

THEOREM 2. *The canonical embeddings $k\langle X\rangle \longrightarrow D_k\langle X\rangle \longrightarrow D_k\langle\langle X\rangle\rangle$ extend to the corresponding embeddings of free fields*

$$k\{X\} \longrightarrow D_k\{X\} \longrightarrow D_k\{\{X\}\}.$$

We first prove a lemma. Denote by $*$ the free product over k of k-algebras. Recall that a homomorphism of rings is called *honest* if it preserves full matrices. Note that $D_k\langle X\rangle = D * k\langle X\rangle$.

LEMMA 3. *The natural homomorphism*

$$k\langle X\rangle \longrightarrow D_k\langle X\rangle$$

is honest.

PROOF. Write $R = k\langle X\rangle$; we must show that the natural homomorphism $R \longrightarrow R * D$ is honest. Now R is a fir with universal field of fractions $F(R)$ say, and we have natural homomorphisms

$$R \longrightarrow R * D \longrightarrow F(R) * D.$$

Any full matrix over R is invertible over $F(R)$, hence invertible over $F(R) * D$ and so is full over $R * D$. ∎

PROOF OF THEOREM 2. It is enough to show that the two first embeddings are honest.

Suppose that $A(x)$ over $k\langle X\rangle$ is full, but not full over $D_k\langle X\rangle$. Then $A = P(x)Q(x)$, for some P, Q over $D_k\langle X\rangle$, of size $n \times r$, $r \times n$ respectively, with $r < n$. By Lemma 3 and the specialization lemma, $A(a_x)$ is invertible over D for some choice of values a_x in D; but $A(a_x) = P(a_x)Q(a_x)$ gives a contradiction.

Suppose now that A over $D_k\langle X\rangle$ is full. Without loss of generality, we may assume that A is invertible over $D_k\langle\langle X\rangle\rangle$ (by applying a translation t such that $t(A)$ is invertible). Certainly, A is then full over $D_k\langle\langle X\rangle\rangle$, which proves the theorem.

Note that the second assertion is also a consequence of the inertia theorem ([C1], Theorem 2.9. 15 p. 133). ■

The latter result shows that if an element f of $D_k\{X\}$ is S-defined, then, under the identification of $D_k\{X\}$ with a subfield of $D_k\{\{X\}\}$, f is actually in $D_k\langle\langle X\rangle\rangle$. This will allow us to transfer the results of Section 2 to elements of $D_k\{X\}$.

For $k\{X\}$, a little more work is needed.

LEMMA 4. *The natural mapping $D * k\{X\} \longrightarrow D_k\{X\}$ is honest, and in particular injective.*

This implies that the universal field of fractions of $D * k\{X\}$ is $D_k\{X\}$ (a fact we shall not use here).

PROOF. Let A be a full matrix over $D * k\{X\}$. Then, by Cramer's rule ([C1] Proposition 7.1.3), A is stably associated to a full matrix A_1 over $D * k\langle X\rangle = D_k\langle X\rangle$. Then A_1 is invertible in $D_k\{X\}$. Hence the image of A in $D_k\{X\}$ is invertible, being stably associated to an invertible matrix. ■

COROLLARY 5. *For D and k as before, D and $k\{X\}$ are linearly disjoint over k in $D_k\{X\}$. Let $f, f_1, \ldots, f_n$ belong to $k\{X\}$. Then the following conditions are equivalent:*

(i) f is in the left D-space generated by $f_1, \ldots, f_n$.

(ii) $t(f)$ is in the left D-space generated by $t(f_1), \ldots, t(f_n)$, for some translation t over D.

(iii) f is in the k-space generated by $f_1, \ldots, f_n$.

PROOF. By Lemma 4, the natural homomorphism $D \otimes_k k\{X\} \longrightarrow D_k\{X\}$ is injective, which proves the linear disjointness. Now, the equivalence of (i) and (ii) is clear, because t is an automorphism of $D_k\{X\}$ over D, and (i) is equivalent to (iii) by linear disjointness. ■

4. Normal forms in free fields.

4.1 *The free field $D_k\{X\}$.*

THEOREM 1. *Let (λ, M, ρ) be a minimal representation of $f \in D_k\{X\}$, and (λ', M', ρ') be another representation of f. Then (λ', M', ρ') is equivalent to a representation (λ_1, M_1, ρ_1), which has the block decomposition*

$$\lambda_1 = (\times, \lambda, 0), \quad M_1 = \begin{pmatrix} \times & 0 & 0 \\ \times & M & 0 \\ \times & \times & \times \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 \\ \rho \\ \times \end{pmatrix}.$$

In particular, minimal representations of f are equivalent.

PROOF. The matrix $M \oplus M'$ is full, so there exists a translation t such that $t(M)$ and $t(M')$ are both invertible over $D_k\langle\langle X\rangle\rangle$. Hence, without loss of generality, we may assume that M and M' are both invertible over $D_k\langle\langle X\rangle\rangle$. Then for some invertible matrices Q, Q' over D, the matrices QM and $Q'M'$ have the form $QM = 1 - N$, $Q'M' = 1 - N'$, where N, N' are linear matrices without constant terms. Then $(\lambda, QM, Q\rho)$ and $(\lambda', Q'M, Q'\rho')$ are S-representations equivalent to the original representations.

Hence, without loss of generality, we may assume that the original representations are S-representations, and in particular, f is in $D_k\langle\langle X\rangle\rangle$. Then (λ, M, ρ) is minimal as S-representation, hence the theorem follows from Proposition 2.12. ∎

Similarly to Proposition 2.10 and Corollary 2.13, we also obtain the following result.

COROLLARY 2. *Let π be a representation of f. Then π is minimal if and only if its left family is left D-linearly independent and its right family is right D-linearly independent. In this case $L(\pi)$ (resp. $R(\pi)$) depends only on f.*

4.2 *The free field $k\{X\}$.* Recall that k is infinite. Let D be a skew field, with center k, and of infinite dimension over k (if $|X| \geq 2$, one may take $D = k\{X\}$).

THEOREM 3. *Any two minimal representations of $f \in k\{X\}$ are equivalent.*

We first prove three lemmas.

LEMMA 4. *Let $M = (N, C)$ be an n by n invertible matrix over a field, where N is n by $n-1$ and C is the last column of M. Then some $n-1$ by $n-1$ submatrix of N is invertible.*

This is well-known.

LEMMA 5. *Let π be a minimal representation of an element f in $k\{X\}$. Then its left and right families are both k-linearly independent.*

PROOF. If the left family of π is not independent then, by taking an equivalent representation, we may suppose that this left family is $f_1, \ldots, f_{n-1}, 0$. Then, for $\pi = (\lambda, M, \rho)$ and $M = (p_{ij})$, we have

$$\sum_{j=1}^{n-1} p_{ij} f_j = \rho_i, \quad i = 1, \ldots, n.$$

By Lemma 4, some $n-1$ by $n-1$ submatrix of $(p_{ij})_{1\leq i\leq n, 1\leq j\leq n-1}$ is invertible in $k\{X\}$, $N = (p_{ij})_{1\leq i,j\leq n-1}$ say. Then $\big((\lambda_1, \ldots, \lambda_{n-1}), N, (\rho_1, \ldots, \rho_{n-1})^T\big)$ is a representation of f, of smaller dimension than π. ∎

LEMMA 6. *Let π, π' be two representations of the same element which have the same left and right families. Suppose that these two families are linearly independent. Then $\pi = \pi'$.*

PROOF. By Theorem 3.2, we may apply a translation t over D, and obtain two representations $t(\pi)$ and $t(\pi')$ of an element of $D_k\{X\}$ whose matrices are invertible in $D_k\langle\langle X\rangle\rangle$. By Corollary 3.5, their left and right families are left and right D-independent. Moreover, they are still equal. Now, $t(\pi)$ and $t(\pi')$ are respectively equivalent to the representations $\big(t(\lambda), Qt(M), Qt(\rho)\big)$, and $\big(t(\lambda'), Q't(M'), Q't(\rho')\big)$, where Q and Q' are chosen in $\mathrm{GL}_n(D)$

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in such a way that $Qt(M) = 1 - N$, $Q't(M') = 1 - N'$, with N, N' linear. The left family of the first representation is $\big(Qt(M)\big)^{-1}\, Qt(\rho) = t(M)^{-1}t(\rho)$, and similarly for the other. Hence, these two left families are equal, and D-independent.

These representations are therefore S-representations, with left D-linearly independent equal families. We thus may apply Lemma 2.9, and we deduce that $t(\lambda) = t(\lambda')$, $Qt(M) = Q't(M')$, $Qt(\rho) = Q't(\rho')$. Thus $\lambda = \lambda'$, $QM = Q'M'$ and $Q\rho = Q'\rho'$.

Furthermore, we know that the right families of π and π' are equal, and right D-linearly independent by Corollary 3.5. Hence $\lambda M^{-1} = \lambda' M'^{-1}$. Since $\lambda M^{-1}Q^{-1} = \lambda(QM)^{-1} = \lambda'(Q'M')^{-1} = \lambda' M'^{-1}Q'^{-1} = \lambda M^{-1}Q'^{-1}$, we conclude that $Q = Q'$, which completes the proof. ∎

PROOF OF THEOREM 3. Let π, π' be two minimal representations of f. By Lemma 5, their left and right families are k-linearly independent. Hence, in $D_k\{X\}$, they are also D-linearly independent, by Corollary 3.5. Thus π, π' are minimal as representations in $D_k\{X\}$, by Corollary 2. The same result implies that $L(\pi) = L(\pi')$ and $R(\pi) = R(\pi')$, in $D_k\{X\}$. Hence the same holds in $k\{X\}$, by Corollary 3.5. Thus, we may replace π' by an equivalent representation π'' such that the left (resp. right) families of π and π'' coincide. Then Lemma 6 shows that $\pi = \pi'$. ∎

We can also characterize minimal representations.

PROPOSITION 7. *A representation π of $f \in k\{X\}$ is minimal if and only if its left and right families are k-linearly independent. In this case, $L(\pi)$ and $R(\pi)$ depend only on f.*

PROOF. The necessity has already been established in Lemma 5. Conversely, if the two families are k-independent, they are also D-independent by Corollary 3.5, so that π is minimal as representation in $D_k\{X\}$. A fortiori it is minimal in $k\{X\}$. The last assertion follows from Theorem 3. ∎

Finally, we show how general representations are related to minimal ones.

PROPOSITION 8. *Let $\pi = (\lambda, M, \rho)$ and $\bar\pi = (\bar\lambda, \bar M, \bar\rho)$ be two representations of $f \in k\{X\}$, the second being minimal. Then the first one is equivalent to a representation having the block form*

$$(\times, \bar\lambda, 0), \quad \begin{pmatrix} \times & 0 & 0 \\ \times & \bar M & 0 \\ \times & \times & \times \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \bar\rho \\ \times \end{pmatrix}.$$

The proof follows the proof of Proposition 2.12, by replacing $1 - M$ by M, D by k, Lemma 2.11 (ii) by the lemma below, and Proposition 2.10 by Proposition 7.

LEMMA 9. *Let $\pi = (\lambda, M, \rho)$ be a representation of f with left family $0, \ldots, 0, f_{p+1}, \ldots, f_n$ (p zeros), where the latter elements are k-linearly independent. Then π is equivalent to a representation (λ', M', ρ') having the $p + (n-p)$ block form.*

$$M' = \begin{pmatrix} \times & 0 \\ \times & \times \end{pmatrix}, \quad \rho' = \begin{pmatrix} 0 \\ \times \end{pmatrix}.$$

PROOF. Let t be a translation over D such that $t(M)$ is invertible in $D_k\langle\langle X\rangle\rangle$. Then, let Q be an invertible matrix over D such that $Qt(M) = 1 - N$, N linear in $D_k\langle X\rangle$.

Then $\big(t(\lambda), Qt(M), Qt(\rho)\big)$ is an S-representation. Its left family is by Theorem 2 and Corollary 3.5 of the form $(0,\ldots,0,g_{p+1},\ldots,g_n)^T$, where $g_{p+1},\ldots,g_n$ are left D-linearly independent series in $D_k\langle\langle X\rangle\rangle$.

Hence Lemma 2.10(ii) shows that one has the block form

$$Qt(M) = \begin{pmatrix} \times & 0 \\ \times & \times \end{pmatrix}, \quad Qt(\rho) = \begin{pmatrix} 0 \\ \times \end{pmatrix}.$$

Hence QM and $Q\rho$ have the same decomposition. Since they are obtained from M and ρ by row operations over D, and since the entries of M and ρ are in $k\langle X\rangle$, we may perform some row operations over k which will produce the same rectangle of 0's. Hence, there exists Q' over k, invertible, such that $Q'M$ and $Q'\rho$ have the same block decomposition, and we take $(\lambda', M', \rho') = (\lambda, Q'M, Q'\rho)$. ∎

4.3 *An example.* Proposition 8 means that if π and $\bar\pi$ are both representations of $f \in k\{X\}$, $\bar\pi$ being minimal, then one may, by performing elementary row and column operations over k on π, decompose π into block form such that $\bar\pi$ appears as a central block. More precisely, if $\pi = (\lambda, M, \rho)$, the row operations must be performed on M and ρ, and the column operations on M and λ. We illustrate this on an example.

Let $f = (x-y^{-1})^{-1} = x^{-1}+(xyx-x)^{-1}$ (Hua's identity). Then we have, with $f' = y^{-1}f$: $1 = (x-y^{-1})f = xf - f'$ and $f - yf' = 0$. Thus

$$\begin{aligned} xf - f' &= 1 \\ f - yf' &= 0. \end{aligned}$$

This is written

$$\begin{pmatrix} x & -1 \\ 1 & -y \end{pmatrix} \begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which shows that

$$\bar\pi = \left((1,0), \begin{pmatrix} x & -1 \\ 1 & -y \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

is a representation of f. Indeed, the matrix in the middle is full (the assignment $x = y = 0$ makes it invertible), and $\bar\pi$ is necessarily minimal.

Now, consider the other expression for f. Let $g_1 = (xyx - x)^{-1}$, $g_2 = yxg_1$, $g_3 = xg_1$, $g_4 = x^{-1}$. Then

$$\begin{aligned} 1 = (xyx - x)g_1 &= -xg_1 + xg_2 \\ 0 &= g_2 - yg_3 \\ 0 &= xg_1 - g_3 \\ 1 &= xg_4. \end{aligned}$$

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This is written

$$\begin{pmatrix} -x & x & 0 & 0 \\ 0 & 1 & -y & 0 \\ x & 0 & -1 & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let M be this 4×4 matrix, $\rho = (1001)^T$ and $\lambda = (1001)$. Since $f = g_1 + g_4$, $\pi = (\lambda, M, \rho)$ is a representation of f.

Then we perform a chain of elementary operations on π (the notation is self explanatory):

$$(1001)\begin{pmatrix} -x & x & 0 & 0 \\ 0 & 1 & -y & 0 \\ x & 0 & -1 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{C_4 - C_1} (1000)\begin{pmatrix} -x & x & 0 & x \\ 0 & 1 & -y & 0 \\ x & 0 & -1 & -x \\ 0 & 0 & 0 & x \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_4} (1000)\begin{pmatrix} -x & x & 0 & 0 \\ 0 & 1 & -y & 0 \\ x & 0 & -1 & -x \\ 0 & 0 & 0 & x \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{C_2 + C_1} (1100)\begin{pmatrix} -x & 0 & 0 & 0 \\ 0 & 1 & -y & 0 \\ x & x & -1 & -x \\ 0 & 0 & 0 & x \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\xrightarrow{R_3 + R_4} (1100)\begin{pmatrix} -x & 0 & 0 & 0 \\ 0 & 1 & -y & 0 \\ x & x & -1 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{R_{23}} (1100)\begin{pmatrix} -x & 0 & 0 & 0 \\ x & x & -1 & 0 \\ 0 & 1 & -y & 0 \\ 0 & 0 & 0 & x \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

The latter representation has a $1 + 2 + 1$ triangular block decomposition, and $\bar{\pi}$ appears as its central block. This gives also a proof (rather lengthy) of Hua's identity.

The previous example raises the question whether the proofs given here are constructive. We shall not discuss this here, but refer the reader to [C2] Section 6.4, for the word problem in free fields.

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