INTRODUCTION TO VON NEUMANN ALGEBRAS AND CONTINUOUS GEOMETRY

Israel Halperin

(received June 22, 1960)

1. What is a von Neumann algebra? What is a factor (i) of type I, (ii) of type II, (iii) of type III? What is a projection geometry? And finally, what is a continuous geometry?

The questions recall some of the most brilliant mathematical work of the past 30 years, work which was done by John von Neumann, partly in collaboration with F. J. Murray, and which grew out of von Neumann's analysis of linear operators in Hilbert space.[1)]

2. Continuous geometries were discovered by von Neumann and most of our present knowledge of these geometries is due to him. The first continuous geometries which he found were projection geometries of certain rings of operators in a separable Hilbert space[2)] (see definition 3 below for the definition of a projection geometry). Roughly speaking, continuous geometries which are projection geometries are a generalization of complex-projective geometry somewhat in the way that Hilbert space is a generalization of finite dimensional Euclidean space. But to be more precise, we need some definitions.

DEFINITION 1. Suppose $\mathcal{R}$ is a ring of bounded linear operators on a Hilbert space H. Then $\mathcal{R}$ is called a von Neumann algebra if

(i) $\mathcal{R}$ contains the multiplication operators $c1$[2a)] on H for every complex number c.

See p. 286 for footnotes.

Canad. Math. Bull. vol. 3, no. 3, September 1960

https://doi.org/10.4153/CMB-1960-034-5 Published online by Cambridge University Press

(ii) $\mathcal{R}$ contains the adjoint operator T^* for each T in $\mathcal{R}$.

(iii) $\mathcal{R}$ contains every T_0 in the strong closure of $\mathcal{R}$.[3]
Von Neumann defined, for any family $\mathcal{F}$ of bounded linear operators on H, the commutant $\mathcal{F}'$ to consist of those bounded linear operators B on H such that $BA = AB$ and $BA^* = A^*B$ for all A in $\mathcal{F}$. He showed that $\mathcal{F}'$ is always a von Neumann algebra and he proved that $\mathcal{F}$ is a von Neumann algebra if and only if $(\mathcal{F}')' = \mathcal{F}$.

DEFINITION 2. A von Neumann algebra $\mathcal{R}$ is called a factor if the only operators in $\mathcal{R}$ which are also in $\mathcal{R}'$ are the multiplication operators cI on H.[4]

Von Neumann showed that a von Neumann algebra is a factor if and only if it is irreducible as a ring.[5]

DEFINITION 3. Suppose $\mathcal{R}$ is a ring of operators on a Hilbert space H. Then the projection geometry of $\mathcal{R}$, denoted $\mathcal{L} = \mathcal{L}(\mathcal{R})$, is defined to be the system 0, M, N, ..., H of those non-empty closed linear subspaces of H which have projection (i.e., orthogonal projection) operators in $\mathcal{R}$.[6] $\mathcal{L}$ is to be considered as an ordered system, ordered by the relation of inclusion $N \subset M$.[7]

If $\mathcal{R}$ is a von Neumann algebra then its projection geometry $\mathcal{L}$ is a lattice, in fact a complete lattice[8] which is orthocomplemented.[9] Furthermore, on this lattice $\mathcal{L}$ there can be defined in a natural way, a relation of equivalence, as follows: M, N are called equivalent (written $M \approx N$) if there exists in $\mathcal{R}$ an operator T which maps M isometrically onto N.[10] Von Neumann and Murray showed:

(i) If M, N in $\mathcal{L}$ are such that some T in $\mathcal{R}$ maps M in a (1,1) way into a set dense in N then $M \approx N$.[11]

(ii) The relation $\approx$ is unrestrictedly additive for orthogonal families, that is, if $M_\alpha \approx N_\alpha$ for each α and the M_α are mutually orthogonal and the N_α are mutually orthogonal, then $\bigcup M_\alpha \approx \bigcup N_\alpha$.

(iii) If M, N in $\mathcal{L}$ are such that $M \approx N_1$ for some $N_1 \subset N$ and $N \approx M_1$ for some $M_1 \subset M$, then $M \approx N$.

https://doi.org/10.4153/CMB-1960-034-5 Published online by Cambridge University Press

(iv) If $\mathcal{R}$ is a <u>factor</u> then for every pair M, N in $\mathcal{L}$ either $M \approx N_1$ for some $N_1 \subset N$ or $N \approx M_1$ for some $M_1 \subset M$.

Finally if $\mathcal{R}$ is a factor, there exists a function $d(M)$, with real finite or infinite non-negative values, defined for the elements M in $\mathcal{L}$, such that: $d(0) = 0$; $d(M) \leq d(N)$ if $M \approx N_1$ for some $N_1 \subset N$; $d(M) = d(N)$ if and only if $M \approx N$; $d(\bigcup M_\alpha) = \sum d(M_\alpha)$ if the M_α are mutually orthogonal.[12)]

3. In analysing the structure of a projection geometry two definitions are useful.

DEFINITION 4. An element $M \neq 0$ in $\mathcal{L}$ is called an <u>atom</u> if for every N in $\mathcal{L}$, the relations $0 \neq N \subset M$ imply that N coincides with M.

DEFINITION 5. An element M in $\mathcal{L}$ is called <u>finite</u> if $N \approx M$, $N \subset M$ imply that N coincides with M. If M is not finite it is called <u>infinite</u>.

The factors $\mathcal{R}$ are classified in types as follows:

(i) $\mathcal{R}$ is said to be of type I if $\mathcal{L}$ contains an atom.

(ii) $\mathcal{R}$ is said to be of type II if $\mathcal{L}$ does not contain any atom, but $\mathcal{L}$ does contain some non-zero finite element.

(iii) $\mathcal{R}$ is said to be of type III if $\mathcal{L}$ does not contain any non-zero finite element.

4. <u>Factors of type</u> I. If $\mathcal{L}$ has atoms, then any two atoms are necessarily equivalent and any non-zero M in $\mathcal{L}$ can be expressed as the union of an orthogonal family of atoms: $M = \sum \oplus (M_\alpha ; \alpha \in I)$ with the M_α orthogonal atoms. It is not difficult to prove that the cardinality of I is uniquely determined by M; $d(M)$ can be defined to be the cardinality of I (this amounts to "normalizing" the function $d(M)$ by the additional condition that $d(\text{atom})$ should be 1). It is easy to prove that this function $d(M)$ satisfies the conditions listed at the end of § 2. If $a = d(H)$, where a is a finite integer or infinite, we say $\mathcal{R}$ is of type I_a.

In the case that H is a separable Hilbert space, a must be one of $1, 2, \ldots, n, \ldots$ or $\aleph_0$. Murray and von Neumann wrote I_∞ to denote $I_{\aleph_0}$.

https://doi.org/10.4153/CMB-1960-034-5 Published online by Cambridge University Press

A factor $\mathcal{R}$ has finite linear dimension as a vector space[13] (this must be the case whenever H has finite dimension) if and only if $\mathcal{R}$ is of type I_n for some finite n. In this case $\mathcal{L}$, considered as an ordered system, can be identified with the ordered system of all linear subsets of a complex-projective geometry, the atoms of $\mathcal{L}$ being identified with the points of the projective geometry (the normalized function d(M) is then the projective geometry dimension increased by 1). If H has finite dimension then d(M) differs from the dimension of M, considered as a Hilbert space (see footnote 2), by a finite multiplicative factor; but if H has infinite dimension then all non-zero M in $\mathcal{L}$ have infinite dimension as Hilbert spaces.

The case I_a occurs if $\mathcal{R}$ is the ring of <u>all</u> bounded linear operators on an H of dimension a. Murray and von Neumann obtained the important result that every $\mathcal{R}$ of type I_a is "essentially" such a ring of all bounded linear operators on an H of dimension a. To be precise, they defined the tensor product $H_1 \otimes H_2$[14] of two Hilbert spaces H_1, H_2 and they showed: a factor $\mathcal{R}$ on a space H is of type I_a if and only if H can be expressed as a tensor product $H_1 \otimes H_2$, with H_1 of dimension a, in such a way that $\mathcal{R}$ is identified with the ring of those bounded linear operators on H which depend only on the factor H_1.[15]

5. <u>Factors of type</u> II. If $\mathcal{L}$ has no atoms then every M in $\mathcal{L}$ can be expressed as the union of two orthogonal equivalent elements; in other words, each M can be decomposed orthogonally into two equivalent parts, each of which can be called a half-M element.

If now $\mathcal{L}$ has no atoms but F is a non-zero finite element, the function d(M) can be constructed in the following way, which amounts to "normalizing" d(M) by the additional condition that d(F) = 1.

Let $F_1 = F$ and let $F_{\frac{1}{2}}$ denote a half-F_1 element. By induction, let $F_{1/2^{n+1}}$ denote a half - $F_{1/2^n}$ element for each n = 1, 2, ...

Then an element M in $\mathcal{L}$ is finite if and only if it can be expressed as a finite or countable union of orthogonal elements

$$M = \sum\nolimits^{\oplus}{}_{i=1}^{r} F_i' \oplus \sum\nolimits^{\oplus}{}_{n=1}^{\infty} F_{1/2^n}'$$

https://doi.org/10.4153/CMB-1960-034-5 Published online by Cambridge University Press

where r is finite, $r = 0, 1, 2, \ldots$, each of $F_1', \ldots, F_r'$ is equivalent to F_1, and each $F_{1/2^n}'$ is either 0 or equivalent to $F_{1/2^n}$; the value of $r + \sum_{n=1}^{\infty} (1/2^n)'$ where $(1/2^n)' = 0$ or $1/2^n$ according as $(F_{1/2^n})'$ is 0 or is equivalent to $F_{1/2^n}$, will be uniquely determined by M. Choose this value to be $d(M)$ if M is finite.

If M is infinite, then M can be expressed as the union of a countable or non-countable family I of orthogonal elements each equivalent to F_1, the cardinality of I being uniquely determined by M. Choose $d(M)$ in this case to be the cardinality of I.

The function $d(M)$ so defined for all M in $\mathcal{L}$ will, in fact, be defined uniquely and will satisfy the conditions given at the end of § 2. This construction of the function does depend on the particular choice of finite element F as normalizing element, but a different choice for F will merely change the function $d(M)$ by multiplying it by a finite non-zero positive constant; in particular, if M is infinite the value of $d(M)$ is independent of the choice of F.

If H itself is finite it is convenient to choose H to be F. Then the values of $d(M)$ are precisely all x in the interval $0 \leq x \leq 1$. In this case we say $\mathcal{R}$ is of type II_1.

If H is infinite, and $d(H) = a$ then the values of $d(M)$ are precisely all real $x \geq 0$ together with all infinite cardinals $\leq a$. In this case we say $\mathcal{R}$ is of type II_a. If H is a separable Hilbert space then a factor of type II must be II_1 or $II_{\aleph_0}$. Murray and von Neumann wrote II_∞ for $II_{\aleph_0}$.

6. Factors of type III. Suppose now that all non-zero elements in $\mathcal{L}$ are infinite. It is not difficult to prove that there exists in $\mathcal{L}$ a non-zero element M_0 such that $0 \neq N \subset M_0$ implies that $N \approx M_0$. Then every non-zero element M in $\mathcal{L}$ can be expressed as the union of a family I of orthogonal elements each equivalent to this M_0: $M = \sum \oplus_{\alpha \in I} M_\alpha$, $M_\alpha \approx M_0$ for all α. If M is $\neq 0$ and is not equivalent to M_0 then it is not difficult to prove that the cardinality of I is uniquely determined. Define $d(M: M_0)$ to be the cardinality of I. If $M = M_0$ the cardinality of I is not uniquely determined; define a to be the supremum of all cardinalities of such I (necessarily $a \geq \aleph_0$).

Now if H itself is equivalent to M_o, the function d(M) can (and must) be defined by: d(0) = 0; for all $M \neq 0$, d(M) = b for some fixed $b \geq a$. In this case $\mathcal{R}$ is said to be of type III_a. If H is a separable Hilbert space, every factor of type III must be of type $III_{\aleph_0}$; Murray and von Neumann wrote III_∞ for this type $III_{\aleph_0}$.

If M_o can actually be expressed as the union of a family I of a orthogonal elements each equivalent to M_o (that is, if the cardinality a is attained) but H is not equivalent to M_o then d(M) can (and must) be defined by: d(0) = 0; d(M) = a if M is equivalent to M_o; $d(M) = d(M: M_o)$ if M is not equivalent to M_o. In this case, let $b = d(H: M_o)$. Then d(M) takes on as values: 0 and every cardinal $\aleph$ satisfying $a \leq \aleph \leq b$. In this case the factor $\mathcal{R}$ is said to be of type $III_{(a,\ b)}$.

Finally, suppose if possible that H is <u>not</u> equivalent to M_o and that a is <u>not</u> attained (necessarily $a > \aleph_0$ and a has the property that it <u>cannot</u> be the sum of a set of cardinals $\sum_{\alpha \in J} a_\alpha$ with each $a_\alpha < a$ and cardinality of $J < a$). Then there will be an element M_1 in $\mathcal{L}$ such that $d(M_1 : M_o) = a$ and yet M_1 is not equivalent to M_o. But for any function d(M) with the properties listed at the end of § 2, it is easy to see that $d(M_o)$ would equal $d(M_1)$. <u>Thus if a is not attained and H is not equivalent to M_o there can be no such function d(M).</u> If $d(H: M_o) = b$ (necessarily $b > a > \aleph_0$), $\mathcal{R}$ is said to be of type $III_{(\acute{a},\ b)}$, the sloped mark indicating that a is not attained. It is not yet clear whether this type $III_{(\acute{a},\ b)}$ actually occurs.

7. <u>Examples of factors</u>. As stated at the end of § 4, for each finite or infinite a(a = 1, 2, ..., $\aleph_0$, ...) and for each Hilbert space H of dimension divisible by a, there exists a factor of type I_a on H. Two factors of type I_a with the same a are isomorphic under a suitable isomorphism of their H-spaces if their H spaces have the same dimension; in any case the factors are isomorphic as rings with a *-operation (a factor is always defined with respect to a particular Hilbert space H but two factors can be *-ring isomorphic even though they are defined on Hilbert spaces of different dimension).

The construction of factors of types II and III is a more difficult matter. Von Neumann and Murray first constructed examples of factors II_1 and $II_{\aleph_0}$ on a separable Hilbert space by using measure-theoretic methods [7]. In a later paper [8] von Neumann defined tensor products $\prod \otimes H_\alpha$ with an infinite number of factor spaces H_α and used an infinite tensor

https://doi.org/10.4153/CMB-1960-034-5 Published online by Cambridge University Press

product $\prod\otimes H_\alpha$ with each H_α a 4-dimensional Euclidean space to construct a wide variety of factors. It turned out that among these tensor product factors, were (i) factors of type II_1 on spaces H of arbitrary dimension $\geq \aleph_0$, (ii) factors of type II_a with fixed but arbitrarily given a $\geq \aleph_0$, on spaces H of arbitrary dimension $\geq$ a, and (iii) factors of type III_a for fixed but arbitrarily given a on spaces H of arbitrary dimension $\geq$ a.[15a)]

The factors constructed by von Neumann with the help of infinite tensor products are not difficult to describe, but the verification of their types, as given by von Neumann, is involved. In fact, von Neumann actually made the verification only for the II_1 factor on a separable Hilbert space by identifying the factor on a tensor product space with a certain II_1 measure-theoretic factor constructed in a previous Murray-von Neumann paper (the $II_{\aleph_0}$ case then follows easily).

Later, von Neumann [9] established the existence of a factor of type $III_{\aleph_0}$ on a separable Hilbert space, again by measure-theoretic methods, and asserted that some of the factors previously constructed by him on infinite tensor product spaces could be identified with some of these (measure-theoretic) factors $III_{\aleph_0}$.

The next two sections are devoted to a discussion of these tensor product factors, avoiding the more involved measure-theoretic apparatus.

8. Infinite tensor product spaces. Suppose H_α is a fixed Hilbert space for each α in a set of indices J (assume each H_α has dimension ≥ 2), and suppose a fixed unit vector ϕ_α has been selected in each H_α . Now let $\prod'\otimes H_\alpha$ consist of all finite formal sums $v = \sum_i (\prod\otimes f_\alpha{}^i)$, where for each α in J and for each i, $f_\alpha{}^i$ is in the corresponding H_α and, with at most a finite number of exceptions, $f_\alpha{}^i = \phi_\alpha$. If $w = \sum_j (\prod\otimes g_\alpha{}^j)$ define $(v|w)$ to be $\sum_{i,j} \prod_\alpha (f_\alpha{}^i | g_\alpha{}^j)$ (note that $\sum_{i,j}$ is a finite sum, and for each i, j, the product $\prod_\alpha (f_\alpha{}^i | g_\alpha{}^j)$ is a finite product since, with a finite number of exceptions, $(f_\alpha{}^i | g_\alpha{}^j) = (\phi_\alpha | \phi_\alpha) = 1$). Note that the vector $g = \prod\otimes \phi_\alpha$ is in $\prod'\otimes H_\alpha$.

In $\prod'\otimes H_\alpha$, for each complex number c identify cv with $\sum_i \prod\otimes (f_\alpha{}^i)'$ where for each i, $(f_\alpha{}^i)' = cf_\alpha{}^i$ for one of the α and $(f_\alpha{}^i)' = f_\alpha{}^i$ for all other α; also identify v and w if

$(v-w|v-w) = 0$. With these identifications $\Pi'\otimes H_\alpha$ becomes a linear space with inner product; its completion is called the tensor product $\Pi\otimes H_\alpha$. This tensor product depends not only on the H_α but also on the particular choice of the ϕ_α. However, different choices of the ϕ_α give tensor product spaces which are isomorphic, all having the same dimension[15b) (von Neumann called this space an incomplete direct product; he wrote $\Pi\otimes H_\alpha$ to denote a much larger space which depends only on the H_α, not on a choice of ϕ_α and he showed that his larger space decomposed in a unique way into orthogonal parts each of which is an incomplete direct product or tensor product as defined above).

Each bounded linear operator T on a single factor space H_{α_0} determines a bounded linear operator $\bar{T}$ on the space $H = \Pi\otimes H_\alpha$ in the following way. If $v = \Pi\otimes f_\alpha$ with $f_\alpha = \phi_\alpha$ for all but a finite number of α, let $\bar{T}v = \Pi\otimes f'_\alpha$ where $f'_\alpha = f_\alpha$ if $\alpha \neq \alpha_0$ and $f'_{\alpha_0} = Tf_{\alpha_0}$. Now $\bar{T}$ is uniquely defined by linearity for each v in $\Pi'\otimes H_\alpha$ (the corresponding $\bar{T}v$ is also in $\Pi'\otimes H_\alpha$) and by continuity $\bar{T}$ is defined uniquely on all of H.

Let $\mathcal{B}$ denote the set of all bounded linear operators on H and for fixed α let $\mathcal{B}_\alpha$ denote the set of all bounded linear operators on H_α. If $\mathcal{F}_\alpha$ is a subset of $\mathcal{B}_\alpha$ let $\bar{\mathcal{F}}_\alpha$ denote the set of all $\bar{T}$ as T varies over $\mathcal{F}_\alpha$. If for each α, $\mathcal{F}_\alpha$ is a subset of $\mathcal{B}_\alpha$ let $\Pi'\otimes\bar{\mathcal{F}}_\alpha$ denote the set of those operators from $\mathcal{B}$ which can be expressed as $\Sigma_i(\Pi_\alpha \bar{T}_\alpha^{\,i})$ with Σ a finite sum and each $\Pi_\alpha \bar{T}_\alpha^{\,i}$ containing only a finite number of factors and each $\bar{T}_\alpha^{\,i}$ in $\bar{\mathcal{F}}_\alpha$; and let $\Pi\otimes\bar{\mathcal{F}}_\alpha$ denote the strong closure[16) of $\Pi'\otimes\bar{\mathcal{F}}_\alpha$.

It is not difficult to prove the following.

(i) If T_1 and T_2 are in different $\bar{\mathcal{B}}_\alpha$ then $T_1T_2 = T_2T_1$.

(ii) For each fixed α, $(\bar{\mathcal{B}}_\alpha)'' = \bar{\mathcal{B}}_\alpha$, i.e. $\bar{\mathcal{B}}_\alpha$ is a von Neumann algebra.

(iii) The only von Neumann algebra which contains all the $\bar{\mathcal{B}}_\alpha$ is $\mathcal{B}$.

(iv) For fixed α, $\bar{\mathcal{F}}_\alpha$ is a von Neumann algebra (respectively, a factor) on H if and only if $\mathcal{F}_\alpha$ is a von Neumann algebra (respectively a factor) on H_α. If $\mathcal{F}_\alpha$, $\bar{\mathcal{F}}_\alpha$ are factors, they are of the same type.

(v) If for each α, $\mathcal{F}_\alpha$ is a von Neumann algebra on H then $\Pi \otimes \overline{\mathcal{F}}_\alpha$ is a von Neumann algebra on H and it is the smallest one which contains all $\overline{\mathcal{F}}_\alpha$.

(vi) If for each α, $\mathcal{F}_\alpha$ is a factor on H_α then $\Pi \otimes \overline{\mathcal{F}}_\alpha$ is a factor on H.

(vi) immediately presents a challenging problem, as yet not completely solved, namely to determine the type of the factor $\Pi \otimes \overline{\mathcal{F}}_\alpha$, given the factor $\mathcal{F}_\alpha$ in each H_α. Von Neumann emphasized that (vi) was also important because it opened up the possibility of constructing complicated factors on H by starting from known factors on the individual H_α. In fact von Neumann constructed factors of the form $\Pi \otimes \overline{\mathcal{F}}_\alpha$ of types II_1, $II_{\aleph_0}$ and $III_{\aleph_0}$ respectively by using spaces H_α, each an Euclidean space of dimension 4, and in each H_α a factor $\mathcal{F}_\alpha$ of type I_2.

We shall discuss these examples of von Neumann in the next section.

9. <u>Factors with a trace function</u>. Let $\mathcal{R}$ be any ring of operators on a Hilbert space H such that $\mathcal{R}$ contains the identity operator on H and $\mathcal{R}$ contains A* if A is in $\mathcal{R}$. Then a function θ with complex numbers (finite!) as values, is called a <u>finite trace</u> on $\mathcal{R}$ if

(i) $\theta(A)$ is defined for each A in $\mathcal{R}$,

(ii) $\theta(A + B) = \theta(A) + \theta(B)$ for all A, B in $\mathcal{R}$,

(iii) $\theta(A^*A)$ is real and > 0 for every $A \neq 0$ in $\mathcal{R}$,

(iv) $\theta(1) = 1$,

(v) $\theta(AB) = \theta(BA)$ for all A, B in $\mathcal{R}$.

Suppose now that $\mathcal{R}$ is known to be a factor and that it possesses a trace. <u>Then it must be of type</u> II_1 <u>or</u> I_n <u>for some finite</u> n, and if P_M denotes the orthogonal projection operator on a closed linear subspace M then the function $\theta(P_M)$ can be taken as d(M). To see this, recall that $M \approx N$ means that there exists an operator T in $\mathcal{R}$ which maps M isometrically onto N. Let $W = P_N T P_M$. Then $W^*W = P_M$, $WW^* = P_N$ so $\theta(P_M) = \theta(W^*W) = \theta(WW^*) = \theta(P_N)$; i.e. $M \approx N$ implies

https://doi.org/10.4153/CMB-1960-034-5 Published online by Cambridge University Press

that $\theta(P_M) = \theta(P_N)$. From this fact and (ii), (iii), and (iv) it follows easily that $d(M) = k\,\theta(P_M)$ for some finite positive constant k. This implies that H is finite so that $\mathcal{R}$ is necessarily of type II_1 or I_n for some finite n.

Now make the following definition.

DEFINITION 6. Let $\mathcal{R}$ be a ring of operators on a space H such that $\mathcal{R}$ contains the identity operator on H and $\mathcal{R}$ contains A^* if A is in $\mathcal{R}$. Then a unit vector g in H is called a <u>trace-vector</u> for $\mathcal{R}$ if $(ABg\,|\,g) = (BAg\,|\,g)$ for all A, B in $\mathcal{R}$.

It is easy to see that if $\mathcal{R}$ possesses a trace-vector g then $(Ag\,|\,g)$ is a finite trace on R; hence if $\mathcal{R}$ is a factor it must be of type I_n or II_1.

Finally, suppose that for each α, $\mathcal{R}_\alpha$ is a factor on H_α and that ϕ_α is a trace-vector in H_α for $\mathcal{R}_\alpha$ (the ϕ_α are the vectors in H_α involved in the construction of $\Pi\otimes H_\alpha$). Let g be the vector $\Pi\otimes\phi_\alpha$ in $\Pi\otimes H_\alpha$. Then it is easy to prove that $(ABg\,|\,g) = (BAg\,|\,g)$ for all A, B in $\Pi'\otimes\bar{\mathcal{R}}_\alpha$ and hence for all A, B in $\mathcal{R} = \Pi\otimes\bar{\mathcal{R}}_\alpha$.

This shows that $\mathcal{R}$ must then be a factor of type I_n or II_1. It is easy to prove that $\mathcal{R}$ is of type I_n if and only if all $\mathcal{R}_\alpha$ are of type I_{a_α} for finite a_α such that $\Pi_\alpha a_\alpha$ is finite (necessarily = n; then $\mathcal{R}_\alpha$ is of type I_1 for all but a finite number of α). <u>In all other cases, $\mathcal{R}$ is necessarily of type II_1. This indicates a method of constructing II_1 factors on a space H of arbitrary dimension</u> $a \geq \aleph_0$. Namely, first choose a set J of indices α so that J has cardinality a. Then choose for each α some space H_α of finite dimensionality $a_\alpha \geq 2$ and choose a unit vector ϕ_α in H_α; $\Pi\otimes H_\alpha$ will then have dimension a. Next, choose if possible in each H_α a factor $\mathcal{R}_\alpha$ of some type I_{n_α} with $n_\alpha \geq 1$ so that ϕ_α is a trace-vector in H_α for $\mathcal{R}_\alpha$ and so that the set J' of α with $n_\alpha > 1$ has cardinality a' with $\aleph_0 \leq a' \leq a$. Then $\mathcal{R} = \Pi\otimes\bar{\mathcal{R}}_\alpha$ will be a factor of type II_1 on the a-dimensional space $\Pi\otimes H_\alpha$ as desired.

To complete the selection of H_α, ϕ_α and $\mathcal{R}_\alpha$, proceed as follows: Let a_α be a product $n_\alpha m_\alpha$ with $n_\alpha \leq m_\alpha$ and let H_α be itself a tensor product $H_\alpha = E_\alpha\otimes F_\alpha$ with E_α an Euclidean space with a complete orthonormal basis $h_1, \ldots, h_{n_\alpha}$ and F_α an Euclidean space with a complete orthonormal basis $k_1, \ldots, k_{m_\alpha}$. Let $\mathcal{R}_\alpha$ be the ring of bounded linear operators on H_α which depend only on E_α and let

$$\phi_\alpha = 1/\sqrt{n_\alpha}\ \{(h_1 \otimes k_1) + (h_2 \otimes k_2) + \ldots + (h_{n_\alpha} \otimes k_{n_\alpha})\} .$$

It is easily verified that ϕ_α is a trace-vector in H_α for $\mathcal{R}_\alpha$ and so $\Pi \otimes \mathcal{R}_\alpha$ is a II_1 factor on $\Pi \otimes H_\alpha$. The simplest case of this situation is when all E_α and F_α are 2-dimensional Euclidean spaces and J has cardinality $\aleph_0$. Note that in the general construction the II_1 factor on $\Pi \otimes H_\alpha$ depends on the cardinality of J'.

To construct a II_b factor on a space H of dimension a (assuming $a \geq b \geq \aleph_0$) proceed as follows. Let $\mathcal{R}_1$ be a II_1 factor on a space H_1 of dimension b, let $\mathcal{B}_2$ be the ring of all bounded linear operators on a separable space H_2, and let $\mathcal{R}_3$ be the I_1 factor consisting of multiplication operators cI on a space H_3 of dimension a. Then $\mathcal{R}_1 \otimes \mathcal{B}_2 \otimes \mathcal{R}_3$ is a factor of type II_b on the space $H = H_1 \otimes H_2 \otimes H_3$ of dimension a.

The construction of factors of type III is easy to describe using tensor products. Again let J be an infinite set of indices α and for each α let H_α be a tensor product $E_\alpha \otimes F_\alpha$ where E_α, F_α have orthonormal complete bases h_1, h_2 and f_1, f_2 respectively. Let $\mathcal{R}_\alpha$ be the ring of bounded linear operators on H_α which depend only on E_α.

But now let $\phi_\alpha = c_1(h_1 \otimes f_1) + c_2(h_2 \otimes f_2)$ with $0 < c_1 < c_2$ and $c_1^2 + c_2^2 = 1$. The fact that $c_1 < c_2$ prevents $g = \Pi \otimes \phi_\alpha$ from being a trace-vector for $\mathcal{R} = \Pi \otimes \mathcal{R}_\alpha$ on $\Pi \otimes H_\alpha$, and von Neumann indicated that this $\mathcal{R}$ is in fact a factor of type III. However, the verification of this fact and construction of factors $III_{(a,b)}$ for $\aleph_0 \leq a \leq b$ will not be given in the present article.

10. Projection geometries. For every von Neumann algebra $\mathcal{R}$ on a space H the projection geometry $\mathcal{L}$ is a kind of generalization of complex-projective geometry (in the case $\mathcal{R}$ is a factor of type I_n, $\mathcal{L}$ is actually a projective geometry). It has already been observed that $\mathcal{L}$ is a complete, orthocomplemented lattice with a special congruence relation. It is not difficult to show that $\mathcal{L}$ is irreducible as a lattice if and only if $\mathcal{R}$ is a factor.

https://doi.org/10.4153/CMB-1960-034-5 Published online by Cambridge University Press

Now $\mathcal{L}$ is not in general a distributive lattice, that is

$$(*) \quad M \cap (N \cup Q) = (M \cap N) \cup (M \cap Q)$$

is not true, in general. In fact, if $\mathcal{R}$ is a factor, the only M for which (*) holds for all N, Q are: $M = 0$ and $M = H$. Thus, even the projective geometry lattices are not distributive.

But if $\mathcal{L}$ is a projective geometry, $\mathcal{L}$ does satisfy a restricted form of (*); namely (*) with the restriction $M \supset N$ (this restricted relation (*) is called the modular law). It turns out that for a general factor $\mathcal{R}$, $\mathcal{L}$ satisfies the modular law if and only if $\mathcal{R}$ is of type I_n with a finite n or of type II_1.

Von Neumann observed that if $\mathcal{R}$ is of type I_n or II_1 then $\mathcal{L}$ is not only modular, but it satisfies the continuity conditions:

$$(**) \quad M_1 \subset M_2 \subset \dots \subset M_\alpha \subset \dots \text{ implies } \bigcup(M_\alpha \cap N) = (\bigcup M_\alpha) \cap N \quad \text{for all } N,$$

$$M_1 \supset M_2 \supset \dots \supset M_\alpha \supset \dots \text{ implies } \bigcap(M_\alpha \cup N) = (\bigcap M_\alpha) \cup N \quad \text{for all } N,$$

(Kaplansky later proved in [3], see also [1], that every orthocomplemented complete modular lattice necessarily satisfies (**)).

Von Neumann considered the II_1 projection geometries to be a natural and very important generalization of classical projective geometry and he succeeded in characterizing them by lattice theoretic axioms (this manuscript has never been published but an abstract will be included in the forthcoming Collected Works of J. von Neumann). However, these projection geometries of II_1 factors are closely associated with the complex number system and von Neumann found it convenient to axiomatize a wider class of lattice-geometries which included the I_n and II_1 projection geometries but no other projection geometries. This wider class of geometries he called continuous geometries. His axioms for a continuous geometry L were: L should be a complete lattice which is modular, satisfies (**) and is complemented (for a lattice to be complemented means: for each a in L there exists at least one a' such that $a \cup a' = 1$ and $a \cap a' = 0$). Of course, if a lattice is orthocomplemented it is complemented.

A detailed account of continuous geometries is given in von Neumann's forthcoming Princeton lectures [13]. The first main theorem of von Neumann for continuous geometry was: if the continuous geometry is irreducible then there exists a unique dimension function d(a) defined for all a in L with the properties: $0 \leq d(a) \leq 1$ for each a in L; $d(0) = 0$; $d(1) = 1$; $d(a \cup b) = d(a) + d(b)$ if $a \cap b = 0$.

REFERENCES

1. I. Amemiya and I. Halperin, Complemented modular lattices, Canad. J. Math. 11 (1959), 481-520.

2. J. Dixmier, Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann), Gauthier-Villars, Paris, 1957.

3. Irving Kaplansky, Any orthocomplemented complete modular lattice is a continuous geometry, Ann. of Math. 61 (1955), 524-541.

4. J. von Neumann, Allgemeine Eigenwerttheorie hermitescher Funktionaloperatoren, Math. Ann. 102 (1929), 49-131.

5. J. von Neumann, Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren, Math. Ann. 102 (1929), 370-427.

6. J. von Neumann, On a certain topology for rings of operators, Ann. of Math. 37 (1936), 111-115.

7. F.J. Murray and J. von Neumann, On rings of operators, Ann. of Math. 37 (1936), 116-229.

8. J. von Neumann, On infinite direct products, Compositio Math. 6 (1938), 1-77.

9. J. von Neumann, On rings of operators, III, Ann. of Math. 41 (1940), 94-161.

10. F.J. Murray and J. von Neumann, On rings of operators, IV, Ann. of Math. 44 (1943), 716-808.

https://doi.org/10.4153/CMB-1960-034-5 Published online by Cambridge University Press

11. J. von Neumann, On some algebraical properties of operator rings, Ann. of Math. 44 (1943), 709-715.

12. J. von Neumann, On rings of operators. Reduction theory, Ann. of Math. (1949), 401-485.

13. J. von Neumann, 5 Notes in Proc. Nat. Acad. Sci. U.S.A., 1936-37, and Continuous Geometry, mimeographed lecture notes, The Institute for Advanced Study, Princeton, 1935-37, published by Princeton University Press, 1960.

14. L. H. Loomis, The lattice theoretic background of the dimension theory, Mem. Amer. Math. Soc. (1955).

15. S. Maeda, Dimension functions on certain general lattices, J. Sci. Hiroshima Univ. 19 (1955), 211-237.

FOOTNOTES

1. See the references [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

2. In modern terminology, a Hilbert space H is any vector space with the complex numbers as scalars, possessing an inner product $(x|y)$ and complete in the corresponding metric: distance $(x,y) = \|x - y\| = (x - y \mid x - y)^{\frac{1}{2}}$. It is usually assumed that H contains at least one non-zero vector. The dimension of H is the cardinality of a complete orthonormal basis; it may be finite (Euclidean space), countable (separable Hilbert space) or non-countable (hyper-Hilbert space).

2a. The multiplication operator cI is, of course: $cI(x) = cx$ for all x in H.

3. If $\mathcal{F}$ is a set of bounded linear operators, then a bounded linear operator T_o on H is said to be in the strong closure of $\mathcal{F}$ if for every given finite set of vectors $x_1, \ldots, x_m$ in H and arbitrarily given $\varepsilon > 0$ there exists a T in $\mathcal{F}$ such that $\|(T_o - T)\, x_i\| < \varepsilon$ for $i = 1, \ldots, m$. The condition (iii) holds automatically whenever (i) holds if H has finite dimension, more generally if $\mathcal{R}$ has finite linear dimension as a vector space (this means: $\mathcal{R}$ consists of the linear combinations of a fixed finite set of operators).

4. In other words: the only operators in $\mathcal{R}$ which permute with all members of $\mathcal{R}$, are the multiplication operators cl on H.

5. Irreducibility of a ring $\mathcal{R}$ with identity element means that it is not possible to find two subrings $\mathcal{R}_1$, $\mathcal{R}_2$ of $\mathcal{R}$, neither consisting only of the zero element in $\mathcal{R}$, such that $T_1T_2 = T_2T_1 = 0$ whenever T_1 is in R_1 and T_2 is in R_2, and such that each T in $\mathcal{R}$ can be expressed as a sum $T = T_1 + T_2$ with T_1 in $\mathcal{R}_1$ and T_2 in $\mathcal{R}_2$.

6. The zero in $\mathcal{L}$, denoted 0, consists of only the zero vector in H.

7. $N \subset M$ (equivalently: $M \supset N$) means that N is part or all of M, that is, every vector in N is also in M.

8. An ordered system M, N, ... is called a complete lattice if: for every family $\{M_\alpha\}$ in the system there is a least M in the system such that $M \supset M_\alpha$ for all α (this M is called the union of the M_α and denoted $\bigcup M_\alpha$) and a greatest M in the system such that $M \subset M_\alpha$ for all α (this M is called the intersection of the M_α and denoted $\bigcap M_\alpha$). In the case of a projection geometry $\mathcal{L}$, the union $\bigcup M_\alpha$ is actually the closed linear subspace of H generated by the M_α and the intersection $\bigcap M_\alpha$ is actually the set-intersection of the M_α; 0 is the least element (the zero) in $\mathcal{L}$, and H is the greatest element (the unit) in $\mathcal{L}$.

9. A lattice with zero 0 and unit 1 is said to be <u>orthocomplemented</u> if for each M in the lattice there is assigned an element M' such that $M \cup M' = 1$, $M \cap M' = 0$ and $(M')' = M$. For a projection geometry, M' can be taken to be the orthogonal complement of M with respect to H.

10. That is, (i) for each x in M, Tx is in N and $\|Tx\| = \|x\|$, and (ii) N is the set of all Tx when x varies over M.

11. Define W' to be the operator which, for each x in M, maps $(T^*T)^{\frac{1}{2}}x$ onto Tx and such that $Wx = 0$ for each x orthogonal to M. Then W' is defined on a set dense in H and is bounded and linear. By continuity W' has an extension W defined on all of H and it is not difficult to see that W is in $\mathcal{R}$ and W maps M <u>isometrically</u> onto N.

https://doi.org/10.4153/CMB-1960-034-5 Published online by Cambridge University Press

12. To be precise, such a function d(M) exists if $\mathcal{R}$ is a factor (cf. Definition 5) of type I or II. But among the factors of type III there may be exceptional cases in which such a function d(M) fails to exist (see the last sentence of § 6 below). It is not yet known whether such exceptional cases do in fact exist. The original discussion of Murray and von Neumann assumed H to be a separable Hilbert space and for this H there is no difficulty with exceptional cases. Von Neumann remarked in a later paper that separability of H was not involved essentially in the discussion of factors but he was probably aware of the possible difficulty connected with exceptional cases among the factors of type III for non-separable Hilbert space.

13. See footnote 3 above.

14. Let $H_1 \otimes' H_2$ consist of all finite formal sums $v = \Sigma_i (f_i \otimes g_i)$ with all f_i in H_1 and g_i in H_2. If $w = \Sigma_j (h_j \otimes k_j)$, define $(v|w)$ to be $\Sigma_{ij}(f_i|h_j)(g_i|k_j)$. Now identify cv with $\Sigma_i (cf_i \otimes g_i)$ for every complex number c and identify v with w if $(v - w | v - w) = 0$. With these identifications $H_1 \otimes' H_2$ becomes a vector space with inner product (a pre-Hilbert space) and its completion is called the tensor product, denoted $H_1 \otimes H_2$.

15. A bounded linear operator T on $H_1 \otimes H_2$ is said to depend only on H_1 if there exists a bounded linear operator T_1 on H_1 such that $T(f \otimes g) = (T_1 f) \otimes g$ for all f in H_1 and g in H_2. Then T_1 determines Tv by linearity for all v in $H_1 \otimes' H_2$ and by continuity for all v in $H_1 \otimes H_2$.

15a. A simple construction of factors II, using groups, is given in [10].

15b. Let the Hilbert-space dimension of a Hilbert space H be denoted by D(H). Then $D(\Pi \otimes H_\alpha) = \Pi D(H_\alpha)$ if $\Pi D(H_\alpha)$ is finite; otherwise $D(\Pi \otimes H_\alpha) = \Sigma D(H_\alpha)$.

16. See footnote 3. above.

Queen's University

https://doi.org/10.4153/CMB-1960-034-5 Published online by Cambridge University Press