Introduction. In this paper we find a new bound for the function $g(h)$, for which $|A(G)|_p \geq p^h$ whenever $|G| \geq p^{\rho(h)}$, $G$ a finite $p$-group. The existence of such a function was first conjectured by W. R. Scott in 1954, who proved that $g(2) = 3$. In 1956 Ledermann and Neumann proved that in the general case of finite groups $g(h) \leq (h - 1)^2 \cdot p^{h-1} + h$ [10]. Since then, J. A. Green, J. C. Howarth and K. H. Hyde have reduced this bound considerably. The best (least) bound to date for finite $p$-groups was obtained by K. H. Hyde [9]. He proved that $g(h) = h(h - 3) + 3$ for $h \geq 5$ and $g(h) = h + 1$ for $h \leq 4$. For finite non-abelian $p$-groups, we improve this bound to: $g(h) = \frac{1}{2}h^2$ for $h \geq 13$, $g(h) = 2h - 5$ for $5 < h \leq 8$, $g(h) = h$ for $h \leq 5$ and for $8 < h \leq 12$ we prove that $g(9) = 14$, $g(10) = 17$, $g(11) = 20$, $g(12) = 23$.

The following notation is used: $G$ is taken to be a finite non-abelian $p$-group with commutator subgroup $G'$ and center $Z$. The order of $G$ is denoted by $|G|$ and $|H|_p$ is the largest power of $p$ dividing $|H|$. Hom $(G, Z)$ is the set of all homomorphisms of $G$ into $Z$ and $A(G), A_c(G), I(G)$ are the groups of automorphisms, central automorphisms, inner automorphisms of $G$ respectively. $G$ is called a $PN$-group if it has no non-trivial abelian direct factor. We denote the lower and the upper central series of $G$ by

\[ G = L_0 > L_1 = G' > \ldots > L_c = 1 \quad \text{and} \]
\[ G = Z_c > Z_{c-1} > \ldots > Z_1 = Z > Z_0 = 1. \]

Throughout this paper $c$ is the class of $G$ and we take the invariants of $G/G'$ to be $m_1 \geq m_2 \geq \ldots \geq m_t \geq 1$ and the invariants of $Z$ to be $k_1 \geq k_2 \geq \ldots \geq k_s \geq 1$, where $t$ and $s$ are the numbers of invariants of $G/G'$ and $Z$ respectively. For non-cyclic $p$-groups $G$, $t \geq 2$, as $G/G'$ is cyclic, if and only if $G$ is cyclic. Also we take $|G/G'| = p^m$ and $|Z| = p^k$.

The cyclic group of order $p^r$ is denoted by $C_{p^r}$.

It has been conjectured that for finite non-cyclic $p$-groups of order greater than $p^2$, $g(h) \leq h$. This has been established for abelian $p$-groups, for $p$-groups of class two and for some other special classes of finite $p$-groups. I believe that in the general case the above conjecture is not valid and that $g(h) > h$.

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For $c = 2$ we have $g(h) \leq h$ [5]. Therefore we shall assume that $c > 2$, whenever $g(h) > h$.

**Lemma 1.** If $G$ is a $PN$-group, then $|A_\epsilon(G)| = p^a$, where

$$a = \sum_{j, t} \min (m_j, k_i) \quad \text{and}$$

(i) $a \geq jk + (t - j)s$, if $m_j \geq k_i$ for some $j, t \geq j \geq 1$.

(ii) $a \geq im + (s - i)t$, if $k_i \geq m_1$ for some $i, s \geq i \geq 1$. In particular, if $k_i \geq m_1 > k_{i+1}$ then $a \geq im + k - (k_1 + \ldots + k_i) + (t - 1)(s - i)$, and if $k_i \geq m_1$, then $a = sm$.

**Proof.** Since $G$ is a $PN$-group, $|A_\epsilon(G)| = |\text{Hom}(G, Z)|$ [1]. Hence

$$|A_\epsilon(G)| = |\text{Hom}(G, Z)| = |\text{Hom}(G/L_1, Z)|$$

$$= \prod \text{Hom}(C_{p^{m_j}}, C_{p^{k_i}}) = p^a,$$

where

$$a = \sum_{j, t} \min (m_j, k_i).$$

Therefore

$$a \geq jk + \sum_{x=j+1, i=1}^{t, s} \min (m_x, k_i) \geq jk + (t - j)s \quad \text{for } m_j \geq k_i.$$

Similarly $a \geq im + (s - i)t$ for $k_i \geq m_1$. If $k_i \geq m_1 > k_{i+1}$,

$$a \geq im + \sum_{j=i+1}^{s} k_f + \sum_{j=2, j=i+1}^{s} \min (m_j, k_f) \geq im + k - (k_1 + \ldots + k_i) + (t - 1)(s - i).$$

For $k_i \geq m_1$, $\min (m_j, k_i) = m_j$, so that $a = ms$.

**Lemma 2.** Let $G$ be a $PN$-group of class $c > 2$. Then $|A_\epsilon(G)| \cdot p^{c-1}$ is a factor of $|A(G)|$.

**Proof.** Since $G/Z_{c-1}$ is not cyclic and $|Z_i/Z_{i-1}| \geq p, i = 1, \ldots, c - 1,

$$|G/Z_2| \geq p^{c-1} \quad \text{and}$$

$$|A(G)| \geq |A_\epsilon(G) \cdot I(G)| = |A_\epsilon(G)| \cdot |I(G)|/|A_\epsilon(G) \cap I(G)|$$

$$= |A_\epsilon(G)| \cdot |G/Z_2| \geq |A_\epsilon(G)| \cdot p^{c-1}.$$ 

From Lemmas 1 and 2 we get:

**Lemma 3.** If $G$ is a $PN$-group of class $c > 2$, then

$$|A(G)| \geq p^{a+c-1} \geq p^{2s+c-1},$$

where $s$ is the number of invariants of $Z$. 

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Lemma 4. If $G$ is a 2-generator finite $p$-group of class $c$, then
\[ Z_{c-1} \subseteq \Phi(G) \quad \text{and} \quad \exp(G/Z_{c-1}) = \exp(L_{c-1}). \]

Proof. If $Z_{c-1} \nsubseteq \Phi(G)$, we can find two generators $a$ and $b$ for $G$ with $a \in Z_{c-1}$. Then all $(c - 1)$-fold commutators in $a$ and $b$ are 1 and so $G$ has class less than $c$, a contradiction. For $a_0, a_1, \ldots, a_{c-1} \in G$,
\[ [a_0, a_1, \ldots, a_{c-1}]^n = [a_0^n, a_1, \ldots, a_{c-1}] \]
for any positive integer $n$. This implies that $\exp(G/Z_{c-1}) = \exp(L_{c-1}).$

Lemma 5. If $m_1 \geq m_2 \geq \ldots \geq m_t \geq 1$ are the invariants of $G/L_1$, then
\[ \exp G \leq p^{m_1 + m_2 (c-1)}. \]

For $t = 2$ and $c > 2$,
\[ \exp Z \leq \exp Z_{c-2} \leq p^{m_1 + m_2 (c-1)-2}. \]

Proof. By [2] $p^{m_2} \geq \exp(L_1/L_2) \geq \ldots \geq \exp(L_{c-1}/L_c)$. So $\exp L_1 \leq p^{m_2(c-1)}$ and hence $\exp G \leq p^{m_1 + m_2 (c-1)}$.

Let $t = 2$. Then $G$ can be generated by two elements. From Lemma 4 we have
\[ \exp(G/Z_{c-1}) = \exp L_{c-1} = p^n \text{ (say)}. \]

Since $G/Z_{c-1}$ is not cyclic, $|G/Z_{c-1}| \geq p^{n+1}$ and so $|G/Z_{c-2}| \geq p^{n+2}$. Also
\[ |L_1/L_2| \leq p^{m_2} \quad \text{and} \quad |G/L_2| = |G/L_1| \cdot |L_1/L_2| \leq p^{m_1 + 2m_2}. \]

But $L_2 \leq Z_{c-2}$. So
\[ |Z_{c-2}/L_2| = |G/L_2|/|G/Z_{c-2}| \leq p^{m_1 + 2m_2 - n - 2}. \]

Therefore,
\[ \exp Z \leq \exp Z_{c-2} \leq |Z_{c-2}/L_2| \cdot \exp L_2 \leq p^{m_1 + m_2 (c-1)-2}, \]
as $\exp L_2 \leq p^{m_2(c-3)+n}$.

The following is an immediate consequence of Lemma 8.5 in [10].

Lemma 6. If $G$ is a finite $p$-group, $|G/Z| = p^b$ and $k_1 \geq k_2 \geq \ldots \geq k_s \geq 1$ are the invariants of $Z$, then $A(G)$ has a $p$-subgroup $F$ of outer automorphisms which is isomorphic to $F \cong F_1 \times F_2 \times \ldots \times F_s$, where $|F_i| = \sup (1, p^{k_i-b})$, and $|F| \geq |Z| \cdot p^{-b_k}$. We also need the following result by W. Gaschütz [6].

Lemma 7. Every finite non-abelian $p$-group has an outer automorphism of order $p^i$ for some $i \geq 1$.

Remark 1. K. G. Hummel [7] (generalized by J. Buckley [4]) showed that if $K$ is a maximal subgroup of $G$ and $Z \nsubseteq K$, then $p|A(K)|_p$ divides
If \( g(h) \) is a strictly increasing integer function, then \( g(h) - 1 \geq g(h - 1) \) and so, inductively, we may assume that \( Z \leq K \) for every maximal subgroup \( K \) of \( G \). This means we may assume that \( G \) is a \( PN \)-group and \( Z \leq \Phi(G) \), where \( \Phi(G) \) is the Frattini subgroup of \( G \).

**Theorem 1.** Let \( G \) be a finite \( p \)-group of class \( c > 2 \). If \( |G| \geq p^h \), then \( |A(G)| \geq p^h \), where \( h \) is an integer with \( h \leq 5 \).

*Proof.* Since \( c > 2 \), by Lemma 3 the only case to consider is \( c = 3 \), \( h = 5 \), \( s = 1 \). For \( |Z| = p \), Lemma 7 gives \( |A(G)| \geq p |I(G)| = |G| \geq p^h \).

Let \( |Z| > p \). If \( m = 2 \), Lemma 5 gives \( k_1 \leq c - 2 = p \), where \( \exp Z = p^{k_1} \). Then \( Z \) is not cyclic, a contradiction. If \( m \geq 3 \), Lemma 1 gives \( a \geq 3 \) and by Lemma 3 we get \( |A(G)| \geq p^3 = p^h \).

**Theorem 2.** Let \( G \) be a finite \( p \)-group of class \( c > 2 \). If \( |G| \geq p^{a+b} \), then \( |A(G)| \geq p^h \), where \( h \) is an integer, \( 5 < h \leq 8 \) and \( g(h) = 2h - 5 \).

*Proof.* Let \( |G/Z| = p^b \). If \( b \geq h - 1 \), then by Lemma 7 \( |A(G)| \geq p^b \geq p^h \). So we take \( b \leq h - 2 \). Then

\[
(1) \quad k \geq g(h) - (h - 2) = h - 3,
\]
where \( |Z| = p^k \). If \( k_1 \leq m_1 \), by Lemma 1(i), \( a \geq k + 1 \geq h - 2 \) and Lemma 3 gives \( |A(G)| \geq p^h \). Thus we take \( k_1 > m_1 \). Then Lemma 1(iii) gives

\[
a \geq m + (s - 1)t.
\]

By Lemma 3 we may assume that

\[
(2) \quad m + (s - 1)t + c \leq h \leq 8;
\]

otherwise we have nothing to show. Since \( m \geq 2 \), \( t \geq 2 \), \( c \geq 3 \) from (2) we get \( s \leq 2 \).

(a) \( s = 2 \). Then \( m + t + c \leq h \), \( h = 7 \) or \( 8 \). For \( h = 7 \), \( m = 2 \), \( t = 2 \), \( c = 3 \), \( k \geq h - 3 = 4 \). By Lemma 5, \( k_1 \leq c - 2 = 1 \) and so \( s = k \geq 4 \), a contradiction.

Let \( h = 8 \). Then \( k \geq 5 \), \( k_1 > 2 \), \( m \leq 3 \). For \( m = 2 \), \( k_1 \leq c - 2 \leq 2 \). For \( m = 3 \), \( c = 3 \) and \( t = 2 \). Then \( G/G' \) has type \( (p^2, p) \) and Lemma 5 gives \( k_1 \leq c - 1 = 2 \). In both cases we have a contradiction.

(b) \( s = 1 \). Then \( k_1 = k \geq h - 3 \), \( m + c \leq h \leq 8 \). So \( m \leq 5 \) and \( c \leq h - 2 \leq 6 \). Consider the following subcases:

(b1) \( m = 2 \). Then \( k_1 \leq c - 2 \), \( h - 3 \leq k_1 \leq c - 2 \). So \( c \geq h - 1 \), a contradiction.

(b2) \( m = 3 \). If \( t = 2 \), \( k_1 \leq c - 1 \) and so \( h - 3 \leq c - 1 \), which gives \( c \geq h - 2 \). But \( m + c \leq h \) gives \( c \leq h - 3 \). If \( t = 3 \), then \( G' = \Phi(G) \geq Z \) and \( \exp G' \geq p^{c-1} \). Hence \( k_1 \leq c - 1 \), a contradiction.

(b3) \( m = 4 \). Then \( c \leq h - 4 \). If \( m_1 = 1 \), then \( G' = \Phi(G) \geq Z \), \( \exp G' \geq p^{c-1} \) and so \( k_1 \leq c - 1 \). So \( h - 3 \leq k_1 \leq c - 1 \) and \( c \geq h - 2 \).
Take \( m_1 > 1 \). Then \( G/G' \) has type \((p, p^2)\), \((p, p, p^2)\) or \((p^2, p^2)\). In the first case Lemma 5 gives \( k_1 \leq c \). In the second case \( \exp Z \leq \exp \Phi(G) \leq p^4 \), so that \( k_1 \leq c \). Then in both cases \( c \geq h - 3 \), a contradiction.

Let \( G/L_1 \) have type \((p^2, p^3)\). Then \( L_1/L_2 \) is cyclic of order at most \( p^2 \). For \( c = 3 \), \( h \geq 7 \), \( |L_2| \geq p^4 \) and \( \exp L_2 \leq p^2 \). This is a contradiction, as \( L_2 \subseteq Z \) and \( Z \) is cyclic. But \( 3 \leq c \leq h - 4 \leq 4 \). So we may assume that \( c = 4 \) and \( h = 8 \). Since

\[
|A(G)|_p \geq |A_c(G)||G/Z_2| \quad \text{and} \quad |A_c(G)| \geq p^4
\]

we get \( |G/Z_2| = p^3 \). So \( |G/Z_3| = p^2 \), \( |Z_3/Z_2| = p \), \( |L_1Z/L_1| \leq p \) and \( |L_2Z/L_2| \leq p^2 \). Let \( |L_2Z/L_2| \leq p \). Since \( \exp L_3 = \exp (G/Z_3) = p \) and \( \exp (L_2/L_3) \leq p^2 \) we get

\[
\exp Z \leq \exp (L_2Z) \leq p^4,
\]

which contradicts (1). So \( |L_2Z/L_2| = p^2 \). Since \( L_2 \subseteq Z_2 \) and \( L_2 < L_1 \leq C_G(Z_2) \), \( L_2 \subseteq Z(Z_2) \). Hence \( L_2Z \subseteq Z(Z_2) \). This gives that \( Z_2 \) is abelian, as \( |Z_3/Z_2| \leq p \). Now \( L_1 \nsubseteq Z_2 \). Pick an element \( x \in L_1 \) with \( x \notin Z_2 \). Since \( x \in Z_3 \setminus Z_2 \) and \( |Z_3/Z_2| = p \) we get that \( Z_3 = \langle x, Z_2 \rangle \). Hence \( Z_3 \) is abelian as \( x \in C_G(Z_2) \). Let \( a \in G \), \( b \in L_1 \). Then

\[
[a^p, b] = [a, b^p] = [a, b]^p \mod L_3.
\]

But \( a^p, b \) are both elements of \( Z_3 \) which is abelian. So \( [a^p, b] = 1 \). Therefore \( [a, b]^p \in L_3 \vee a \in G \) and \( \forall b \in L_1 \). This implies that \( \exp L_2/L_3 = p \).

Then

\[
\exp Z \leq \exp (L_2Z) \leq p^4,
\]

a contradiction.

\( (b_4) \quad m = 5 \). Then \( h = 8 \), \( c = 3 \).

Let \( m_2 = 1 \). If \( t = 2 \), by Lemma 5, \( k_1 \leq 4 \). If \( t > 2 \), \( \exp \Phi(G) \leq p^4 \) and again \( k_1 \leq 4 \), a contradiction. So we take \( m_2 > 1 \) and \( G/G' \) has type either \((p^3, p^2)\) or \((p^3, p^2, p)\). In the first case \( |L_1/L_2| \leq p^2 \) and so \( |L_2| \geq p^4 \). But \( \exp L_2 \leq p^2 \) and \( L_2 \) is cyclic. This is a contradiction. In the second case \( |L_1/L_2| \leq p^4 \) and so \( |G/L_2| \leq p^9 \), which gives that \( |L_2| \geq p^2 \). Since \( |A_c(G)| \geq p^3 \), by the proof of Lemma 2 we get \( |G/Z_2| = p^3 \). Also \( L_1 \subseteq Z(Z_2) \), as \( Z_2 \subseteq C_G(L_1) \). Let \( x \in L_2 \subseteq Z \). Then \( x \) is a product of commutators of the form \( [a, b] \) and \( [a, b]^{-1} = [b, a] \) with \( a \in G \), \( b \in L_1 \).

But \( [a, b] \) and \( [b, a] \) commute with both \( a \) and \( b \), so \( [a, b]^p = [a^p, b] = 1 \) and \( [b, a]^p = [b, a]^p = 1 \), as \( a^p \in Z_2 \) and \( b \in L_1 \subseteq Z(Z_2) \). This gives \( x^p = 1 \vee x \in L_2 \), as \( L_2 \) is abelian. Therefore \( \exp L_2 = p \). But \( L_2 \) is cyclic of order greater than \( p \). This is a contradiction.

**Theorem 3.** Let \( G \) be a finite \( p \)-group of class \( c > 2 \).
(i) If \(|G| \geq p^{14}\) then \(|A(G)|_p \geq p^9\),
(ii) If \(|G| \geq p^{17}\) then \(|A(G)|_p \geq p^{10}\),
(iii) If \(|G| \geq p^{20}\) then \(|A(G)|_p \geq p^{11}\) and
(iv) If \(|G| \geq p^{23}\) then \(|A(G)|_p \geq p^{12}\).

**Proof.** We give the proof of the case (iv), which is the more complicated. The proofs of the other cases are of the same pattern and are therefore omitted.

Let \(|G/Z| = p^b\). If \(b \geq 11\), \(|A(G)|_p \geq p \cdot p^b \geq p^{12}\). Therefore we take \(b \leq 10\). So

\[(1) \quad k \geq 23 - 10 = 13.\]

If \(Z\) is cyclic, by Lemma 6 we get

\[|A(G)| \geq |F| \cdot |I(G)| \geq p^k \cdot p^{-b} \cdot p^b = p^k \geq p^{13}.\]

Assume that \(Z\) is not cyclic and so \(s > 1\). If \(k_1 \leq m_1\), Lemma 1 gives \(a \geq k + s \geq 13\). Take \(k_1 > m_1\). By Lemma 3 it is enough to show that \(a + c - 1 \geq 12\). Therefore we may assume that

\[(2) \quad a + c \leq 12.\]

Since \(k_1 > m_1\), Lemma 1 gives \(a \geq m + (s - 1)t\) and so

\[(3) \quad m + (s - 1)t + c \leq 12,\]

which gives \(s \leq 4\).

(a) \(s = 4\). Then \(m + 3t + c \leq 12\), \(t = 2\), \(m \leq 3\), \(c \leq 4\). By Lemma 5 we get \(k_1 \leq c - 1 \leq 3\). Then \(s \geq \frac{1}{3}k > 4\).

(b) \(s = 3\). Then (3) gives \(m + 2t + c \leq 12\), \(m \leq 5\), \(c \leq 6\) and \(k_1 \geq 5\). For \(m = 2\), \(k_1 \leq c - 2 \leq 4\), a contradiction. For \(m = 3\) and \(t = 2\), \(k_1 \leq c - 1 \leq 4\), as \(c \leq 5\) in this case. For \(m = 3\) and \(t = 3\), \(c = 3\), \(k_1 \leq c = 3\), a contradiction. For \(m = 4\), \(c \leq 4\) and Lemma 5 gives \(k_1 \leq 2c - 2 \leq 6\). Then \(k_2 \geq 4\) and Lemma 1(i) gives \(a \geq 10\). This is impossible as \(a + c \leq 12\). For \(m = 5\), \(c = 3\), \(t = 2\). Then \(k_1 \leq 2c - 1 \leq 5\). So \(k_2 \geq 4\) and by Lemma 1, \(a \geq 12\), a contradiction.

(c) \(s = 2\). Then \(m + t + c \leq 12\), \(m \leq 7\), \(c \leq 8\) and \(k_1 \geq 7\). For \(m = 2\), \(k_1 \leq c - 2 \leq 6\), a contradiction. For \(m = 3\) and \(t = 3\), \(c \leq 6\) and \(k_1 \leq c \leq 6\). For \(m = 3\) and \(t = 2\), \(c \leq 7\) and \(k_1 \leq c - 1 \leq 6\). For \(m = 4\), \(c \leq 6\) and \(k_1 \leq 2c - 2 \leq 10\). So \(k_2 \geq 3\) and by Lemma 1, \(a \geq 8\) which together with (2) gives \(c \leq 4\). Then \(k_1 \leq 2c - 2 \leq 6\). For \(m = 5\), \(c \leq 5\) and Lemma 5 gives \(k_1 \leq 2c \leq 10\). Then \(k_2 \geq 3\) and \(a \geq 9\), which gives \(c = 3\). So \(k_1 \leq 2c = 6\). Hence in all the above cases we have a contradiction, as \(k_1 \geq 7\).

For \(m = 6\), \(c \leq 4\) and \(k_1 \leq 3c - 2 \leq 10\). So \(k_2 \geq 3\), \(a \geq 10\), a contradiction.

For \(m = 7\), \(c = 3\), \(t = 2\). So \(k_1 \leq 3c - 1 = 8\), \(k_2 \geq 5\) and \(a \geq 13\), a contradiction.
Theorem 4. Let $G$ be a finite $p$-group of class $c > 2$ and $g(h) = h^2/6$, where $h$ is an integer, $h \geq 13$. If $|G| \geq p^{g(h)}$, then $|A(G)|_p \geq p^h$.

Proof. By Remark 1, we shall assume that $G$ is a $PN$-group. Let $|G/Z| = p^b$. If $b \leq h - 1$, Lemma 7 gives $|A(G)|_p \geq p|I(G)| = p^{b+1} \geq p^h$. Take $b \leq h - 2$. Then

(1) \[ k \geq g(h) - (h - 2) = h^2/6 - h + 2 > h. \]

If $k \geq h$ Lemma 6 gives

$$|A(G)|_p \geq |F_1|:|I(G)| \geq p^h.$$ 

So $k \leq h - 1$. If $k_1 = h - 1 = k_2$,

$$|A(G)|_p \geq |F_1||F_2||I(G)| \geq p^{2b-b-2} \geq p^h,$$

as $b \leq h - 2$. Therefore we may assume that

(2) \[ k_1 \leq h - 1 \quad \text{and} \quad k_1 \leq h - 2 \quad \text{for} \quad i \geq 2 \]

Then

$$ (h - 2)(s - 1) \geq k - k_1 \geq \frac{1}{6}h^2 - h + 2 - (h - 1) = \frac{1}{6}(h - 10)(h - 2) - \frac{1}{3}. $$

Since $s$ is an integer we get

(3) \[ s - 1 \geq (h - 10)/6. \]

Let $|A_c(G)| = p^a$. By Lemma 3 it is enough to show that $a \geq h - c + 1$. So we take

(4) \[ h \geq a + c. \]

If $k_1 \leq m_1$, by Lemma 1(i) we get $a \geq k + s > h$, a contradiction. So $k_1 > m_1$ and applying Lemma 1(ii) we get

(5) \[ a \geq im + t(s - i) \quad \text{for} \quad k_i \geq m_1, \]

(6) \[ a \geq im + k - (k_1 + \ldots + k_i) + (t - 1)(s - i) \quad \text{for} \quad k_i \geq m_1 > k_{i+1}. \]

Next applying Lemma 5 we get: For $m = 6$, $k_1 \leq 3c - 2$ if $t = 2$, and $k_1 \leq 2c + 1 \leq 3c - 2$ if $t > 2$. So

(7) \[ k_1 \leq 3c - 2 \quad \text{for} \quad m = 6. \]

Also,

(8) \[ k_1 \leq 2c \quad \text{for} \quad m = 5, \]

(9) \[ k_1 \leq 2c - 2 \quad \text{for} \quad m = 4, \]

(10) \[ k_1 \leq c \quad \text{for} \quad m = 3 \quad \text{and} \]

(11) \[ k_1 \leq c - 2 \quad \text{for} \quad m = 2. \]

Consider the following cases.
(a) $m \geq 5$. Let $k_i \geq m_1 > k_{i+1}$ and $m \geq 6$. By (4) $h \geq 6i + 5$. Then for $i > 1$,

$$0 \leq 6i - 11 = (3i - 1)^2 - 9i^2 + 12i - 12 \leq \frac{1}{6}h^2 - h + 2 - h + 1 - (i - 1)(h - 2) + 1$$

$$\geq \frac{1}{6}h^2 - h(i + 1) + 8i + 2 \geq h - 2 \geq h - c + 1.$$

For $i = 1$ this inequality reduces to $h^2 - 18h + 72 \geq 0$, which is valid for $h \geq 13$.

From (1), (2) and (6) we have

$$a \geq 6i + k - (k_1 + \ldots + k_i) + 1 \geq 6i + \frac{1}{6}h^2 - h + 2 - h + 1 - (i - 1)(h - 2) + 1 \geq \frac{1}{6}h^2 - h(i + 1) + 8i + 2 \geq h - 2 \geq h - c + 1.$$

Next let $m = 5$. Then (8) gives

$$2cs \geq k \geq \frac{1}{6}h^2 - h + 2.$$

First let $k_i \geq m_1 > k_{i+1}$. Then from (4) and (6), $h \geq 5i + c + 2 > 4i + c + 2$. For $i > 1$,

$$0 < (2i - c + 1)^2 + 12i^2 - 6i - 9 = (4i + c - 4)^2 - 24 + 30i - 12ci + 6c$$

$$\leq (h - 6)^2 - 24 + 30i - 12ci + 6c = h^2 - 12h + 12 + 30i - 12ci + 6c.$$

So

$$h^2 - 12h + 12 + 30i - 12ci + 6c > 0.$$

For $i = 1$ this inequality reduces to $h^2 - 12h + 42 - 6c > 0$, which is valid for $h \geq 13$, $h \geq 6 + c$. Therefore (6) gives

$$a \geq 5i + k - (k_1 + \ldots + k_i) + 1 \geq 5i + \frac{1}{6}h^2 - h + 2 - 2ci + 1 \geq h - c + 1$$

by (13). Now let $k_1 \geq m_1$. Then by (4), (5) and Lemma 1 we get

$$h \geq ms + c \quad \text{and} \quad a = ms.$$

For $m \geq 7$, (3) gives

$$a \geq 7s \geq \frac{1}{5}(h - 10) + 7 \geq h - 2 \geq h - c + 1,$$

as $h \geq 7s + c \geq 17$ since (3) implies $s > 1$. Similarly for $m = 6$,

$$a = 6s \geq h - 10 + 6 \geq h - c + 1,$$

unless $c \leq 4$. For $c \leq 4$, (7) gives $k_1 \leq 10$ so that $10s \geq k$ and
\[ h \geq 6s + c \geq 15. \text{ Hence} \\
60s \geq 6k \geq h^2 - 6h + 12 \geq 10(h - 2). \]

Thus \( a = 6s \geq h - 2 \geq h - c + 1. \)

Finally take \( m = 5. \) By (14), \( h \geq 10 + c. \) Here \( 5h^2 - 6h(5 + 2c) + 12c^2 - 12c + 60 \geq 0, \) since the discriminant \( D = -96c^2 + 96c - 300 \) of the left side of the inequality is negative. So by (12)

\[ 10cs \geq 5k \geq \frac{3}{8}h^2 - 5h + 10 \geq 2c(h - c + 1). \]

Hence \( a = 5s \geq h - c + 1. \)

(b) \( m = 4. \) Let \( k_i \geq m_1 > k_{i+1}. \) By (9) \( k_1 \leq 2c - 2 \) and so

(15) \[ 2s(c - 1) \geq k \geq \frac{1}{6}h^2 - h + 2. \]

From (4) and (5) we get \( h \geq 4i + c + 2. \) So substituting in (6),

\[ a \geq 4i + \frac{1}{6}h^2 - h + 2 - i(2c - 2) + 1 = \frac{1}{6}h^2 - h + 6i - 2ci + 3 \geq h - c + 1 \]

by (13). Let \( k, l \geq m_1. \) Then \( a = 4s \) and \( h \geq 4s + c. \) For \( h \geq 17, \) (3) gives \( s \geq 3. \) So \( h \geq 12 + c. \) Therefore

\[ h^2 - 6h + 12 \geq 3(c - 1)(h - c + 1) \text{ or} \]
\[ h^2 - 3h(c + 1) + 3c^2 - 6c + 15 \geq 0, \]

since if the discriminant \( D = -3c^2 + 42c - 51 \) of the left side of the inequality is not negative, then \( c \leq 12 \) and

\[ 2h \geq 24 + 2c = 3(c + 1) + (21 - c) \geq 3(c + 1) + \sqrt{D}. \]

For \( c = 3 \) or \( 4 \) this inequality reduces to \( h^2 - 12h + 24 \geq 0, h^2 - 15h + 39 \geq 0, \) which are valid for \( h \geq 13. \) Substituting in (15) we get

\[ 4s(c - 1) \geq 2k \geq \frac{1}{6}h^2 - 2h + 4 \geq (c - 1)(h - c + 1). \]

This gives \( a = 4s \geq h - c + 1 \) for \( h \geq 17 \) or \( c \leq 4. \) Let \( 16 \geq h \geq 13, \) \( c > 4. \) From (4) and (5), \( c \leq 8. \) Then \( a = 4s \geq h - c + 1, \) unless \( c = 8, h = 16; c = 7, h = 15, 16; c = 6, h = 14, 15, 16; c = 5, h = 13, 14, 15, 16. \) For these cases by substituting in (15) we get \( s \geq 3, \) so again \( a = 4s \geq h - c + 1. \)

(c) \( m = 3. \) Let \( k_i \geq m_1 > k_{i+1}. \) Then \( t = 2 \) and Lemma 5 gives \( k_1 \leq c - 1. \) From (4) and (6), \( h \geq 3i + c + 2. \) Then for all \( i, \)

\[ 0 < \frac{1}{6}(9i^2 + c^2 - 2c - 8) = \frac{1}{6}(3i + c - 4)^2 - 4 - ic + 4i + c \leq \frac{1}{6}(h - 6)^2 - 4 - ic + 4i + c = \frac{1}{6}h^2 - 2h - ic + 4i + c + 2. \]

Substituting in (6),

\[ a \geq 3i + \frac{1}{6}h^2 - h + 2 - ic + 4i + c + 3 \geq h - c + 1. \]
Let \( k_s \geq m_1 \). From (10), \( k_1 \leq c \). Then \( cs \geq k \geq \frac{1}{6}h^2 - h + 2 \) so that
\[
3cs \geq \frac{1}{2}h^2 - 3h + 6 \geq c(h - c + 1),
\]
since \( h^2 - 2h(c + 3) + 2c^2 - 2c + 12 \geq 0 \). In fact, if the discriminant
\[
D = -4c^2 + 32c - 12
\]
of the left side of the inequality is not negative, then \( c \leq 7 \) and \( h > 12 = (3 + c) + (9 - c) \geq 3 + c + \frac{1}{2}\sqrt{D} \). Hence \( a = 3s \geq h - c + 1 \).

(d) \( m = 2 \). From (11), \( k_1 \leq c - 2 \) so that
\[
(c - 2)s \geq k.
\]
Here \( h^2 - 3ch + 3c^2 - 9c + 18 \geq 0 \) for \( h \geq 15 \), or for \( h \geq 13 \) provided \( c \leq 6 \) or \( c \geq 10 \). In fact, if the discriminant \( D = -3c^2 + 36c - 72 \) of the left side of the inequality is not negative, then \( c \leq 9 \) and
\[
2h \geq 30 = 3c + 3(10 - c) \geq 3c + \sqrt{D}.
\]
Similarly for \( h \geq 13 \), if \( D \geq 0 \) then \( c \leq 9 \) and
\[
2h > 24 = 3c + 3(8 - c) \geq 3c + \sqrt{D}
\]
provided \( c \leq 6 \). From (16),
\[
2(c - 2)s \geq 2k \geq \frac{1}{3}h^2 - 2h + 4.
\]
Therefore
\[
2(c - 2)s \geq \frac{1}{3}h^2 - 2h + 4 \geq ch - c^2 + 3c - 2 - 2h = (c - 2)(h - c + 1),
\]
which gives \( a = 2s \geq h - c + 1 \), except when \( c = 7, 8, 9 \) and \( h = 13, 14 \). For these cases direct substitution of the values of \( h \) and \( c \) in (17) gives \( a = 2s \geq h - c + 1 \).

Remark 2. I think that the bound \( g(h) = 2h - 5 \), \( 5 < h \leq 8 \) is the best possible. But the bound \( g(h) = h^2/6 \), \( h \geq 13 \) is definitely not the best. For example, using a similar technique, we can take \( g(18) = 52 \) instead of \( (18)^2/6 = 54 \). Even for large values of \( h \), \( g(h) = h^2/6 \) can be reduced.

References


*Koridallos, Athens,
Greece*