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MATHEMATICAL NOTES

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NOTE ON "PARACOMPACTNESS IN SMALL PRODUCTS"

BY
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In [1], Willard proves the following

LEMMA. *If a regular paracompact space X has a dense Lindelöf subspace, then X is Lindelöf.*

Willard notes that the above is a generalization of the standard theorem: A separable paracompact space is Lindelöf. Actually, it is a standard fact ([2, p. 24]) that a separable metacompact space is Lindelöf. Moreover, one discovers that if a separable space X is such that each open cover of X has a point-countable open refinement, then X is Lindelöf.

The above Lemma was used in [1] to obtain a very short proof of the main theorem. We are unable to prove Willard's lemma if "paracompact" is replaced by "metacompact". We did, however, succeed in removing the regularity requirement from the hypotheses and in weakening slightly the paracompactness requirement. It is also possible to relax the requirement that the space possesses a dense Lindelöf subspace.

In [3], it is mentioned that a crucial concept in the investigation is an idea which generalizes simultaneously the notion of the Lindelöf property and that of separability. Following [3], we call a space *weakly Lindelöf* if each open cover of the space contains a countable subcollection with a dense union. It is clear that if a space is either Lindelöf or separable, it has necessarily a dense Lindelöf subspace and that if a space has a dense Lindelöf subspace, it is necessarily weakly Lindelöf. As an example of a non-Lindelöf space which has a dense Lindelöf subspace, we mention the Sorgenfrey plane ([2, p. 103, Example 84]). The purpose of this note is to prove the following theorem, in which no separation axiom is assumed.

THEOREM. *If X is such that each open cover of X has a locally countable open refinement and X is weakly Lindelöf, then X is Lindelöf.*

Proof. Suppose X is not Lindelöf. Then there exists an open cover $\mathcal{U}$ of X such that $\mathcal{U}$ has no countable subcover. Let $\mathcal{V}$ be a locally countable open refinement of

$\mathscr{U}$. For each $x \in X$, let O_x be an open set containing x such that O_x meets at most countably many sets of $\mathscr{V}$. Let $\mathscr{V}_x = \{V \in \mathscr{V} : V \text{ meets } O_x\}$. Since $\{O_x : x \in X\}$ is an open cover of X, there exists a countable subcollection $\{O_{x_i} : i \in N\}$ such that $D = \bigcup \{O_{x_i} : i \in N\}$ is dense in X. Since the collection $\mathscr{V}_0 = \bigcup \{\mathscr{V}_{xi} : i \in N\}$ is a countable subcollection of $\mathscr{V}$, we see that $\mathscr{V}_0$ is not a cover of X. Select a nonempty set $V_* \in \mathscr{V} - \mathscr{V}_0$. Since D is dense in X, V_* meets D. That is, V_* meets O_{x_j} for some $j \in N$. This implies $V_* \in \mathscr{V}_{x_j}$ and so $V_* \in \mathscr{V}_0$, a contradiction.

COROLLARY. *Let X be a space such that each open cover of X has a locally countable open refinement. The following conditions on X are equivalent:*

(1) *X has a dense Lindelöf subspace*

(2) *X is weakly Lindelöf*

(3) *X is Lindelöf*

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