SOME GENERALIZATIONS OF BURNSIDE'S THEOREM

N. A. WIEGMANN

1. Introduction. Burnside's Theorem in the theory of group representations states that a necessary and sufficient condition that a semigroup of matrices of degree n over the complex field be irreducible is that the semigroup contain n^2 linearly independent matrices. In the course of dealing with sets of matrices with coefficients in a division ring, Brauer **(1)** obtained a generalization of this theorem which concerned irreducible semigroups with elements in a division ring. In the present work irreducible semigroups of matrices with elements in the field of real quaternions are considered and generalizations of Burnside's Theorem of a more specific nature are obtained by using certain properties of matrices with such elements.

The following facts and terminology may be briefly noted. Let $\mathfrak{A}$ be a semigroup (relative to multiplication) of quaternion matrices of degree n. By the l-rank (i.e., left rank) of $\mathfrak{A}$ is meant the maximum number of left linearly independent matrices in $\mathfrak{A}$; r-rank has a corresponding meaning. If every matrix A of $\mathfrak{A}$ is of either form

$$\begin{bmatrix} A_1 & O \\ A_3 & A_4 \end{bmatrix} \text{ or } \begin{bmatrix} A_1 & A_2 \\ O & A_4 \end{bmatrix}$$

where A_1 is $m \times m$, $m < n$, the semigroup is said to be decomposed. If $\mathfrak{A}$ is such that there exists a non-singular quaternion matrix P such that the set $P\mathfrak{A}P^{-1}$ is decomposed, then $\mathfrak{A}$ is said to be reducible; if not $\mathfrak{A}$ is said to be irreducible. According to a result **(1**, 4.4**B)** of Brauer's, Schur's Lemma holds for semigroups of matrices with elements in a division ring: If $\mathfrak{A}$ and $\mathfrak{B}$ are irreducible semigroups which are intertwined by a matrix P, then P is either 0 or non-singular. This result could be obtained in the real quaternion case by paralleling Schur's proof while using the result **(3**, Theorem 10**)** that for every quaternion matrix P there exist unitary quaternion matrices U and V such that UPV is a real diagonal matrix with non-negative diagonal elements. It can also be shown that the following is true: If two semigroups $\mathfrak{A}$ and $\mathfrak{B}$ of quaternion matrices are intertwined by a matrix P and if $\mathfrak{B}$ is irreducible, then either $P = 0$ or $\mathfrak{A}$ contains $\mathfrak{B}$ as an irreducible component.

2. The form of a matrix commutative with an irreducible representation.

THEOREM 1. *If $\mathfrak{A}$ is an irreducible semigroup of quaternion matrices, then any matrix M which commutes with every element of $\mathfrak{A}$ is of the form $M = P^{-1}(kI)P$ where k is a complex number and P is a non-singular quaternion matrix.*

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Let M be a quaternion matrix with the given property. It is known **(3,** Theorem 1**)** that there exists a non-singular matrix P such that $PMP^{-1} = J$ is in Jordan normal form with complex elements along the diagonal.

If M is non-derogatory **(3)**, then each A in $\mathfrak{A}$ is such that $PAP^{-1} = B_1 \dot{+} B_2 \dot{+} \ldots \dot{+} B_m$ where each B_i is triangular **(3**, pp. 195**)**, but this contradicts the assumption on $\mathfrak{A}$.

If M is derogatory, J is of the form kI, where k is a complex number, or it is not. If not, $J = J_1 \dot{+} J_2 \dot{+} \ldots \dot{+} J_t$ where this form may be assumed to be such that J_i contains only and all diagonal elements λ_i which are characteristic roots of M, where $\lambda_i \neq \lambda_j$ for $i \neq j$ and where $\lambda_i \neq \bar{\lambda}_j$, for $i \neq j$, the latter according to the Jordan form designated for M in **(3)**. Now either J has one characteristic root or it has more than one; i.e., either $t = 1$, above, or $t > 1$ in J_t. These cases are now considered.

Let $J = J_1 \dot{+} J_2 \dot{+} \ldots \dot{+} J_t$, $t > 1$. Let A be any element of $\mathfrak{A}$ and let $PAP^{-1} = X = (x_{ij}.)$ It will be shown that

$$X = \begin{bmatrix} X_1 & X_2 \\ 0 & X_3 \end{bmatrix},$$

where X_1 has the same order as that of J_1. Let j be the order of J_1 and let the first block (in the direct sum which J_2 represents) of the form

$$\begin{bmatrix} \lambda_2 & 1 & 0 & \\ & \lambda_2 & & \\ & & \cdots & 1 \\ 0 & & & \lambda_2 \end{bmatrix}$$

be of order k. Then we have a series of relations of the form

(i) $\quad \lambda_2 x_{sp} + x_{s+1,p} = x_{sp}\lambda_1, \quad \lambda_2 x_{j+k,p} = x_{j+k,p}\lambda_1,$

or

(ii) $\quad \lambda_2 x_{sp} + x_{s+1,p} = x_{s,p-1} + x_{sp}\lambda_1, \quad \lambda_2 x_{j+k,p} = x_{j+k,p-1} + x_{j+k,p}\lambda_1,$

where, for a fixed p chosen from $p = 1, 2, \ldots, j$, either (i) or (ii) holds (but not both); where for $p = 1$, (i) holds; and where if $k > 1$, $j < s < j + k$, and where if $k = 1$, only the second relations in (i) and (ii) hold. Now one of two cases occurs:

(a) At least one λ_i of M is real and there is no loss in generality in assuming λ_1 is such a real root. It follows from the second relation in (i) that $x_{j+k,1} = 0$, and from the first that $x_{s1} = 0$ for $j < s < j + k$. For any other p for which (i) holds, the same is true so that $x_{sp} = 0$ for $j < s \leqslant j + k$ for these p. Consider the first p for which (ii) holds; then in (ii), $x_{s,p-1} = 0 = x_{j+k,p-1}$ and so $x_{j+k,p} = 0$ from the second relation, and from the first $\lambda_2 x_{j+k-1,p} + 0 = 0 + x_{j+k-1,p}\lambda_1$ so that $x_{j+k-1,p} = 0$ and, in turn, $x_{sp} = 0$ for $j < s \leqslant j + k$ for this p. If the next p for which (ii) holds is treated in a similar fashion it follows that $x_{sp} = 0$ for $j < s \leqslant j + k$, and $1 \leqslant p \leqslant j$.

(b) If all λ_i are non-real complex, it follows from the second relation in (i) that $x_{j+k,1} = 0$ (since $\lambda_2 \neq \lambda_1$ and $\bar{\lambda}_2 \neq \lambda_1$) and from the first relation that $x_{s1} = 0$ for $j < s < j + k$. For any p for which (i) is true, the same result holds. Considering the first p for which (ii) holds, it follows as in (a) that all $x_{sp} = 0$ for this p and the same range of s so that $x_{sp} = 0$ here, also, for $j < s \leqslant j + k$ and $1 \leqslant p \leqslant j$. Now for any subsequent such blocks in J_2 or for any such blocks in $J_3, \ldots, J_t$ a similar procedure applies so that $x_{lp} = 0$ for $1 \leqslant p \leqslant j$ and for $j < l \leqslant n$ (where X is $n \times n$). So X has the above form which contradicts the assumption of the theorem.

Let $J = J_1$ so that only λ_1 appears along the diagonal where there must be an element 1 to the right of at least one λ_1. Let $J_1 = J_{11} \dotplus J_{22} \dotplus \ldots \dotplus J_{kk}$ where each J_{ii} is either the single element λ_1 or a matrix with λ_1 along the diagonal and 1 above each such λ_1 after the first, and where J_{11} is definitely of the latter form. Let X_{ii} have the same order as J_{ii} and let

$$X = \begin{bmatrix} X_{11} & X_{12} & \ldots & X_{1k} \\ & & \ldots & \\ X_{k1} & X_{k2} & \ldots & X_{kk} \end{bmatrix},$$

so that, from the assumptions, X_{11}, X_{12}, X_{21}, and X_{22} at least appear and are not vacuous and $k < n$.

Now it will be shown that it follows from $JX = XJ$ that each X_{ij} is a triangular matrix with zeros below the main diagonal; it is to be noted here that when some X_{ij}, $i \neq j$, is not square, the latter statement will be taken to mean that any element $x_{ij} = 0$ when $i > j$. Then it follows that $J_{ii}X_{ij} = X_{ij}J_{jj}$ $(i, j = 1, 2, \ldots, k)$. When $i = j$, since J_{ii} is non-derogatory, X_{ii} is in triangular form $(i = 1, 2, \ldots, k)$. When $i \neq j$, consider for example X_{12} as a typical case: $J_{11}X_{12} = X_{12}J_{22}$, and let these be of order $s \times s$, $s \times t$, and $t \times t$ respectively. Equating the elements in the first column (and dropping the subscript on λ_1):

(i) $$\lambda x_{i1} + x_{i+1,1} = x_{i1}\lambda\ , \quad 1 \leqslant i < s\ , \quad \lambda x_{s1} = x_{s1}\lambda\ ;$$

and from the other columns there result:

(ii) $$\lambda x_{lj} + x_{l+1,j} = x_{l,j-1} + x_{lj}\lambda\ , \quad \lambda x_{sj} = x_{s,j-1} + x_{sj}\lambda\ , \quad 1 \leqslant l < s \quad \text{and} \quad 1 < j \leqq t\ ,$$

where the following is to be noted: if X_{ij} consists of one column, only (i) applies, and if X_{ij} consists of only one row, the second relations only in (i) and (ii) apply. (In the latter case X_{ij} is already in triangular form.) If λ is real, from the first relation in (i), $x_{i+1,1} = 0$ for $1 \leqslant i < s$; from the first relation in (ii) $x_{l+1,j} = x_{l,j-1}$ for $1 \leqslant l < s$, $1 < j \leqslant t$. Together these show $x_{ij} = 0$ for $i > j$. If λ is non-real complex, from the second relation in (i) x_{s1} must be complex and from the first $x_{s1} = 0$ and also, in turn,

$$x_{s-1,1} = x_{s-2,1} = \ldots = x_{21} = 0.$$

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Since $x_{s1} = 0$, from the second relation in (ii) x_{s2} is complex and from the first $x_{s2} = 0$; similarly $x_{s-1,2} = \ldots = x_{32} = 0$. Continuing with all elements x_{sp} in the last row of X_{12}, it will be seen that $x_{ij} = 0$ for $i > j$.

For each A in $\mathfrak{A}$, $PAP^{-1} = X$ is of the above type, sectioned as above, and having each X_{ij} in triangular form as described. Form the matrices

$$\begin{bmatrix} X_{11} \\ \cdot \\ \cdot \\ \cdot \\ X_{k1} \end{bmatrix}, \begin{bmatrix} X_{12} \\ \vdots \\ \cdot \\ \cdot \\ X_{k2} \end{bmatrix}, \ldots, \begin{bmatrix} X_{1k} \\ \cdot \\ \cdot \\ \cdot \\ X_{kk} \end{bmatrix}.$$

Let X_{ij} have row order r_{ij} and column order c_{ij}; then the first columns of each of the above matrices have zeros in all the same positions and have, possibly, non-zero elements only in the 1st, the $(r_{11} + 1)$st, the $(r_{11} + r_{21} + 1)$ st, $\ldots$, the $(r_{11} + r_{21} + \ldots + r_{k-1,1} + 1)$st positions. By column operations on the right of X, the following columns of X may be interchanged:

the 2nd and the $(c_{11} + 1)$st,
the 3rd and the $(c_{11} + c_{12} + 1)$st,
the 4th and the $(c_{11} + c_{12} + c_{13} + 1)$st,

$\cdot \qquad \cdot$

$\cdot \qquad \cdot$

$\cdot \qquad \cdot$

the kth and the $(c_{11} + c_{12} + \ldots + c_{1,k-1} + 1)$st.

(Note that $c_{11} > 1$.) The resultant matrix is such that the first k columns have all-zero rows except for the 1st, the $(r_{11} + 1)$st, $\ldots$, the $(r_{11} + r_{21} + \ldots + r_{k-1,1} + 1)$st. Now

$$r_{11} = c_{11},\ r_{21} = c_{12},\ r_{31} = c_{13},\ \ldots,\ r_{k-1,1} = c_{1,k-1}$$

so that if the same operations as above are now performed on the left on the rows, a similarity transformation will have been applied to X with the result that a matrix is obtained such that the first k columns have all-zero rows beyond the kth row; therefore, X is similar to a matrix

$$\begin{bmatrix} X_1 & X_2 \\ 0 & X_3 \end{bmatrix},$$

where X_1 is $k \times k$, $k < n$.

It follows then that $M = P^{-1}(kI)P$ where k is complex and P is quaternion.

3. Generalizations of Burnside's Theorem. If $\mathfrak{A}$ is a semigroup consisting of square matrices of degree n with quaternion elements, there may exist a non-singular (quaternion) matrix P such that $P\mathfrak{A}P^{-1} = \mathfrak{A}_1$ contains only matrices with complex elements, or there may not exist such a P. An example of the former may be obtained by forming $P^{-1}\mathfrak{A}_1P$ where $\mathfrak{A}_1$ is any complex semigroup and P is a non-singular quaternion matrix. An example of the latter (other than the set of all $n \times n$ quaternion matrices) is the set, $\mathfrak{A}$, of all $n \times n$ unitary matrices with quaternion elements. This set is closed under matrix multiplication. There exists no P such that $P\mathfrak{A}P^{-1} = \mathfrak{C}$ is a complex set; this can be seen as follows: P cannot be complex since then $\mathfrak{A}$ would be complex. Also, in the notation of (**3**, p. 191) no $P = P_1 + jP_2$, P_1 and P_2 complex, $P_2 \neq 0$, can provide this. For if so, let $\mathfrak{U}$ be the set of all $n \times n$ complex unitary matrices; this is a semigroup of complex matrices which is irreducible under all complex similarity transformations. Since $\mathfrak{U}$ is a subset of $\mathfrak{A}$, P must be such that $P\mathfrak{U} = \mathfrak{C}_1 P$ where $\mathfrak{C}_1$ is composed solely of complex matrices. From $(P_1 + jP_2)\mathfrak{U} = \mathfrak{C}_1(P_1 + jP_2)$ it follows that $P_1\mathfrak{U} = \mathfrak{C}_1 P_1$ and $jP_2\mathfrak{U} = j\bar{\mathfrak{C}}_1 P_2$ (where $\bar{\mathfrak{C}}_1$ consists of the complex conjugate of each matrix in $\mathfrak{C}_1$). From the first relation and from results in complex theory, either $P_1 = 0$ or $\mathfrak{C}_1$ contains $\mathfrak{U}$ as an irreducible component and since both $\mathfrak{C}_1$ and $\mathfrak{U}$ are $n \times n$ in dimension, $\mathfrak{C}_1$ is irreducible also, relative to the complex field. By Schur's Lemma in complex theory either $P_1 = 0$ or P_1 is non-singular. The latter must hold for if $P_1 = 0$, $P_2\mathfrak{U} = \bar{\mathfrak{C}}_1 P_2$ and since $\mathfrak{U}$ and $\bar{\mathfrak{C}}_1$ are irreducible, either $P_2 = 0$ also (not possible) or P_2 is non-singular in which case $\mathfrak{A} = P^{-1}\mathfrak{C}P = P_2^{-1}j^{-1}\mathfrak{C}jP_2 = P_2^{-1}\bar{\mathfrak{C}}P_2$ is complex. Therefore P_1 is non-singular. From $jP_2\mathfrak{U} = j\bar{\mathfrak{C}}_1 P_2$, either $P_2 = 0$ (not possible) or P_2 is non-singular. Then $P = P_1 + jP_2$ where both P_1 and P_2 are non-singular. But this is not possible because, for example, since $(ij)I$ and $2^{-\frac{1}{2}}(i+j)I$ are elements of $\mathfrak{A}$, they must be similar under the P^{-1}, P transformation to complex matrices C_1 and C_2, respectively. From the first relation it follows that

$$\bar{P}_2 P_1^{-1} = P_1 \bar{P}_2^{-1},$$

and from this and the second, the contradictory fact that $C_2 = 0$ would result.

THEOREM 2. *Let $\mathfrak{A}$ be an irreducible semigroup of quaternion matrices. Then $M = P(kI)P^{-1}$, where k is non-real complex, commutes with each element of $\mathfrak{A}$ if and only if $P^{-1}\mathfrak{A}P = \mathfrak{C}$ is a complex set.*

If $MA = AM$ where A is any element of $\mathfrak{A}$, then

$$(kI)(P^{-1}AP) = (P^{-1}AP)(kI)$$

and, since k is non-real complex, $P^{-1}AP$ is complex for any A in $\mathfrak{A}$. The converse is immediate.

It is now desirable to separate irreducible semigroups of matrices with quaternion elements into two classes: these which are similar to a complex semigroup under some quaternion similarity transformation, and those which are not similar to a complex semigroup under any such transformation. These cases are considered in turn.

THEOREM 3. *Let $\mathfrak{A}$ be a semigroup of quaternion matrices of degree n which is not similar to a complex set. If $\mathfrak{A}$ is irreducible, then $\mathfrak{A}$ has l-rank n^2; and, conversely, if every semigroup similar to $\mathfrak{A}$ has l-rank n^2, then $\mathfrak{A}$ is irreducible.*

This will be shown in two ways; in the first use is made of a known theorem while in the second a direct proof is given.

Let $\mathfrak{A}$ be a given irreducible semigroup as described. Using the notation and terminology of Brauer (**1**, pp. 513, 520), let $\mathfrak{C}(\mathfrak{A})$ denote the commuting ring of $\mathfrak{A}$, i.e., the set of all quaternion matrices P which intertwine a set $\mathfrak{A}$ of square matrices with itself. This means that for each A of $\mathfrak{A}$, $AP = PA$. It has been seen above that any matrix which commutes with each element of an irreducible semigroup is of the form $P^{-1}(cI)P$. Because of the given nature of $\mathfrak{A}$ c is real. Therefore any $M = P^{-1}(cI)P = cI$ is a real scalar matrix so that here $\mathfrak{C}(\mathfrak{A})$ is the set of all real scalar matrices. Theorem (9.2A) of Brauer's work states the following: let G be an irreducible semigroup of degree n. If G has l-rank k and $\mathfrak{C}(\mathfrak{A})$ has r-rank v, then $n^2 \leqslant kv$. In this instance the r-rank v of $\mathfrak{C}(\mathfrak{A})$ is obviously 1 so that $n^2 \leqslant k$. On the other hand $k \leqslant n^2$, so that $k = n^2$.

This same result can be obtained directly as in the complex case as follows. Let $A = (a_{\kappa\lambda})$ be any quaternion matrix of an irreducible semigroup $\mathfrak{A}$ of order n as given. There may exist n^2 quaternion numbers $k_{\lambda\kappa}$, $\lambda = 1, 2, \ldots, n;$ $\kappa = 1, 2, \ldots, n$ such that for each A in $\mathfrak{A}$

$$\sum_{\kappa,\lambda=1}^{n} a_{\kappa\lambda} k_{\lambda\kappa} = 0;$$

if $K = (k_{\lambda\kappa})$ and if

$$\chi(AK) = \sum_{\kappa,\lambda=1}^{n} a_{\kappa\lambda} k_{\lambda\kappa},$$

this can be expressed, as usual, as $\chi(AK) = 0$. If there exists more than one such K-matrix, any right linear combination of them will also be such a K-matrix.

LEMMA. *If $\mathfrak{A}$ has l-rank r where $\mathfrak{A}$ is of degree n, then the number of right linearly independent K-matrices is $n^2 - r$.*

For if

$$A_\rho = (a_{\kappa\lambda}^{(\rho)}), \qquad \rho = 1,2, \ldots, r,$$

is a set of left linearly independent matrices of $\mathfrak{A}$, for a given K-matrix, then

$$\sum_{\lambda,\kappa=1}^{n} a_{\kappa\lambda}^{(\rho)} k_{\lambda\kappa} = 0, \qquad \rho = 1,2, \ldots, r.$$

This is a right system of r homogeneous linear equations in n^2 unknowns, $k_{\lambda\kappa}$. The r rows of the $r \times n^2$ coefficient matrix are left linearly independent and so (see **2**, p. 41, for example) there exist $n^2 - r$ right linearly independent solutions.

Now if $\mathfrak{A}$ is irreducible and not similar to a complex set, it will be shown that there can exist no system of non-zero K matrices. Let us assume the contrary. As in the complex case the following may be noted first of all: For

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any A in $\mathfrak{A}$, AK is itself a K-matrix; if $K_1, \ldots, K_m$ is a (right linearly independent) basis for all K-matrices, each A in $\mathfrak{A}$ determines an $m \times m$ matrix $R = (r_{\sigma\rho})$ from

$$AK_\rho = \sum_{\sigma=1}^{m} K_\sigma r_{\sigma\rho}, \qquad \rho = 1,2, \ldots, m;$$

the set $\mathfrak{R}$ of all such matrices determined from a given basis is a semigroup of degree m which is homomorphic to $\mathfrak{A}$; under a change of a K-basis, the set $\mathfrak{R}$ becomes a similar set $P^{-1}\mathfrak{R}P$ where $P = (p_{ij})$ is a non-singular matrix and

$$L_\rho = \sum_{\sigma=1}^{m} K_\sigma p_{\sigma\rho} \qquad \rho = 1,2, \ldots, m,$$

is the new K-basis; and the K-basis may be chosen so that $\mathfrak{R}$ is of the form

$$\begin{bmatrix} \mathfrak{R}_1 & \mathfrak{R}_2 \\ 0 & \mathfrak{R}_4 \end{bmatrix}$$

where $\mathfrak{R}_1$ is irreducible and of degree m_1 where $1 \leqslant m_1 \leqslant m$. If a right linearly independent set $K_1, \ldots, K_m$ existed, then for any A in $\mathfrak{A}$,

$$AK_\rho = \sum_{\sigma=1}^{m_1} K_\sigma r_{\sigma\rho}, \qquad \rho = 1,2, \ldots, m_1$$

(where only $\mathfrak{R}_1$ is utilized). Let

$$K_\rho = (k_{\kappa\lambda}^{(\rho)})$$

so that

$$\sum_{\lambda=1}^{n} a_{\kappa\lambda} k_{\lambda\mu}^{(\rho)} = \sum_{\sigma=1}^{m_1} k_{\kappa\mu}^{(\sigma)} r_{\sigma\rho} \qquad \rho = 1,2, \ldots, m_1; \ \kappa = 1,2, \ldots, n; \ \mu = 1,2, \ldots, n.$$

Let

$$P_u = (k_{iu}^{(j)}), \ u = 1,2, \ldots, n; \ i = 1,2, \ldots, n; \ j = 1,2, \ldots, m_1.$$

Then $AP_u = P_u R$ for all A in $\mathfrak{A}$ and all corresponding R in $\mathfrak{R}_1$, and for $u = 1, 2, \ldots, n$. The conditions of Schur's Lemma are met so that either a given $P_u = 0$ or is non-singular. If all $P_u = 0$ for $u = 1, 2, \ldots, n$, then

$$K_1, K_2, \ldots, K_{m_1}$$

are all zero matrices; but this contradicts their linear independence. If not all $P_u = 0$, take any such non-singular P_u so that $A = P_u R P_u^{-1}$; if the proper change of K-basis is made, R can be taken to be the same as A. For each A in $\mathfrak{A}$, $AP_u = P_u A$ and since the conditions of Theorem 1 are met, each non-singular $P_u = Q_u(c_u I)Q_u^{-1}$ where c_u is complex. From Theorem 2, since $\mathfrak{A}$ is not similar to a complex set under any quaternion similarity transformation, $P_u = c_u I$ $(u = 1, 2, \ldots, n)$ where c_u is real. Then it follows that each K_ρ $(\rho = 1, 2, \ldots, m_1 = n)$, is such that each row except the ρth is zero

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and this row is of the form $[c_1, c_2, \ldots, c_n]$ for each. For any A of $\mathfrak{A}$ the matrix AK_ρ has for its jth diagonal position the element $a_{j\rho}\, c_j$ $(\rho = 1, 2, \ldots, m_1 = n)$. Now $\chi(AK_\rho) = 0$ $(\rho = 1, 2, \ldots, n)$, so that if A^T denotes the transpose of any A in $\mathfrak{A}$,

$$A^T \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix},$$

and since all c_i are real, the transpose of the above yields $[c_1, c_2, \ldots, c_n] \,.\, A = 0 \,.\, [c_1, c_2, \ldots, c_n] = [0, 0, \ldots, 0]$. Now $\mathfrak{A}$ is irreducible so by the corollary to Schur's Lemma, either $P = 0$ or the representation 0 contains $\mathfrak{A}$ as an irreducible component. Since the latter is not possible, each $c_i = 0$; but this means that each $K\rho = 0$ which contradicts the linearly independent character of the $K\rho$.

The converse of the theorem follows directly.

In connection with the latter proof, it can be verified that the following generalization is true: let $\mathfrak{G}$ be an abstract semigroup; let $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \ldots$ be a finite number of irreducible semigroups of quaternion matrices of degrees $m_1, m_2, m_3, \ldots$, respectively, which are homomorphic to $\mathfrak{G}$ (relative to matrix multiplication) such that no two semigroups are similar to each other, and such that none is similar to a semigroup of complex matrices. Then there exists no set of non-zero matrices, $K, L, M, \ldots$ such that $\chi(AK) + \chi(BL) + \chi(CM) + \ldots = 0$ simultaneously for all sets of matrices $A, B, C, \ldots$ which correspond to the same element of $\mathfrak{G}$ and belong to $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \ldots$, respectively. A proof may be used which parallels that of the complex case and depends on the direct proof of Theorem 3 above. Also, as in this case, one can then state the following: Let $\mathfrak{A}$ and $\mathfrak{B}$ be irreducible semigroups of quaternion matrices which are homomorphic to a semigroup $\mathfrak{G}$ and are not similar to complex semigroups; if the traces of the elements of $\mathfrak{A}$ and $\mathfrak{B}$ which correspond to the same element of $\mathfrak{G}$ are the same, then $\mathfrak{A}$ and $\mathfrak{B}$ are similar. Since $\chi(A) - \chi(B) = 0$ for any A and B of $\mathfrak{A}$ and $\mathfrak{B}$, respectively, which correspond to the same element of $\mathfrak{G}$, the only alternative is that $\mathfrak{A}$ and $\mathfrak{B}$ are similar.

Consider next the case where $\mathfrak{A}$ is an irreducible semigroup of quaternion matrices of degree n and $P\mathfrak{A}P^{-1} = \mathfrak{K}$ is a complex semigroup; let $\bar{\mathfrak{K}}$ denote the set obtained from $\mathfrak{K}$ by taking the ordinary complex conjugate of each matrix of $\mathfrak{K}$. If Q is a complex matrix such that $CQ = Q\bar{C}$ for each C in $\mathfrak{K}$, $\mathfrak{K}$ and $\bar{\mathfrak{K}}$ will be said to be *interjoined* by Q. There may exist such a Q which is non-singular or there may not. It is convenient to consider these cases separately. In this connection the following may be noted:

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THEOREM 4. *Let $\mathfrak{A}$ be an irreducible semigroup and let $P^{-1}\mathfrak{A}P = \mathfrak{K}$ be a complex set. If $\mathfrak{K}$ and $\bar{\mathfrak{K}}$ are interjoined by means of a non-singular complex matrix, then every complex semigroup similar to $\mathfrak{A}$ has this property.*

Let $P^{-1}\mathfrak{A}P = \mathfrak{K}_1$ and $Q^{-1}\mathfrak{A}Q = \mathfrak{K}_2$ be complex. Then $\mathfrak{A} = P\mathfrak{K}_1P^{-1} = Q\mathfrak{K}_2Q^{-1}$ or $\mathfrak{K}_1P^{-1}Q = P^{-1}Q\mathfrak{K}_2$; i.e., $P^{-1}Q = M = M_1 + jM_2$ (where M_1 and M_2 are complex matrices) intertwines $\mathfrak{K}_1$ and $\mathfrak{K}_2$. Since $\mathfrak{K}_1(M_1+jM_2) = (M_1+jM_2)\mathfrak{K}_2$, $\mathfrak{K}_1M_1 = M_1\mathfrak{K}_2$ and $\bar{\mathfrak{K}}_1M_2 = M_2\mathfrak{K}_2$. Since $\mathfrak{K}_1$, $\mathfrak{K}_2$, and $\bar{\mathfrak{K}}_1$ are irreducible and all matrices involved are complex, M_1 and M_2 are either 0 or non-singular (except that both cannot be 0) and $Q = P(M_1 + jM_2)$. Now assume there exists a complex matrix S such that $SCS^{-1} = \bar{C}$ for each C in $\mathfrak{K}_1$. Then there exists a matrix K in $\mathfrak{K}_2$ such that at least one of $CM_1 = M_1K$ and $\bar{C}M_2 = M_2K$ holds where M_1 and M_2 are fixed. If the former, then $\bar{M}_1\bar{K}\bar{M}_1^{-1} = \bar{C} = SCS^{-1} = SM_1KM_1^{-1}S^{-1}$; if the latter, $\bar{M}_2\bar{K}\bar{M}_2^{-1} = C = S^{-1}\bar{C}S = S^{-1}M_2KM_2^{-1}S$. Since each K in $\mathfrak{K}_2$ can be accounted for in this way; the desired result is obtained.

A result of Brauer's, of use in what follows, states the following: If $\mathfrak{G}$ is an irreducible semigroup of degree n, let E_i denote the row $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with ith component 1; let h be the largest number of indices $u_1, u_2, \ldots, u_h$ with $1 \leqslant u_i \leqslant n$ such that conditions $\sum E_uC_u = 0$, C_u in $\mathfrak{C}(\mathfrak{G})$, u ranging over $u_1, \ldots, u_h$, imply $C_{u_i} = 0$ for all u_i. Then the l-rank of $\mathfrak{G}$ is equal to nh (**1**, pp. 531–532).

If $P^{-1}\mathfrak{A}P = \mathfrak{K}$ is a complex semigroup, if $\mathfrak{A}$ is irreducible, and if M is a matrix in $\mathfrak{C}(\mathfrak{A})$, then $P^{-1}\mathfrak{A}PP^{-1}MP = P^{-1}MPP^{-1}\mathfrak{A}P$ or $\mathfrak{K}N = N\mathfrak{K}$ where $N = P^{-1}MP = M_1 + jM_2$, M_1 and M_2 complex. Then $\mathfrak{K}M_1 = M_1\mathfrak{K}$ and $\bar{\mathfrak{K}}$ and $\mathfrak{K}$ are interjoined by M_2.

(a) If no non-singular complex matrix interjoins $\mathfrak{K}$ and $\bar{\mathfrak{K}}$, then $M_2 = 0$ and M_1 is any complex scalar matrix. Therefore $M = P(k_iI)P^{-1}$ and $\mathfrak{C}(\mathfrak{A})$ consists of all matrices $P(k_iI)P^{-1}$ where P is fixed and k_i is any real or complex number. Let $P = P_1 + jP_2 = (p_{ij})$ where P_1 and P_2 are complex matrices, and let the rank of the $n \times 2n$ matrix $[P_1, P_2]$ be r. Let $u_1, u_2, \ldots, u_r$ denote a set of natural numbers between 1 and n such that the correspondingly numbered rows of $[P_1, P_2]$ are linearly independent. Form the expression $\sum E_uP(k_uI)P^{-1} = 0$ where the summation is over the above $u_1, u_2, \ldots, u_r$ and k_u is any complex number; this is equivalent to $\sum E_uP(k_uI) = 0$. Since E_uP is the uth row of P, this is equivalent to the system of n linear homogeneous equations $T \cdot \alpha = 0$ in r (complex) unknowns,

$$k_{u_1}, \ k_{u_2}, \ \ldots, \ k_{u_r},$$

where T is the $n \times r$ matrix such that the element in the ith row and jth column is $p_{u_j i}$ and α is a column vector whose transpose is

$$[k_{u_1}, \ k_{u_2}, \ \ldots, \ k_{u_r}];$$

let $T = T_1 + jT_2$ where T_1 and T_2 are complex. Since the α vector is to be complex, $T \cdot \alpha = 0$ is equivalent to

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \cdot \alpha = 0 .$$

Since the coefficient matrix is of rank r, only the zero solution is possible, i.e., $P(k_u I)P^{-1} = 0$. If any number of rows greater than r is taken a like set of equations results but not all solutions are necessarily zero so that not all $P(k_n I)P^{-1}$ are zero. Therefore, $h = r$.

THEOREM 5. *If $\mathfrak{A}$ is irreducible of degree n, and if $P^{-1}\mathfrak{A}P = \mathfrak{K}$ is complex where no non-singular complex matrix interjoins $\mathfrak{K}$ and $\bar{\mathfrak{K}}$, then $\mathfrak{A}$ has l-rank rn where r is the rank of the matrix $[P_1, P_2]$ where $P = P_1 + jP_2$, P_1 and P_2 complex.*

(b) If $P^{-1}\mathfrak{A}P = \mathfrak{K}$ is interjoined with $\bar{\mathfrak{K}}$ under a non-singular complex matrix, it is convenient to consider separately the cases in which $\mathfrak{K}$ is real and in which $\mathfrak{K}$ is not real.

If $\mathfrak{K}$ is real, then in the above $\mathfrak{K}M_1 = M_1\mathfrak{K}$ and $\mathfrak{K}M_2 = M_2\mathfrak{K}$ so that both M_1 and M_2 are complex scalar matrices. In this case $\mathfrak{C}(\mathfrak{A})$ consists of all matrices of the form $P(k_i + jl_i)IP^{-1}$ (where k_i and l_i are any complex numbers and P is fixed), i.e., of the form $P(q_i I)P^{-1}$ where q_i is any quaternion element. Consider the matrix P^T and let r be its rank (i.e., the number of columns in every maximal set of right linearly independent columns or the number of rows in every maximal set of left linearly independent rows); choose a maximal set of r right linearly independent columns of P^T and let $u_1, u_2, \ldots, u_r$ denote the corresponding column numbers. Form the sum $\sum E_u P(q_u I)P^{-1} = 0$, as before, over this set of u_i; this is equivalent to $\sum E_u P(q_u I) = 0$, or, as before, to the set of equations $T \,.\, \alpha = 0$ where T has as its columns the above mentioned set of r right linearly independent columns of P^T and α is now a column vector of r (quaternion) unknown components. Such a system has only the zero solution; and, as before, if any set of u-indices larger in number than r were taken, non-zero solutions could be obtained.

THEOREM 6. *If $\mathfrak{A}$ is irreducible of degree n and if $P^{-1}\mathfrak{A}P = \mathfrak{K}$ is real, then $\mathfrak{A}$ has l-rank rn where r is the rank of P^T.*

If $\mathfrak{K}$ is not real but is interjoined with $\bar{\mathfrak{K}}$ by means of a non-zero complex matrix, then (with reference to the paragraph above (a)) $M_1 = kI$ is complex scalar and an M_2 which is non-singular exists so that $P^{-1}MP = kI + jM_2$, M_2 non-singular. Let S be any other non-singular complex matrix interjoining $\mathfrak{K}$ and $\bar{\mathfrak{K}}$. Then $\bar{C} = SCS^{-1} = M_2CM_2^{-1}$ for any C in $\mathfrak{K}$ so that $M_2^{-1}SC = CM_2^{-1}S$ for any C in $\mathfrak{K}$; since $\mathfrak{K}$ is irreducible, $M_2^{-1}S = lI$, l complex, so that $S = lM_2$. In this case $\mathfrak{C}(\mathfrak{A})$ consists of all matrices of the form $P(k_iI + jl_iM_2)P^{-1}$ where P and M_2 are fixed and k_i and l_i are complex scalars. (It may be noted that M_2 is itself a non-zero complex scalar if and only if $\mathfrak{K}$ is real). Let $M_2 = Z = (z_{ij})$ for simplicity. Let $P = P_1 + jP_2$, as before, and form $S = [P_1,P_2]$; if t rows of S are chosen to form a matrix S_1, S_1

may be considered to be of the form $S_1 = [Q, R]$ where Q and R are composed of corresponding rows of P_1 and P_2, respectively. Let s be the maximum number of rows of S which are linearly independent and form $[Q, R]$ such that

$$\begin{bmatrix} Q & R \\ -\bar{R}Z & \bar{Q}Z \end{bmatrix}$$

is of rank $2s$. Then consider

$$\sum_u E_u P(k_u I + j l_u Z) P^{-1} = 0$$

where the summation is to be taken over the numbers of the s rows of S chosen as above, $u_1, u_2, \ldots u_s$. After discarding the P^{-1} on the right and noting that Z, k_i and l_i are in the complex field, it can be seen that this relationship is equivalent to the set of $2n$ equations in $2s$ (complex) unknowns

$$\begin{bmatrix} Q^T & -Z^T\bar{R}^T \\ R^T & Z^T\bar{Q}^T \end{bmatrix} \cdot \alpha = 0$$

where

$$\alpha^T = [k_{u_1}, \ldots, k_{u_s}, l_{u_1}, \ldots, l_{u_s}].$$

Since the coefficient matrix is of rank $2s$, only the 0 solution is possible; i.e., if the above relation is to hold, all the matrices from $\mathfrak{C}(\mathfrak{A})$ must be 0 matrices. It is evident that s is the largest number of u_i such that this is true.

THEOREM 7. *If $\mathfrak{A}$ is irreducible of degree n, if $P^{-1}\mathfrak{A}P = \mathfrak{K}$ is non-real complex, and if $\mathfrak{K}$ and $\bar{\mathfrak{K}}$ are interjoined by means of a non-singular complex Z, then $\mathfrak{A}$ has l-rank sn where, when $P = P_1 + jP_2$, s is the maximum number of linearly independent rows of $[P_1, P_2]$ which form $[Q, R]$ such that*

$$\begin{bmatrix} Q & R \\ -\bar{R}Z & \bar{Q}Z \end{bmatrix}$$

is of rank $2s$.

REFERENCES

1. R. Brauer, *On sets of matrices with coefficients in a division ring*, Trans. Amer. Math. Soc., *49* (1941), 502–547.
2. E. H. Moore, *General analysis*, Part I, Memoirs of the American Philosophical Society, I (1935).
3. N. A. Wiegmann, *Some theorems on matrices with real quaternion elements*, Can. J. Math., *7* (1955), 191–201.

Catholic University, Washington, D.C.