ON THE NUMBER OF AUTOMORPHISMS OF A
FINITE \( p \)-GROUP

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Introduction. In this paper we find a new bound for the function 
\( g(h) \), for which \( |A(G)|_p \geq p^h \) whenever \( |G| \geq p^{g(h)} \), \( G \) a finite \( p \)-group. 

The existence of such a function was first conjectured by W. R. Scott in 1954, who proved that \( g(2) = 3 \). In 1956 Ledermann and Neumann proved that in the general case of finite groups \( g(h) \leq (h - 1)^2 \cdot p^{h-1} + h \) \[10\]. Since then, J. A. Green, J. C. Howarth and K. H. Hyde have reduced this bound considerably. The best (least) bound to date for finite \( p \)-groups was obtained by K. H. Hyde \[9\]. He proved that 
\[ g(h) = \frac{1}{2}h(h - 3) + 3 \] for \( h \geq 5 \) and 
\[ g(h) = h + 1 \] for \( h \leq 4 \). For finite non-abelian \( p \)-groups, we improve this bound to: 
\[ g(h) = \frac{1}{2}h^2 \] for \( h \geq 13 \), 
\[ g(h) = 2h - 5 \] for \( 5 < h \leq 8 \), 
\[ g(h) = h \] for \( h \leq 5 \) and for \( 8 < h \leq 12 \) we prove that 
\[ g(9) = 14, \ g(10) = 17, \ g(11) = 20, \ g(12) = 23. \]

The following notation is used: \( G \) is taken to be a finite non-abelian \( p \)-group with commutator subgroup \( G' \) and center \( Z \). The order of \( G \) is denoted by \( |G| \) and \( |H|_p \) is the largest power of \( p \) dividing \( |H| \). Hom \((G, Z)\) is the set of all homomorphisms of \( G \) into \( Z \) and \( A(G), \ A_c(G), \ I(G) \) are the groups of automorphisms, central automorphisms, inner automorphisms of \( G \) respectively. \( G \) is called a PN-group if it has no non-trivial abelian direct factor. We denote the lower and the upper central series of \( G \) by 
\[ G = L_0 > L_1 = G' > \ldots > L_\epsilon = 1 \] and \[ G = Z_\epsilon > Z_{\epsilon-1} > \ldots > Z_1 = Z > Z_0 = 1. \]

Throughout this paper \( \epsilon \) is the class of \( G \) and we take the invariants of \( G/G' \) to be \( m_1 \geq m_2 \geq \ldots \geq m_t \geq 1 \) and the invariants of \( Z \) to be \( k_1 \geq k_2 \geq \ldots \geq k_s \geq 1 \), where \( t \) and \( s \) are the numbers of invariants of \( G/G' \) and \( Z \) respectively. For non-cyclic \( p \)-groups \( G, t \geq 2, \) as \( G/G' \) is cyclic, if and only if \( G \) is cyclic. Also we take \( |G/G'| = p^m \) and \( |Z| = p^k \).

The cyclic group of order \( p^r \) is denoted by \( C_{p^r} \).

It has been conjectured that for finite non-cyclic \( p \)-groups of order greater than \( p^2 \), \( g(h) \leq h \). This has been established for abelian \( p \)-groups, for \( p \)-groups of class two and for some other special classes of finite \( p \)-groups. I believe that in the general case the above conjecture is not valid and that \( g(h) > h \).

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For \( c = 2 \) we have \( g(h) \leq h \) [5]. Therefore we shall assume that \( c > 2 \), whenever \( g(h) > h \).

**Lemma 1.** If \( G \) is a \( PN \)-group, then \( |A_\epsilon(G)| = p^a \), where

\[
|A_\epsilon(G)| = |\text{Hom}(G, Z)| = |\text{Hom}(G/L_1, Z)| = \prod_{j, i} |\text{Hom}(C_{p^{m_j}}, C_{p^{k_i}})| = p^a,
\]

where

\[
a = \sum_{j, i} \min (m_j, k_i).
\]

Therefore

\[
a \geq jk + \sum_{x=j+1, \ell=1}^{t, i} \min (m_x, k_i) \geq jk + (t - j)s \quad \text{for } m_j \geq k_i.
\]

Similarly \( a \geq im + (s - i)t \) for \( k_i \geq m_1 \). If \( k_i \geq m_1 > k_{i+1}, \)

\[
a \geq im + \sum_{j=1, f=i+1}^{s} k_f + \sum_{j=1, j=i+1}^{t, s} \min (m_j, k_f) \geq im + k - (k_1 + \ldots + k_i) + (t - 1)(s - i).
\]

For \( k_s \geq m_1 \), \( \min (m_j, k_i) = m_j \), so that \( a = ms \).

**Lemma 2.** Let \( G \) be a \( PN \)-group of class \( c > 2 \). Then \( |A_\epsilon(G)| \cdot p^{c-1} \) is a factor of \( |A(G)| \).

**Proof.** Since \( G/Z_{c-1} \) is not cyclic and \( |Z_i/Z_{i-1}| \geq p, i = 1, \ldots, c - 1, \)

\[
|G/Z_2| \geq p^{c-1}
\]

and

\[
|A(G)| \geq |A_\epsilon(G) \cdot I(G)| = |A_\epsilon(G)| \cdot |I(G)|/|A_\epsilon(G) \cap I(G)|
\]

\[
= |A_\epsilon(G)| \cdot |G/Z_2| \geq |A_\epsilon(G)| \cdot p^{c-1}.
\]

From Lemmas 1 and 2 we get:

**Lemma 3.** If \( G \) is a \( PN \)-group of class \( c > 2 \), then

\[
|A(G)| \geq p^{a+c-1} \geq p^{2s+c-1},
\]

where \( s \) is the number of invariants of \( Z \).
**Lemma 4.** If $G$ is a 2-generator finite $p$-group of class $c$, then

$$Z_{c-1} \leq \Phi(G) \quad \text{and} \quad \exp (G/Z_{c-1}) = \exp L_{c-1}.$$  

**Proof.** If $Z_{c-1} \not\leq \Phi(G)$, we can find two generators $a$ and $b$ for $G$ with $a \in Z_{c-1}$. Then all $(c-1)$-fold commutators in $a$ and $b$ are 1 and so $G$ has class less than $c$, a contradiction. For $a_0, a_1, \ldots, a_{c-1} \in G$,

$$[a_0, a_1, \ldots, a_{c-1}]^{p^n} = [a_0^{p^n}, a_1, \ldots, a_{c-1}]$$

for any positive integer $n$. This implies that $\exp (G/Z_{c-1}) = \exp L_{c-1}$.

**Lemma 5.** If $m_1 \geq m_2 \geq \ldots \geq m_t \geq 1$ are the invariants of $G/L_1$, then

$$\exp G \leq p^{m_1 + m_2 (c-1)}.$$  

For $t = 2$ and $c > 2$,

$$\exp Z \leq \exp Z_{c-2} \leq p^{m_1 + m_2 (c-1)-2}.$$  

**Proof.** By [2] $p^{m_2} \geq \exp (L_1/L_2) \geq \ldots \geq \exp (L_{c-1}/L_c)$. So $\exp L_1 \leq p^{m_2 (c-1)}$ and hence $\exp G \leq p^{m_1 + m_2 (c-1)}$.

Let $t = 2$. Then $G$ can be generated by two elements. From Lemma 4 we have

$$\exp (G/Z_{c-1}) = \exp L_{c-1} = p^n \ (\text{say}).$$

Since $G/Z_{c-1}$ is not cyclic, $|G/Z_{c-1}| \geq p^{n+1}$ and so $|G/Z_{c-2}| \geq p^{n+2}$. Also

$$|L_1/L_2| \leq p^{m_2} \quad \text{and} \quad |G/L_2| = |G/L_1| \cdot |L_1/L_2| \leq p^{m_1+2m_2}.$$  

But $L_2 \leq Z_{c-2}$. So

$$|Z_{c-2}/L_2| = |G/L_2| / |G/Z_{c-2}| \leq p^{m_1+2m_2-n-2}.$$  

Therefore,

$$\exp Z \leq \exp Z_{c-2} \leq |Z_{c-2}/L_2| \cdot \exp L_2 \leq p^{m_1 + m_2 (c-1)-2},$$

as $\exp L_2 \leq p^{m_2 (c-3)+n}$.

The following is an immediate consequence of Lemma 8.5 in [10].

**Lemma 6.** If $G$ is a finite $p$-group, $|G/Z| = p^b$ and $k_1 \geq k_2 \geq \ldots \geq k_s \geq 1$ are the invariants of $Z$, then $A(G)$ has a $p$-subgroup $F$ of outer automorphisms which is isomorphic to $F \cong F_1 \times F_2 \times \ldots \times F_s$, where $|F_i| = \sup (1, p^{k_i-b})$ and $|F| \geq |Z| \cdot p^{-b_k}$.

We also need the following result by W. Gaschütz [6].

**Lemma 7.** Every finite non-abelian $p$-group has an outer automorphism of order $p^i$ for some $i \geq 1$.

**Remark 1.** K. G. Hummel [7] (generalized by J. Buckley [4]) showed that if $K$ is a maximal subgroup of $G$ and $Z \nsubseteq K$, then $p|A(K)|$, divides
If \( g(h) \) is a strictly increasing integer function, then \( g(h) - 1 \geq g(h - 1) \) and so, inductively, we may assume that \( Z \leq K \) for every maximal subgroup \( K \) of \( G \). This means we may assume that \( G \) is a \( PN \)-group and \( Z \leq \Phi(G) \), where \( \Phi(G) \) is the Frattini subgroup of \( G \).

**Theorem 1.** Let \( G \) be a finite \( p \)-group of class \( c > 2 \). If \( |G| \geq p^h \), then \( |A(G)|_p \geq p^h \), where \( h \) is an integer with \( 5 \leq h \leq 5 \).

**Proof.** Since \( c > 2 \), by Lemma 3 the only case to consider is \( c = 3 \), \( h = 5 \), \( s = 1 \). For \( |Z| = p \), Lemma 7 gives \( |A(G)|_p \geq p|I(G)| = |G| \geq p^h \).

Let \( |Z| > p \). If \( m = 2 \), Lemma 5 gives \( k_1 \leq c - 2 = p \), where \( \exp Z = p^{k_1} \). Then \( Z \) is not cyclic, a contradiction. If \( m \geq 3 \), Lemma 1 gives \( a \geq 3 \) and by Lemma 4 we get \( |A(G)|_p \geq p^3 = p^h \).

**Theorem 2.** Let \( G \) be a finite \( p \)-group of class \( c > 2 \). If \( |G| \geq p^{\ell(k)} \), then \( |A(G)|_p \geq p^h \), where \( h \) is an integer, \( 5 < h \leq 8 \) and \( g(h) = 2h - 5 \).

**Proof.** Let \( |G/Z| = p^b \). If \( b \geq h - 1 \), then by Lemma 7 \( |A(G)|_p \geq p \cdot p^b \geq p^h \). So we take \( b \leq h - 2 \). Then

\[(1) \quad k \geq g(h) - (h - 2) = h - 3,\]
where \( |Z| = p^k \). If \( k_1 \leq m_1 \), by Lemma 1(i), \( a \geq k + 1 \geq h - 2 \) and Lemma 3 gives \( |A(G)|_p \geq p^h \). Thus we take \( k_1 > m_1 \). Then Lemma 1(iii) gives

\[a \geq m + (s - 1)t.\]

By Lemma 3 we may assume that

\[(2) \quad m + (s - 1)t + c \leq h \leq 8;\]
otherwise we have nothing to show. Since \( m \geq 2 \), \( t \geq 2 \), \( c \geq 3 \) from (2) we get \( s \leq 2 \).

(a) \( s = 2 \). Then \( m + t + c \leq h \), \( h = 7 \) or 8. For \( h = 7 \), \( m = 2 \), \( t = 2 \), \( c = 3 \), \( k \geq h - 3 = 4 \). By Lemma 5, \( k_1 \leq c - 2 = 1 \) and so \( s = k \geq 4 \), a contradiction.

Let \( h = 8 \). Then \( k \geq 5 \), \( k_1 > 2 \), \( m \leq 3 \). For \( m = 2 \), \( k_1 \leq c - 2 \leq 2 \). For \( m = 3 \), \( c = 3 \) and \( t = 2 \). Then \( G/G' \) has type \( (p^2, p) \) and Lemma 5 gives \( k_1 \leq c - 1 = 2 \). In both cases we have a contradiction.

(b) \( s = 1 \). Then \( k_1 = k \geq h - 3 \), \( m + c \leq h \leq 8 \). So \( m \leq 5 \) and \( c \leq h - 2 \leq 6 \). Consider the following subcases:

(b1) \( m = 2 \). Then \( k_1 \leq c - 2 \), \( h - 3 \leq k_1 \leq c - 2 \). So \( c \geq h - 1 \), a contradiction.

(b2) \( m = 3 \). If \( t = 2 \), \( k_1 \leq c - 1 \) and so \( h - 3 \leq c - 1 \), which gives \( c \geq h - 2 \). But \( m + c \leq h \) gives \( c \leq h - 3 \). If \( t = 3 \), then \( G' = \Phi(G) \geq Z \) and \( \exp G' \leq p^{c-1} \). Hence \( k_1 \leq c - 1 \), a contradiction.

(b3) \( m = 4 \). Then \( c \leq h - 4 \). If \( m_1 = 1 \), then \( G' = \Phi(G) \geq Z \), \( \exp G' \leq p^{c-1} \) and so \( k_1 \leq c - 1 \). So \( h - 3 \leq k_1 \leq c - 1 \) and \( c \geq h - 2 \).
Take $m_1 > 1$. Then $G/G'$ has type $(p, p^3)$, $(p, p, p^2)$ or $(p^2, p^3)$. In the first case Lemma 5 gives $k_1 \leq c$. In the second case $\exp Z \leq \exp \Phi(G) \leq p^4$, so that $k_1 \leq c$. Then in both cases $c \geq h - 3$, a contradiction.

Let $G/L_1$ have type $(p^2, p^3)$. Then $L_1/L_2$ is cyclic of order at most $p^2$. For $c = 3$, $h \geq 7$, $|L_2| \leq p^4$ and $\exp L_2 \leq p^2$. This is a contradiction, as $L_2 \leq Z$ and $Z$ is cyclic. But $3 \leq c \leq h - 4 \leq 4$. So we may assume that $c = 4$ and $h = 8$. Since

$$|A(G)|_p \geq |A_c(G)||G/Z_2|$$

we get $|G/Z_2| = p^3$. So $|G/Z_3| = p^2$, $|Z_3/Z_2| = p$, $|L_1Z/L_1| \leq p$ and $|L_2Z/L_2| \leq p^2$. Let $|L_2Z/L_2| \leq p$. Since $\exp L_3 = \exp (G/Z_3) = p$ and $\exp (L_2/L_3) \leq p^2$ we get

$$\exp Z \leq \exp (L_2Z) \leq p^4,$$

which contradicts (1). So $|L_2Z/L_2| = p^2$. Since $L_2 \leq Z_2$ and $L_2 < L_1 \leq C_G(Z_2)$, $L_2 \leq Z(Z_2)$. This gives that $Z_2$ is abelian, as $|Z_2/Z_3Z_2| \leq p$. Now $L_1 \not\leq Z_2$. Pick an element $x \in L_1$ with $x \not\in Z_2$. Since $x \in Z_3 \setminus Z_2$ and $|Z_3/Z_2| = p$ we get that $Z_3 = \langle x, Z_2 \rangle$. Hence $Z_3$ is abelian as $x \in C_G(Z_2)$. Let $a \in G$, $b \in L_1$. Then

$$[a^p, b] = [a, b^p] = [a, b]^p \mod L_3.$$

But $a^p, b$ are both elements of $Z_3$ which is abelian. So $[a^p, b] = 1$. Therefore $[a, b]^p \in L_3 \forall a \in G$ and $\forall b \in L_1$. This implies that $\exp L_2/L_3 = p$. Then

$$\exp Z \leq \exp (L_2Z) \leq p^4,$$

a contradiction.

(b4) $m = 5$. Then $h = 8$, $c = 3$.

Let $m_2 = 1$. If $t = 2$, by Lemma 5, $k_1 \leq 4$. If $t > 2$, $\exp \Phi(G) \leq p^4$ and again $k_1 \leq 4$, a contradiction. So we take $m_2 > 1$ and $G/G'$ has type either $(p^2, p^3)$ or $(p^3, p^3, p)$. In the first case $|L_1/L_2| \leq p^2$ and so $|L_2| \geq p^4$. But $\exp L_2 \leq p^2$ and $L_2$ is cyclic. This is a contradiction. In the second case $|L_1/L_2| \leq p^4$ and so $|G/Z_2| \leq p^9$, which gives that $|L_2| \geq p^2$. Since $|A_c(G)| \geq p^3$ by the proof of Lemma 2 we get $|G/Z_2| = p^3$. Also $L_1 \leq Z(Z_2)$, as $Z_2 \leq C_G(L_1)$. Let $x \in L_2 \leq Z$. Then $x$ is a product of commutators of the form $[a, b]$ and $[a, b]^{-1} = [b, a]$ with $a \in G, b \in L_1$. But $[a, b]$ and $[b, a]$ commute with both $a$ and $b$, so $[a, b]^p = [a^p, b] = 1$ and $[b, a]^p = [b, a^p] = 1$, as $a^p \in Z_2$ and $b \in L_1 \leq Z(Z_2)$. This gives $x^p = 1 \forall x \in L_2$, as $L_2$ is abelian. Therefore $\exp L_2 = p$. But $L_2$ is cyclic of order greater than $p$. This is a contradiction.

**Theorem 3.** Let $G$ be a finite $p$-group of class $c > 2$. 

(i) If $|G| \geq p^{14}$ then $|A(G)|_p \geq p^9$,
(ii) If $|G| \geq p^{17}$ then $|A(G)|_p \geq p^{10}$,
(iii) If $|G| \geq p^{20}$ then $|A(G)|_p \geq p^{11}$ and 
(iv) If $|G| \geq p^{23}$ then $|A(G)|_p \geq p^{12}$.

Proof. We give the proof of the case (iv), which is the more complicated. The proofs of the other cases are of the same pattern and are therefore omitted.

Let $|G/Z| = p^b$. If $b \geq 11$, $|A(G)|_p \geq p \cdot p^b \geq p^{12}$. Therefore we take $b \leq 10$. So

(1) $k \geq 23 - 10 = 13$.

If $Z$ is cyclic, by Lemma 6 we get

$|A(G)| \geq |F| \cdot |I(G)| \geq p^k \cdot p^{b-k} = p^k \cdot p^{13}.$

Assume that $Z$ is not cyclic and so $s > 1$. If $k_1 \leq m_1$, Lemma 1 gives $a \geq k + s > 13$. Take $k_1 > m_1$. By Lemma 3 it is enough to show that $a + c - 1 \geq 12$. Therefore we may assume that

(2) $a + c \leq 12$.

Since $k_1 > m_1$, Lemma 1 gives $a \geq m + (s - 1)t$ and so

(3) $m + (s - 1)t + c \leq 12$,

which gives $s \leq 4$.

(a) $s = 4$. Then $m + 3t + c \leq 12$, $t = 2$, $m \leq 5$, $c \leq 4$. By Lemma 5 we get $k_1 \leq c - 1 \leq 3$. Then $s \geq \frac{1}{3}k > 4$.

(b) $s = 3$. Then (3) gives $m + 2t + c \leq 12$, $m \leq 5$, $c \leq 6$ and $k_1 \geq 5$. For $m = 2$, $k_1 \leq c - 2 \leq 4$, a contradiction. For $m = 3$ and $t = 2$, $k_1 \leq c - 1 \leq 4$, as $c \leq 5$ in this case. For $m = 3$ and $t = 3$, $c = 3$, $k_1 \leq c = 3$, a contradiction. For $m = 4$, $c \leq 4$ and Lemma 5 gives $k_1 \leq 2c - 2 \leq 6$. Then $k_2 \geq 4$ and Lemma 1(i) gives $a \geq 10$. This is impossible as $a + c \leq 12$. For $m = 5$, $c = 3$, $t = 2$. Then $k_1 \leq 2c - 1 \leq 5$. So $k_2 \geq 4$ and by Lemma 1, $a \geq 12$, a contradiction.

(c) $s = 2$. Then $m + t + c \leq 12$, $m \leq 7$, $c \leq 8$ and $k_1 \geq 7$. For $m = 2$, $k_1 \leq c - 2 \leq 6$, a contradiction. For $m = 3$ and $t = 3$, $c \leq 6$ and $k_1 \leq c \leq 6$. For $m = 3$ and $t = 2$, $c \leq 7$ and $k_1 \leq c - 1 \leq 6$. For $m = 4$, $c \leq 6$ and $k_1 \leq 2c - 2 \leq 10$. So $k_2 \geq 3$ and by Lemma 1, $a \geq 8$ which together with (2) gives $c \leq 4$. Then $k_1 \leq 2c - 2 \leq 6$. For $m = 5$, $c \leq 5$ and Lemma 5 gives $k_1 \leq 2c \leq 10$. Then $k_2 \geq 3$ and $a \geq 9$, which gives $c = 3$. So $k_1 \leq 2c = 6$. Hence in all the above cases we have a contradiction, as $k_1 \geq 7$.

For $m = 6$, $c \leq 4$ and $k_1 \leq 3c - 2 \leq 10$. So $k_2 \geq 3$, $a \geq 10$, a contradiction.

For $m = 7$, $c = 3$, $t = 2$. So $k_1 \leq 3c - 1 = 8$, $k_2 \geq 5$ and $a \geq 13$, a contradiction.
Theorem 4. Let $G$ be a finite $p$-group of class $c > 2$ and $g(h) = h^2/6$, where $h$ is an integer, $h \geq 13$. If $|G| \geq p^{g(h)}$, then $|A(G)|_p \geq p^h$.

Proof. By Remark 1, we shall assume that $G$ is a PN-group. Let $|G/Z| = p^b$. If $b \geq h - 1$, Lemma 7 gives $|A(G)|_p \geq p|I(G)| = p^{b+1} \geq p^h$. Take $b \leq h - 2$. Then

\[(1) \quad k \geq g(h) - (h - 2) = h^2/6 - h + 2 > h.\]

If $k \geq h$ Lemma 6 gives

\[|A(G)|_p \geq |F_1 : |I(G)| \geq p^h.\]

So $k \leq h - 1$. If $k = h - 1 = k_2$,

\[|A(G)|_p \geq |F_1| |F_2| |I(G)| \geq p^{g(h-b-2)} \geq p^h,\]

as $b \leq h - 2$. Therefore we may assume that

\[(2) \quad k_1 \leq h - 1 \quad \text{and} \quad k_i \leq h - 2 \quad \text{for} \quad i \geq 2\]

Then

\[(h - 2)(s - 1) \geq k - k_1 \geq \frac{1}{6}h^2 - h + 2 - (h - 1) = \frac{1}{6}(h - 10)(h - 2) - \frac{1}{6}.\]

Since $s$ is an integer we get

\[(3) \quad s - 1 \geq (h - 10)/6.\]

Let $|A_c(G)| = p^a$. By Lemma 3 it is enough to show that $a \geq h - c + 1$. So we take

\[(4) \quad h \geq a + c.\]

If $k_1 \leq m_1$, by Lemma 1(i) we get $a \geq k + s > h$, a contradiction. So $k_1 > m_1$ and applying Lemma 1(ii) we get

\[(5) \quad a \geq im + t(s - i) \quad \text{for} \quad k_i \geq m_1,\]

\[(6) \quad a \geq im + k - (k_1 + \ldots + k_i) + (t - 1)(s - i) \quad \text{for} \quad k_i \geq m_1 > k_{i+1}.\]

Next applying Lemma 5 we get: For $m = 6$, $k_1 \leq 3c - 2$ if $t = 2$, and $k_1 \leq 2c + 1 \leq 3c - 2$ if $t > 2$. So

\[(7) \quad k_1 \leq 3c - 2 \quad \text{for} \quad m = 6.\]

Also,

\[(8) \quad k_1 \leq 2c \quad \text{for} \quad m = 5,\]

\[(9) \quad k_1 \leq 2c - 2 \quad \text{for} \quad m = 4,\]

\[(10) \quad k_1 \leq c \quad \text{for} \quad m = 3 \quad \text{and}\]

\[(11) \quad k_1 \leq c - 2 \quad \text{for} \quad m = 2.\]

Consider the following cases.
(a) \( m \geq 5 \). Let \( k_i \geq m_1 > k_{i+1} \) and \( m \geq 6 \). By (4) \( h \geq 6i + 5 \). Then for \( i > 1 \),
\[
0 \leq 6i - 11 = (3i - 1)^2 - 9i^2 + 12i - 12 \\
\leq (h - 3i - 6)^2 - 9i^2 + 12i - 12 \\
= h^2 - 6h(i + 2) + 48i + 24.
\]
For \( i = 1 \) this inequality reduces to \( h^2 - 18h + 72 \geq 0 \), which is valid for \( h \geq 13 \).

From (1), (2) and (6) we have
\[
a \geq 6i + k - (k_1 + \ldots + k_i) + 1 \\
\geq 6i + \frac{1}{6}h^2 - h + 2 - h + 1 - (i - 1)(h - 2) + 1 \\
\geq \frac{1}{6}h^2 - h(i + 1) + 8i + 2 \geq h - 2 \geq h - c + 1.
\]

Next let \( m = 5 \). Then (8) gives
\[
2cs \geq k \geq \frac{1}{6}h^2 - h + 2.
\]
First let \( k_i \geq m_1 > k_{i+1} \). Then from (4) and (6), \( h \geq 5i + c + 2 > 4i + c + 2 \). For \( i > 1 \),
\[
0 < (2i - c + 1)^2 + 12i^2 - 6i - 9 = (4i + c - 4)^2 - 24 \\
\leq (h - 6)^2 - 24 + 30i - 12ci + 6c \\
= h^2 - 12h + 12 + 30i - 12ci + 6c.
\]
So
\[
h^2 - 12h + 12 + 30i - 12ci + 6c > 0.
\]
For \( i = 1 \) this inequality reduces to \( h^2 - 12h + 42 - 6c > 0 \), which is valid for \( h \geq 13 \), \( h \geq 6 + c \). Therefore (6) gives
\[
a \geq 5i + k - (k_1 + \ldots + k_i) + 1 \\
\geq 5i + \frac{1}{6}h^2 - h + 2 - 2ci + 1 \geq h - c + 1
\]
by (13). Now let \( k_i \geq m_1 \). Then by (4), (5) and Lemma 1 we get
\[
(14) \quad h \geq ms + c \quad \text{and} \quad a = ms.
\]
For \( m \geq 7 \), (3) gives
\[
a \geq 7s \geq \frac{5}{6}(h - 10) + 7 \geq h - 2 \geq h - c + 1,
\]
as \( h \geq 7s + c \geq 17 \) since (3) implies \( s > 1 \). Similarly for \( m = 6 \),
\[
a = 6s \geq h - 10 + 6 \geq h - c + 1,
\]
unless \( c \leq 4 \). For \( c \leq 4 \), (7) gives \( k_i \leq 10 \) so that \( 10s \geq k \) and
$h \geq 6s + c \geq 15$. Hence

$$60s \geq 6k \geq h^2 - 6h + 12 \geq 10(h - 2).$$

Thus $a = 6s \geq h - 2 \geq h - c + 1$.

Finally take $m = 5$. By (14), $h \geq 10 + c$. Here $5h^2 - 6h(5 + 2c) + 12c^2 - 12c + 60 \geq 0$, since the discriminant $D = -96c^2 + 96c - 300$ of the left side of the inequality is negative. So by (12)

$$10cs \geq 5k \geq \frac{5}{6}h^2 - 5h + 10 \geq 2c(h - c + 1).$$

Hence $a = 5s \geq h - c + 1$.

(b) $m = 4$. Let $k_i \geq m_1 > k_{i+1}$. By (9) $k_1 \leq 2c - 2$ and so

$$a \geq 4i + \frac{1}{6}h^2 - h + 2 - i(2c - 2) + 1 = \frac{1}{6}h^2 - h + 6i - 2ci + 3 \geq h - c + 1$$

by (13). Let $k_s \geq m_1$. Then $a = 4s$ and $h \geq 4s + c$. For $h \geq 17$, (3) gives $s \geq 3$. So $h \geq 12 + c$. Therefore

$$h^2 - 6h + 12 \geq 3(c - 1)(h - c + 1) \quad \text{or}$$

$$h^2 - 3h(c + 1) + 3c^2 - 6c + 15 \geq 0,$$

since if the discriminant $D = -3c^2 + 42c - 51$ of the left side of the inequality is not negative, then $c \leq 12$ and

$$2h \geq 24 + 2c = 3(c + 1) + 21 \geq 3(c + 1) + \sqrt{D}.$$ 

For $c = 3$ or 4 this inequality reduces to $h^2 - 12h + 24 \geq 0$, $h^2 - 15h + 39 \geq 0$, which are valid for $h \geq 13$. Substituting in (15) we get

$$4s(c - 1) \geq 2k \geq \frac{1}{6}h^2 - 2h + 4 \geq (c - 1)(h - c + 1).$$

This gives $a = 4s \geq h - c + 1$ for $h \geq 17$ or $c \leq 4$. Let $16 \geq h \geq 13$, $c > 4$. From (4) and (5), $c \leq 8$. Then $a = 4s \geq h - c + 1$, unless $c = 8$, $h = 16$; $c = 7$, $h = 15, 16$; $c = 6$, $h = 14, 15, 16$; $c = 5$, $h = 13, 14, 15, 16$. For these cases by substituting in (15) we get $s \geq 3$, so again $a = 4s \geq h - c + 1$.

(c) $m = 3$. Let $k_i \geq m_1 > k_{i+1}$. Then $t = 2$ and Lemma 5 gives $k_1 \leq c - 1$. From (4) and (6), $h \geq 3i + c + 2$. Then for all $i$,$$

0 < \frac{1}{6}(9i^2 + c^2 - 2c - 8) = \frac{1}{6}(3i + c - 4)^2 - 4 - ic + 4i + c \leq \frac{1}{6}(h - 6)^2 - 4 - ic + 4i + c = \frac{1}{6}h^2 - 2h - ic + 4i + c + 2.$$

Substituting in (6),

$$a \geq 3i + \frac{1}{6}h^2 - h + 2 - i(c - 1) + 1 = \frac{1}{6}h^2 - h - ic + 4i + 3 \geq h - c + 1.$$
Let $k_s \geq m_1$. From (10), $k_1 \leq c$. Then $cs \geq k \geq \frac{1}{6}h^2 - h + 2$ so that

$$3cs \geq \frac{1}{6}h^2 - 3h + 6 \geq c(h - c + 1),$$

since $h^2 - 2h(c + 3) + 2c^2 - 2c + 12 \geq 0$. In fact, if the discriminant $D = -4c^2 + 32c - 12$ of the left side of the inequality is not negative, then $c \leq 7$ and $h > 12 = (3 + c) + (9 - c) \geq 3 + c + \frac{1}{2}\sqrt{D}$. Hence $a = 3s \geq h - c + 1$.

(d) $m = 2$. From (11), $k_1 \leq c - 2$ so that

$$(16) \quad (c - 2)s \geq k.$$  

Here $h^2 - 3ch + 3c^2 - 9c + 18 \geq 0$ for $h \geq 15$, or for $h \geq 13$ provided $c \leq 6$ or $c \geq 10$. In fact, if the discriminant $D = -3c^2 + 36c - 72$ of the left side of the inequality is not negative, then $c \leq 9$ and

$$2h \geq 30 = 3c + 3(10 - c) \geq 3c + c + \sqrt{D}.$$ 

Similarly for $h \geq 13$, if $D \geq 0$ then $c \leq 9$ and

$$2h > 24 = 3c + 3(8 - c) \geq 3c + \sqrt{D}$$

provided $c \leq 6$. From (16),

$$(17) \quad 2(c - 2)s \geq 2k \geq \frac{1}{6}h^2 - 2h + 4.$$ 

Therefore

$$2(c - 2)s \geq \frac{1}{6}h^2 - 2h + 4 \geq ch - c^2 + 3c - 2 - 2h$$

$$= (c - 2)(h - c + 1),$$

which gives $a = 2s \geq h - c + 1$, except when $c = 7, 8, 9$ and $h = 13, 14$. For these cases direct substitution of the values of $h$ and $c$ in (17) gives $a = 2s \geq h - c + 1$.

Remark 2. I think that the bound $g(h) = 2h - 5, 5 < h \leq 8$ is the best possible. But the bound $g(h) = h^2/6, h \geq 13$ is definitely not the best. For example, using a similar technique, we can take $g(18) = 52$ instead of $(18)^2/6 = 54$. Even for large values of $h, g(h) = h^2/6$ can be reduced.

References


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