Canad. Math. Bull. Vol. **40** (3), 1997 pp. 376–384

THE DUAL PAIR $\mathrm{PGL}_3 \times G_2$

BENEDICT H. GROSS AND GORDAN SAVIN

ABSTRACT. Let H be the split, adjoint group of type E_6 over a p-adic field. In this paper we study the restriction of the minimal representation of H to the closed subgroup $\mathrm{PGL}_3 \times G_2$.

1. **Introduction.** Let k be a p-adic field, and G_2 the exceptional simple group of type G_2 over k. Then the product

$$\mathrm{PGL}_3 \times G_2 \tag{1.1}$$

is a dual pair in the split, adjoint group H of type E_6 over k [7]. We want to determine the restriction of the minimal representation [9] of H to this pair.

Let D be a division algebra of rank 3 over k, and $PD^\times$ the inner form of PGL_3 over k associated to D. This group has rank 0, and is independent of the choice of D (as the two division algebras are opposite algebras). The product

$$PD^\times \times G_2 \tag{1.2}$$

is the dual pair in the inner form H_D of H, which has rank 2 over k and is associated to D. We want to determine the restriction of the minimal representation of H_D to this pair.

In this paper we give a conjectural description of these restrictions (Conjecture 3.1), and work out two special cases (Proposition 4.17 and 4.18). As a consequence we reprove a result of Shahidi [10] on generalized principal series of G_2 (Corollary 5.5).

2. **Parameters.** The dual group of PGL_3 and $PD^\times$ is $\mathrm{SL}_3(\mathbb{C})$. Irreducible, admissible representations π of $\mathrm{PGL}_3(k)$ are parametrized by homomorphisms

$$\varphi\colon W(k) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_3(\mathbb{C}) \tag{2.1}$$

satisfying the usual conditions [3]. The component group A_φ of the centralizer of φ is either trivial, or equal to $\mu_3 =$ the center of $\mathrm{SL}_3(\mathbb{C})$. The latter occurs when the resulting 3-dimensional representation of $W(k) \times \mathrm{SL}_2(\mathbb{C})$ is irreducible.

Irreducible, admissible representations π_D of $PD^\times(k) = D^\times/k^\times$ are finite dimensional, and parametrized by the homomorphisms (2.1) with $A_\varphi = \mu_3$. For example, the

Received by the editors March 21, 1996.
AMS subject classification: Primary: 22E35 and 50; Secondary: 11F70.

https://doi.org/10.4153/CMB-1997-045-0 Published online by Cambridge University Press

Steinberg representation St of PGL_3 has parameter φ trivial on $W(k)$ and giving the principal $\mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_3(\mathbb{C})$. This has $A_\varphi = \mu_3$, and corresponds to the trivial (= Steinberg) representation of $PD^\times$.

Let $\mathbb{O}$ be the $\mathbb{Q}$-algebra of Cayley's octonions. Then $G_2(k) = \mathrm{Aut}(\mathbb{O} \otimes k)$ is the exceptional group of type G_2 [5]. The dual group of G_2 is $G_2(\mathbb{C})$. Conjecturally, irreducible representations π' of $G_2(k)$ are parametrized by pairs (φ', χ') where

$$\varphi' \colon W(k) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow G_2(\mathbb{C}) \tag{2.2}$$

and χ' is an irreducible representation of the component group $A_{\varphi'}$ of the centralizer of φ'.

Let ξ be a 3-rd root of unity in $\mathbb{O}$. Then the map

$$g(x) = \xi x \xi^{-1} \tag{2.3}$$

gives an automorphism of order 3 of $\mathbb{O}$, hence an element of order 3 in $G_2(\mathbb{C})$. The centralizer of g in $G_2(\mathbb{C})$ is isomorphic to $\mathrm{SL}_3(\mathbb{C})$. We fix an embedding

$$f \colon \mathrm{SL}_3(\mathbb{C}) \longrightarrow G_2(\mathbb{C}). \tag{2.4}$$

The normalizer of g in $G_2(\mathbb{C})$ contains $\mathrm{SL}_3(\mathbb{C})$ with index 2, and induces the outer automorphism

$$i(A) = {}^tA^{-1} \tag{2.5}$$

of $\mathrm{SL}_3(\mathbb{C})$.

If φ is a parameter for PGL_3 or $PD^\times$ as in (2.1), then the composition $\varphi' = f \circ \varphi$ is a parameter for G_2 as in (2.2). The map f induces a homomorphism

$$f_* \colon A_\varphi \longrightarrow A_{\varphi'}. \tag{2.6}$$

PROPOSITION 2.7. *The map f_* is injective, and has image a normal subgroup of index* 1 *or* 2.

PROOF. This is proved by direct computation, using [4]. The case $A_{\varphi'} = S_3$ occurs precisely when $\mathrm{Im}(\varphi)$, the image of $W(k) \times \mathrm{SL}_2(\mathbb{C})$ under φ, acts irreducibly on $\mathbb{C}^3$ and is contained in $SO_3(\mathbb{C})$. The case $A_{\varphi'} = \mu_2$ occurs when $\mathrm{Im}(\varphi)$ stabilizes a unique line and is contained in $S\big(O_1(\mathbb{C}) \times O_2(\mathbb{C})\big) = O_2(\mathbb{C})$, or when the image is $S\big(O_1(\mathbb{C})^3\big) = \mu_2^2$.

3. **Conjectures.** Let π be an irreducible representation of $\mathrm{PGL}_3(k)$. We define $\Theta(\pi)$ as the set of irreducible representations of π' of $G_2(k)$ such that $\pi \otimes \pi'$ is a quotient of the minimal representation of H. Let π_D be an irreducible representation of $D^\times / k^\times$. We define $\Theta(\pi_D)$ as the set of irreducible representations of π' of $G_2(k)$ such that $\pi_D \otimes \pi'$ is a quotient of the minimal representation of H_D.

https://doi.org/10.4153/CMB-1997-045-0 Published online by Cambridge University Press

CONJECTURE 3.1. *Let* $\varphi\colon W(k)\times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_3(\mathbb{C})$ *be a parameter of* π *or* π_D. *Then*
(1) $\Theta(\pi)$ *is the set of* π' *whose parameters* (φ',χ') *satisfy:* $\varphi' = f\circ\varphi$ *and* $\chi'\circ f_* = 1$.
(2) $\Theta(\pi_D)\cup\Theta(\pi_D^\vee)$ *is the set of* π' *whose parameters* (φ',χ') *satisfy:* $\varphi' = f\circ\varphi$ *and* $\chi'\circ f_* \neq 1$.

A simple consequence of this would be that a representation π' of $G_2(k)$ occurs as a quotient of one of the minimal representations if and only its Langlands parameter φ' is lifted from SL_3. It then occurs in precisely one of the sets $\Theta(\pi)$ or $\Theta(\pi_D)\cup\Theta(\pi_D^\vee)$, depending on the restriction of χ' to the subgroup $f_*(A_\varphi)$ of $A_{\varphi'}$.

Since the minimal representation of H extends to $\mathrm{Aut}(H)$, and the outer automorphism of H fixes G_2 and induces the outer automorphism of PGL_3, we have

$$\Theta(\pi) = \Theta(\pi^\vee). \tag{3.2}$$

This is compatible with Conjecture 3.1, for if φ is the parameter of π, then $i\circ\varphi$ is the parameter of $\pi^\vee$. Furthermore, the two lifted parameters $f\circ\varphi$ and $f\circ i\circ\varphi = i\circ f\circ\varphi$ are equivalent in $G_2(\mathbb{C})$.

4. **Some examples.** We now give some examples of Conjecture 3.1. Recall that for each semi-simple conjugacy class s in $\mathrm{SL}_3(\mathbb{C})$, there is an unramified representation $\pi(s)$ of $\mathrm{PGL}_3(k)$ with Satake parameter s. Similarly, if s' is a semi-simple conjugacy class in $G_2(\mathbb{C})$, there is an unramified representation $\pi(s')$ of $G_2(k)$ with Satake parameter s'. The parameter φ of $\pi(s)$ is trivial on $\mathrm{SL}_2(\mathbb{C})$ and on the inertia subgroup of $W(k)$, and $s = \varphi(Fr)$. Let $s' = f(s)$. Then Conjecture 3.1 predicts that

$$\Theta\big(\pi(s)\big) = \{\pi(s')\}. \tag{4.1}$$

This statement has been checked for tempered $\pi(s)$ in [7]. Recall that $\pi(s)$ is tempered if s is contained in a compact subgroup of $\mathrm{SL}_3(\mathbb{C})$.

Let St be the Steinberg representation of $\mathrm{PGL}_3(k)$, and 1_D the trivial (=Steinberg) representation of $D^\times/k^\times$. These have parameter φ trivial on $W(k)$ and giving the embedding of the principal $\mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_3(\mathbb{C})$. The parameter $\varphi' = f\circ\varphi$ gives the sub-regular $\mathrm{SL}_2(\mathbb{C})$ in $G_2(\mathbb{C})$, with $A_{\varphi'} = S_3$. The corresponding L-packet on $G_2(k)$ has 3 members [8], p. 482

$$\{\pi'_{gen},\ \pi'_I,\ \pi'_{sc}[1]\} \tag{4.2}$$

where π'_{gen} is the unique element with a Whittaker model, and with a 3-dimensional space of Iwahori invariants and was studied by Lusztig [6]. The representation π'_I has a 1-dimensional space of Iwahori invariants; it is square integrable and was studied by Borel [2]. Finally, $\pi'_{sc}[1]$ is unipotent super-cuspidal, and induced from the unipotent cuspidal representation of $G_2(\mathcal{O}_k)$ (pulled back from $G_2(q)$) of dimension $q(q-1)^2(q^3+1)/6(q+1)$. We predict that:

$$\begin{cases} \Theta(\mathrm{St}) = \{\pi'_{gen}, \pi'_{sc}[1]\} \\ \Theta(1_D) = \{\pi'_I\}. \end{cases} \tag{4.3}$$

https://doi.org/10.4153/CMB-1997-045-0 Published online by Cambridge University Press

Now let χ be an unramified cubic character of $k^\times$. We have the twisted representations $\mathrm{St}\otimes\chi$ and $\chi_D = 1_D \otimes \chi$. The corresponding parameter has $A_\varphi = A_{\varphi'} = \mu_3$, and the lifted L-packet on $G_2(k)$ has 3 members [8], p. 482

(4.4) $$\{\pi'_{gen},\ \pi'_{sc}[\xi],\ \pi'_{sc}[\xi^2]\}$$

where π'_{gen} is the unique element with a Whittaker model, and $\pi'_{sc}[\xi^a]$ are unipotent supercuspidal representations of $G_2(O_k)$ (pulled back from $G_2(q)$) of dimension $q(q^2-1)^2/3$. We predict that:

(4.5) $$\begin{cases} \Theta(\mathrm{St}\otimes\chi) = \Theta(\mathrm{St}\otimes\chi^2) = \{\pi'_{gen}\} \\ \Theta(\chi_D) \cup \Theta(\chi_D^2) = \{\pi'_{sc}[\xi], \pi'_{sc}[\xi^2]\}. \end{cases}$$

These predictions are consistent with the following. Let K be the special maximal compact subgroup of H_D with reduction $D_4^3(q)$. Then the minimal K-type of the minimal representation of H_D should be the reflection representation of $D_4^3(q)$, of dimension $q^5 - q^3 + q$. This representation, restricted to $G_2(q)$, is a sum of 3 representations, 2 of which are the unipotent cuspidal of dimension $q(q^2-1)^2/3$.

Let $\tilde{Q}_1$ and $\tilde{Q}_2$ be the two non-conjugated maximal parabolic subgroups of $\mathrm{GL}_3(k) = \mathrm{GL}(W_3)$ stabilizing 1-dimensional space W_1 and 2-dimensional space W_2 in W_3, respectively. We fix $W_1 \subset W_2$. Their Levi factors are $\mathrm{GL}(W_1)\times\mathrm{GL}(W_1^\perp)$ and $\mathrm{GL}(W_2)\times\mathrm{GL}(W_2^\perp)$ respectively, where $W_1^\perp$ and $W_2^\perp$ are annihilators of W_1 and W_2 in W_3^*. The corresponding maximal parabolic subgroups in PGL_3 will be denoted by $Q_1 = L_1U_1$ and $Q_2 = L_2U_2$. We have isomorphisms

(4.6) $$\begin{cases} L_1 \cong \mathrm{GL}(W_1^\perp) \\ L_2 \cong \mathrm{GL}(W_2). \end{cases}$$

The modular characters of L_1 and L_2 are

(4.7) $$\rho_1(g) = |\det g|^{1/2} \text{ and } \rho_2(g) = |\det g|^{1/2}.$$

Let τ be a self-contragredient, super-cuspidal representation of $\mathrm{GL}(W_2)$. Let $\tau_s = \tau \otimes |\det|^s$. Then the generalized principal series of $\mathrm{PGL}_3(k)$

(4.8) $$\begin{cases} \pi_1(s) = \mathrm{Ind}_{Q_1}^{\mathrm{PGL}_3}(\tau_s) \\ \pi_2(s) = \mathrm{Ind}_{Q_2}^{\mathrm{PGL}_3}(\tau_s) \end{cases}$$

are irreducible, and we have isomorphisms

(4.9) $$\begin{cases} \pi_1^\vee(s) = \pi_2(s) \\ \pi_2^\vee(s) = \pi_1(s). \end{cases}$$

The parameter φ of $\pi(0) = \pi_1(0) = \pi_2(0)$ is trivial on $\mathrm{SL}_2(\mathbb{C})$, and factorizes through

(4.10) $$\varphi\colon W(k) \longrightarrow \mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_3(\mathbb{C}),$$

https://doi.org/10.4153/CMB-1997-045-0 Published online by Cambridge University Press

where $W(k) \longrightarrow \mathrm{GL}_2(\mathbb{C})$ is the parameter of τ, and $\mathrm{GL}_2(\mathbb{C})$, is a Levi factor of a maximal parabolic subgroup of $\mathrm{SL}_3(\mathbb{C})$, stabilizing a line in $\mathbb{C}^3$. Let χ_τ be the central character of τ. Note that $\chi_\tau^2 = 1$, since $\tau \cong \tau^\vee$. The image of φ is contained in

$$\begin{cases} \mathrm{SL}_2(\mathbb{C}) \text{ if } \chi_\tau = 1 \\ O_2(\mathbb{C}) \text{ if } \chi_\tau \neq 1. \end{cases} \tag{4.11}$$

Maximal parabolic subgroups of $G_2(k)$ can be defined as stabilizers of non-trivial nil subalgebras of $\mathbb{O}\otimes k$. A nil subalgebra is a subspace consisting of traceless elements with trivial multiplication (*i.e.* the product of any two elements is 0). The possible dimensions are 1 and 2. Fix $V_1 \subset V_2$, a pair of nil-subalgebras. Then $P_1 = M_1N_1$ and $P_2 = M_2N_2$, the stabilizers of V_1 and V_2, are two non-conjugated maximal parabolic subgroups of G_2, with $P_1 \cap P_2$ a Borel subgroup. Let

$$V_3 = \{x \in \mathbb{O} \otimes k \mid \bar{x} = -x, \text{ and } x \cdot V_1 = 0\}. \tag{4.12}$$

We have isomorphisms

$$\begin{cases} M_1 \cong \mathrm{GL}(V_3/V_1) \\ M_2 \cong \mathrm{GL}(V_2). \end{cases} \tag{4.13}$$

The action of the Levi factor of P_1 on V_1 is given by det, and the modular characters are

$$\rho_1'(g) = |\det(g)|^{5/2} \text{ and } \rho_2'(g) = |\det(g)|^{3/2}. \tag{4.14}$$

Let τ be as above, and define a generalized principal series by

$$I_2(s) = \mathrm{Ind}_{P_2}^{G_2}(\tau_s). \tag{4.15}$$

If $s > 0$, then $I_2(s)$ has unique (Langlands') quotient $\pi_2'(s)$; equivalently, $\pi_2'(s)$ is unique submodule of $I_2(-s)$. The parameter φ' of $\pi_2'(s)$ is $f \circ \varphi$, where φ is the parameter of $\pi_1(s)$ or $\pi_2(s)$. Also, using (4.11), it is easy to see that the centralizer $A_{\varphi'}$ of the parameter φ' of $I_2(0)$ is

$$\begin{cases} 1 \text{ if } \chi_\tau = 1 \\ \mu_2 \text{ if } \chi_\tau \neq 1. \end{cases} \tag{4.16}$$

Therefore, $I_2(0)$ should be irreducible unless $\chi_\tau \neq 1$, in which case $I_2(0) = \pi_2' + \pi_{2,gen}'$, where $\pi_{2,gen}'$ is unique generic summand. This was shown by Shahidi [10]. Hence Conjecture 3.1 predicts the following.

PROPOSITION 4.17. *If $s > 0$, then*

$$\Theta\big(\pi_1(s)\big) = \Theta\big(\pi_2(s)\big) = \{\pi_2'(s)\}.$$

Also,

$$\Theta\big(\pi(0)\big) = \begin{cases} \{I_2(0)\} \text{ if } \chi_\tau = 1 \\ \{\pi_2', \pi_{2,gen}'\} \text{ if } \chi_\tau \neq 1. \end{cases}$$

PROOF. In the next section.

Let π' be an irreducible representation of $G_2(k)$. We define $\Theta_H(\pi')$ as the set of irreducible representations of π of $\mathrm{PGL}_3(k)$ such that $\pi \otimes \pi'$ is a quotient of the minimal representation of H. Conjecture 3.1 predicts the following.

PROPOSITION 4.18. *If $s > 0$,*

$$\Theta_H\big(\pi_2'(s)\big) = \{\pi_1(s), \pi_2(s)\}.$$

Also, if $\chi_\tau \neq 1$,

$$\Theta_H(\pi_2') = \Theta_H(\pi_{2,gen}') = \{\pi(0)\}.$$

PROOF. In the next section.

Finally, let $\pi_1'(s)$, $(s > 0)$, be the Langlands' quotient of the other generalized principal series $I_1(s) = \mathrm{Ind}_{P_1}^{G_2}(\tau_s)$. Conjecture 3.1 predicts that $\pi_1'(s)$ does not appear in the restriction of the minimal representations of H and H_D. In particular,

$$\Theta_H\big(\pi_1'(s)\big) = \emptyset. \tag{4.19}$$

5. **Some calculations.** We now proceed to show Proposition 4.17. Assume that $s \geq 0$, and let π' be in $\Theta\big(\pi_1(s)\big)$. Since $\pi_1(s) = \pi_2(-s)$, by Frobenius reciprocity,

$$\mathrm{Hom}_{\mathrm{PGL}_3(k)\times G_2(k)}(\Pi, \pi_2(-s)\otimes\pi') = \mathrm{Hom}_{L_2(k)\times G_2(k)}(\Pi_{U_2}, \tau_{-s+\frac{1}{2}}\otimes\pi') \tag{5.1}$$

where $\frac{1}{2}$ enters through the normalization of parabolic induction. Hence we need to find out for which π', $\tau_{-s+\frac{1}{2}}\otimes\pi'$ is a quotient of Π_{U_2}.

The structure of the $L_2(k)\times G_2(k)$-module Π_{U_2}, is given by [7; Theorem 4.3]. To describe the needed result, we need some additional notation. There exists (see [7]) a maximal parabolic $\mathfrak{Q}_2 = \mathfrak{L}_2\mathfrak{U}_2$ in H whose Levi factor $\mathfrak{L}_2$ is of type D_5, and such that

$$\begin{cases} (\mathrm{PGL}_3 \times G_2)\cap\mathfrak{L}_2 = L_2\times G_2 \\ \mathrm{PGL}_3\cap\mathfrak{U}_2 = U_2. \end{cases} \tag{5.2}$$

Let B be a Borel subgroup of $\mathrm{GL}(W_2)$, stabilizing the line W_1.

PROPOSITION 5.3 [7; THEOREM 4.3]. *Let $\mathrm{GL}_2(k) = \mathrm{GL}(W_2)$ be the Levi factor of Q_2. Then the $\mathrm{GL}_2(k)\times G_2(k)$-module Π_{U_2} has a filtration*

$$0 = \Pi_0 \subset \Pi_1 \subset \Pi_2 \subset \Pi_3 = \Pi_{U_2}$$

such that

(1) $\Pi_1/\Pi_0 \cong |\det|^2 \otimes \mathrm{ind}_{\mathrm{GL}_2\times P_2}^{\mathrm{GL}_2\times G_2}\big(C_c^\infty(\mathrm{GL}_2)\big)$
(2) $\Pi_2/\Pi_1 \cong |\det|^2 \otimes \mathrm{ind}_{B\times P_1}^{\mathrm{GL}_2\times G_2}\big(C_c^\infty(\mathrm{GL}_1)\big)$
(3) $\Pi_3/\Pi_2 = \Pi_{\mathfrak{U}_2} \cong |\det| \otimes \Pi(\mathfrak{L}_2) + |\det|^2 \otimes 1$

Here det is the usual determinant on $\mathrm{GL}(W_2)$, and the induction ind is not normalized. In (1), $C_c^\infty(\mathrm{GL}_2)$ is the regular representation of

$$GL(W_2)\times \mathrm{GL}(V_2).$$

In (2), $C_c^\infty(\mathrm{GL}_1)$ is the regular representation of

$$GL(W_1)\times \mathrm{GL}(V_1).$$

In (3), $\Pi(\mathfrak{L}_2)$ is the minimal representation of $\mathfrak{L}_2$. The center of $\mathfrak{L}_2$, which coincides with the center of $\mathrm{GL}(W_2)$, acts trivially on $\Pi(\mathfrak{L}_2)$.

Next, we need the following

https://doi.org/10.4153/CMB-1997-045-0 Published online by Cambridge University Press

LEMMA 5.4. *If $\chi_\tau \neq 1$ or $s \neq -1/2$, then $\tau_{-s+\frac{1}{2}} \otimes \pi'$ is a quotient of Π_{U_2} if and only if it is a quotient of Π_1.*

PROOF. The center of $\mathrm{GL}(W_2)$ acts on $\tau_{-s+\frac{1}{2}}$ by $\chi_\tau \cdot |\cdot|^{1-2s}$, and on $|\det| \otimes \Pi(\mathfrak{L}_2)$ by $|\cdot|^2$. If $\chi_\tau \neq 1$ or $s \neq -1/2$, then these two central characters are different, hence $\tau_{-s+\frac{1}{2}} \otimes \pi'$ is a quotient of Π_{U_2} if and only if it is a quotient of Π_2. Since τ is a supercuspidal representation, $\tau_{-s+\frac{1}{2}} \otimes \pi'$ is a quotient of Π_2 if and only if it is a quotient of Π_1. This proves the lemma.

By the Peter-Weyl, $|\det|^2 \otimes C_c^\infty(\mathrm{GL}_2)$ has

$$\tau_{-s+\frac{1}{2}} \otimes \tau_{s+\frac{3}{2}}$$

as unique $\mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$-invariant quotient transforming as $\tau_{-s+\frac{1}{2}}$ under the first factor. Hence $\tau_{-s+\frac{1}{2}} \otimes \pi'$ is a quotient of Π_1, if and only if π' is a quotient of $I_2(s)$. Hence we obtain $\Theta\big(\pi_1(s)\big) = \{\pi_2'(s)\}$ if $s > 0$, and the second statement of Proposition 4.17. The statement $\Theta\big(\pi_2(s)\big) = \{\pi_2'(s)\}$ follows from (3.2) and (4.9).

COROLLARY 5.5. *(Shahidi). Assume that $s \neq 0$. If $\chi_\tau \neq 1$, or $\chi_\tau = 1$ and $s \neq \pm 1/2$, then $I_2(s)$ is irreducible.*

PROOF. Assume that $s > 0$. By (4.17) we know that $\pi_1(s) \otimes \pi_2'(s)$ is a quotient of Π. By Frobenius reciprocity, $\tau_{s+\frac{1}{2}} \otimes \pi_2'(s)$ is a quotient of Π_{U_2}, and if $\chi_\tau \neq 1$ or $s \neq \frac{1}{2}$, then it must be a quotient of Π_1, as in Lemma 5.4. Hence $\pi_2'(s)$ is a quotient of $I_2(-s)$. However, $\pi_2'(s)$ is unique submodule of $I'(-s)$. Both are possible only if $I_2(-s)$ is irreducible. Since $I_2(s) \cong I_2(-s)^\vee$, the corollary follows.

We now check Proposition 4.18. Let $s \geq 0$, and let π' be a submodule of $I_2(-s)$. Then, by Proposition 4.17,

$$\{\pi_1(s), \pi_2(s)\} \subseteq \Theta_H(\pi'). \tag{5.6}$$

Let π be in $\Theta_H(\pi')$. By Frobenius reciprocity,

$$\mathrm{Hom}_{\mathrm{PGL}_3(k)\times G_2(k)}\big(\Pi, \pi \otimes I_2(-s)\big) = \mathrm{Hom}_{\mathrm{PGL}_3(k)\times M_2(k)}(\Pi_{N_2}, \pi \otimes \tau_{-s+\frac{3}{2}}), \tag{5.7}$$

where $\frac{3}{2}$ enters through the normalization of parabolic induction. We need to find out for which π, $\pi \otimes \tau_{-s+\frac{3}{2}}$ is a quotient of Π_{N_2}.

The structure of the $\mathrm{PGL}_3(k) \times M_2(k)$-module Π_{N_2}, is given by [7; Theorem 7.6]. To describe the needed result, we need some additional notation. There exists (see [7]) a maximal parabolic $\mathfrak{P}_2 = \mathfrak{M}_2\mathfrak{N}_2$ in H whose Levi factor $\mathfrak{M}_2$ is of type A_5, and such that

$$\begin{cases} (\mathrm{PGL}_3 \times G_2) \cap \mathfrak{M}_2 = M_2 \times \mathrm{PGL}_3 \\ G_2 \cap \mathfrak{N}_2 = N_2. \end{cases} \tag{5.8}$$

Let B be the Borel subgroup of $\mathrm{GL}(V_2)$, stabilizing the line V_1, and $Q = Q_1 \cap Q_2$ the Borel subgroup of PGL_3 stabilizing the line $W_1 \otimes W_2^\perp$.

https://doi.org/10.4153/CMB-1997-045-0 Published online by Cambridge University Press

PROPOSITION 5.9 [7; THEOREM 7.6]. *Let* $\mathrm{GL}_2(k) = \mathrm{GL}(W_2)$ *be the Levi factor of* P_2. *Then the* $\mathrm{PGL}_3(k) \times \mathrm{GL}_2(k)$*-module* Π_{N_2} *has a filtration*

$$0 = \Pi_0 \subset \Pi_1 \subset \Pi_2 \subset \Pi_3 = \Pi_{N_2}$$

such that

(1) $\Pi_1/\Pi_0 \cong \mathrm{ind}_{Q_1\times \mathrm{GL}_2}^{\mathrm{PGL}_3\times GL_2}\big(C_c^\infty(\mathrm{GL}_2)\big)\otimes|\det|^2 + \mathrm{ind}_{Q_2\times \mathrm{GL}_2}^{\mathrm{PGL}_3\times \mathrm{GL}_2}\big(C_c^\infty(\mathrm{GL}_2)\big)\otimes|\det|^2$

(2) $\Pi_2/\Pi_1 \cong \mathrm{ind}_{Q\times B}^{\mathrm{PGL}_3\times \mathrm{GL}_2}\big(C_c^\infty(\mathrm{GL}_1)\big)\otimes|\det|^2$

(3) $\Pi_3/\Pi_2 = \Pi_{\mathfrak{N}_2} \cong \Pi(\mathfrak{M}_2)\otimes|\det|^{\frac{3}{2}} + 1\otimes|\det|^2$.

Here det is the usual determinant on $\mathrm{GL}(V_2)$, and the induction ind is not normalized. In (1), $C_c^\infty(\mathrm{GL}_2)$ is the regular representation of

$$GL(W_1^\perp)\times \mathrm{GL}(V_2) \text{ and } \mathrm{GL}(W_2)\times \mathrm{GL}(V_2)$$

respectively. In (2), $C_c^\infty(\mathrm{GL}_1)$ is the regular representation of

$$GL(W_1\otimes W_2^\perp)\times \mathrm{GL}(V_1).$$

In (3), $\Pi(\mathfrak{M}_2)$ is the minimal representation of $\mathfrak{M}_2$. The center of $\mathfrak{M}_2$, which coincides with the center of $\mathrm{GL}(V_2)$, acts trivially on $\Pi(\mathfrak{M}_2)$.

Similar to Lemma 5.4, one proves:

LEMMA 5.10. *If* $\chi_\tau \neq 1$ *or* $s \neq 0$, *then* $\pi\otimes\tau_{-s+\frac{3}{2}}$ *is a quotient of* Π_{N_2} *if and only if it is a quotient of* Π_1.

By the Peter-Weyl, $C_c^\infty(\mathrm{GL}_2)\otimes|\det|^2$ has

$$\tau_{s+\frac{1}{2}}\otimes\tau_{-s+\frac{3}{2}}$$

as unique $\mathrm{GL}_2(k)\times\mathrm{GL}_2(k)$-invariant quotient transforming as $\tau_{-s+\frac{3}{2}}$ under the second factor. Hence $\pi\otimes\tau_{-s+\frac{3}{2}}$ is a quotient of Π_1, if and only if π is a quotient of (hence isomorphic to) $\pi_1(s)$, or $\pi_2(s)$. Therefore $\Theta_H(\pi')\subseteq\{\pi_1(s),\pi_2(s)\}$, and Proposition 4.18 follows from (5.6).

REFERENCES

1. M. Aschbacher, *The* 27*-dimensional module for* E_6*. I*, Invent. Math. **89**(1987), 159–196.
2. A. Borel, *Admissible representations of semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Invent. Math. **35**(1976), 233–259.
3. B. H. Gross and D. Prasad, *On the decomposition of a representation of* SO_n *when restricted to* SO_{n-1}, Can. J. Math. **44**(1992), 974–1002.
4. B. H. Gross and G. Savin, *Motives with Galois group* G_2, (1996), preprint.
5. N. Jacobson, *Automorphisms of composition algebras*, Rend. Palermo, 1958.
6. G. Lusztig, *Some examples of square integrable representations of semisimple p-adic groups*, Tran. Am. Math. Soc. **277**(1983), 623–653.
7. K. Magaard and G. Savin, *Exceptional* Θ*-correspondences*, Compositio, to appear.
8. M. Reeder, *Iwahori spherical discrete series*, Annales ENS **27**(1994), 463–491.

https://doi.org/10.4153/CMB-1997-045-0 Published online by Cambridge University Press

9. G. Savin, *Dual pair* $G_J \times \mathrm{PGL}_2$*;* G_J *is the automorphism group of the Jordan algebra* J, Invent. Math. **118**(1994), 141–160.

10. F. Shahidi, *Langlands' conjecture on Plancherel measures for p-adic groups*. In: Harmonic Analysis on Reductive Groups, Bowdoin College, Birkhauser, 1991.

Department of Mathematics
Harvard University
Cambridge, MA
USA 02138
e-mail: gross@math.harvard.edu

Department of Mathematics
University of Utah
Salt Lake City, UT
USA 84112
e-mail: savin@math.utah.edu

https://doi.org/10.4153/CMB-1997-045-0 Published online by Cambridge University Press