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Weights of the Mod p Kernels of Theta Operators

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Abstract. Let $\Theta^{[j]}$ be an analogue of the Ramanujan theta operator for Siegel modular forms. For a given prime p, we give the weights of elements of mod p kernel of $\Theta^{[j]}$, where the mod p kernel of $\Theta^{[j]}$ is the set of all Siegel modular forms F such that $\Theta^{[j]}(F)$ is congruent to zero modulo p. In order to construct examples of the mod p kernel of $\Theta^{[j]}$ from any Siegel modular form, we introduce new operators $A^{(j)}(M)$ and show the modularity of $F|A^{(j)}(M)$ when F is a Siegel modular form. Finally, we give some examples of the mod p kernel of $\Theta^{[j]}$ and the filtrations of some of them.

1 Introduction

Serre [23] developed the theory of p-adic and congruences for the elliptic modular forms and produced several interesting results. For q-expansions $f = \sum_{n\geq 0} a_f(n)q^n$ of the elliptic modular forms, the Ramanujan theta operator is defined as

$$\theta = q\frac{d}{dq} : f \longmapsto \theta(f) = \sum_{n\geq 1} n a_f(n) q^n.$$

This operator plays an important role in Serre's paper. In particular, he showed that the filtrations of all elements of the mod p kernel of θ are divisible by p for the case of level 1, where the mod p kernel of θ is the set of all elliptic modular forms f such that $\theta(f) \equiv 0 \bmod p$. Moreover, Katz [15] showed that this property holds for the general level case.

Böcherer and Nagaoka [6] extended the notion of the θ-operator to the case of Siegel modular forms of degree n. For this operator Θ (defined in [6]), several people found examples of elements of the mod p kernel of Θ, *i.e.*, Siegel modular forms F satisfying $\Theta(F) \equiv 0 \bmod p$. The Klingen–Eisenstein series of weight 12 arising from Ramanujan's Δ function is such an example for $p = 23$ [2]. Mizumoto [18] found another example of weight 16 and $p = 31$, which comes from the Klingen–Eisenstein series arising from a cusp form of weight 16. Recently, Kodama and Nagaoka, as well as the authors, constructed families of such examples of weight $\frac{n+p-1}{2}$ (resp. $\frac{n+3p-1}{2}$) and degree n if the weight is even (resp. odd) [5, 16, 21, 22, 25].

A new feature in the case of Siegel modular forms that are not elliptic modular forms is that one should also study vector-valued generalizations $\Theta^{[j]}$ of Θ-operator

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for $0 \le j \le n$; their p-adic properties were given in [8], *e.g.*, $\Theta^{[1]}$ maps a Siegel modular form $\sum_T a_F(T)q^T$ to a formal series $\sum_T Ta_F(T)q^T$ with coefficients in symmetric matrices of size n.

In this paper, we discuss the necessity (as in the one variable cases) of the relation between the weight and the prime p for an element of the mod p kernel of the generalized theta operators $\Theta^{[j]}$ in the case where the weight is small compared with p. We remark that Yamauchi [28] and Weissauer [26] also studied the necessity in the special cases $\Theta^{[1]}$ or Θ. Moreover we construct elements of the mod p kernel of $\Theta^{[j]}$ from arbitrary Siegel modular forms. In order to do this, we introduce an operator $A^{(j)}(M)$ and study its properties (Section 4). Finally, we give some examples of the mod p kernel of $\Theta^{[j]}$ and introduce the filtrations of some of them (Sections 5, 6).

2 Preliminaries

2.1 Siegel Modular Forms

We denote by $\mathbb{H}_n$ the Siegel upper half-space of degree n. We define an action of the symplectic group $\mathrm{Sp}_n(\mathbb{R})$ on $\mathbb{H}_n$ by $gZ = (AZ+B)(CZ+D)^{-1}$ for $Z \in \mathbb{H}_n$, $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$. For a holomorphic function $F\colon \mathbb{H}_n \to \mathbb{C}$ and a matrix $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$, we define a slash operator by $F|_k\, g = j(g,Z)^{-k}F(gZ)$, where $j(g,Z) := \det(CZ+D)$.

Let N be a natural number. In this paper, we deal with three types of congruence subgroups of Siegel modular group $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$ as follows:

$$\begin{aligned}
\Gamma^{(n)}(N) &:= \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n \,\middle|\, B \equiv C \equiv 0_n \bmod N,\ A \equiv D \equiv 1_n \bmod N \right\}, \\
\Gamma_1^{(n)}(N) &:= \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n \,\middle|\, C \equiv 0_n \bmod N,\ A \equiv D \equiv 1_n \bmod N \right\}, \\
\Gamma_0^{(n)}(N) &:= \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n \,\middle|\, C \equiv 0_n \bmod N \right\}.
\end{aligned}$$

Let Γ be one of the above modular groups of degree n with level N. For a natural number k and a Dirichlet character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, the space $M_k(\Gamma,\chi)$ of Siegel modular forms of weight k with character χ consists of all holomorphic functions $F\colon \mathbb{H}_n \to \mathbb{C}$ satisfying $F|_k\, g = \chi(\det D)F(Z)$, for $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$. If $n = 1$, the usual condition for the cusps should be added.

If $k = l/2$ is half-integral, then we assume that the level N of Γ satisfies $4 \mid N$. For $g \in \Gamma_0^{(n)}(4)$, we put $j_{1/2}(g,Z) := \theta^{(n)}(gZ)/\theta^{(n)}(Z)$, where $\theta^{(n)}(Z) := \sum_{X\in\mathbb{Z}^n} e^{2\pi i\, {}^tXZX}$.

Then it is known that $j_{1/2}(g,Z)^2 = \left(\frac{-4}{\det D}\right)\det(CZ+D)$ for $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(n)}(4)$, where $\left(\frac{-4}{*}\right)$ is the Kronecker character for the discriminant -4.

We define a slash operator for a holomorphic function $F\colon \mathbb{H}_n \to \mathbb{C}$ by

$$F|_k g := j_{1/2}(g,Z)^{-l}F(gZ) \quad \text{for } g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma.$$

We denote by $M_k(\Gamma,\chi)$ the space of all holomorphic functions $F\colon \mathbb{H}_n \to \mathbb{C}$ such that $F|_k g = \chi(\det D)F(Z)$ for $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$. For more details on Siegel modular forms of half-integral weight, we refer to [1].

When χ is a trivial character, we simply write $M_k(\Gamma)$ for $M_k(\Gamma,\chi)$. Any $F \in M_k(\Gamma,\chi)$ has a Fourier expansion of the form

$$F(Z) = \sum_{0\le T\in \frac{1}{N}\Lambda_n} a_F(T)q^T, \quad q^T := e^{2\pi i \operatorname{tr}(TZ)}, Z \in \mathbb{H}_n,$$

where $\Lambda_n := \{T = (t_{ij}) \in \mathrm{Sym}_n(\mathbb{Q}) \mid t_{ii},\ 2t_{ij} \in \mathbb{Z}\}$ (the lattice in $\mathrm{Sym}_n(\mathbb{R})$ of half-integral, symmetric matrices). In particular, if Γ satisfies $\Gamma \supset \Gamma_1^{(n)}(N)$, then the Fourier expansion of F is given by the form $F(Z) = \sum_{0\le T\in\Lambda_n} a_F(T)q^T$.

We denote by Λ_n^+ the set of all positive definite elements of Λ_n. For a subring R of $\mathbb{C}$, let $M_k(\Gamma,\chi)_R \subset M_k(\Gamma,\chi)$ denote the R-module of all Siegel modular forms whose Fourier coefficients are in R.

2.2 Vector-valued Siegel Modular Forms

For later use, we briefly introduce the notion of vector-valued Siegel modular forms. Let $\Gamma \subset \Gamma_n$ be one of the subgroups of level N introduced in Subsection 2.1, and let $\rho\colon \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{\mathbb{C}}(V_\rho)$ be a polynomial representation of $\mathrm{GL}_n(\mathbb{C})$. A holomorphic function $F\colon \mathbb{H}_n \to V_\rho$ is said to be a (holomorphic) vector-valued Siegel modular form of automorphy factor ρ and of level Γ if and only if F satisfies the following property: $F(gZ) = \rho(CZ+D)F(Z)$ for all $g = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma$. If $n = 1$, we add the cusp condition.

As in the scalar-valued case, a vector-valued Siegel modular form F has the following Fourier expansion:

$$F(Z) = \sum_{0\le T\in \frac{1}{N}\Lambda_n} a_F(T)q^T, \quad Z \in \mathbb{H}_n,\ a_F(T) \in V_\rho.$$

2.3 Congruences for Modular Forms

Let F_1, F_2 be two formal power series of the forms $F_i = \sum_{0\le T\in\frac{1}{N}\Lambda_n} a_{F_i}(T)q^T$ with $a_{F_i}(T) \in \mathbb{Z}_{(p)}$. We write $F_1 \equiv F_2 \bmod p$, if and only if $a_{F_1}(T) \equiv a_{F_2}(T) \bmod p$ for all $T \in \frac{1}{N}\Lambda_n$ with $T \ge 0$.

Let p be a prime and $\widetilde{M}_k(\Gamma)_{p^l}$ the space of modular forms modulo p^l for Γ defined as $\widetilde{M}_k(\Gamma)_{p^l} := \{\widetilde{F} \mid F \in M_k(\Gamma)_{\mathbb{Z}_{(p)}}\}$, where $\widetilde{F} := \sum_T \widetilde{a_F(T)}q^T$ and $\widetilde{a_F(T)} := a_F(T) \bmod p^l$. We put $\widetilde{M}(\Gamma)_{p^l} := \sum_{k\in\mathbb{Z}_{\ge 0}} \widetilde{M}_k(\Gamma)_{p^l}$.

We also explain a notion of vector-valued modular forms modulo p^l. A naive notion of vector-valued modular forms modulo p^l is sufficient for our purpose. We suppose that the representation ρ is matrix-valued and $V = \mathbb{C}^d$. We call a formal series $\sum_T a(T)q^T$ with $a(T) \in (\mathbb{Z}/p^l\mathbb{Z})^d$ a modular form modulo p^l, if it arises from a $\mathbb{C}^d$-valued modular form with Fourier coefficients in $\mathbb{Z}_{(p)}^d$ by reduction modulo p^l (coefficientwise). Congruences between $\mathbb{C}^d$-valued modular forms can be explained in the same way. Note that this notion depends on choosing a basis for the representation space $V = V_\rho$. All the vector-valued modular forms considered in our paper arise from scalar-valued ones by differential operators and the choice of coordinates will be obvious.

Let ω_N be the filtration of modular forms modulo p introduced by Serre and Swinnerton-Dyer: for a formal power series of the form $F = \sum_{T\in\Lambda_n} a_F(T)q^T$ (not constant

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modulo p) with $a_F(T) \in \mathbb{Z}_{(p)}$, we define

$$\omega_N(F) := \inf\{k \in \mathbb{Z}_{\geq 1} \mid \widetilde{F} \in \widetilde{M}_k(\Gamma_1^{(n)}(N))_p\}.$$

If $F \equiv c \bmod p$ for some $c \in \mathbb{Z}_{(p)}$, then we regard it as $\omega_N(F) = 0$.

The following proposition immediately follows from [7,14].

Proposition 2.1 *Let p be an odd prime, N a positive integer with $p \nmid N$, and $F \in M_k(\Gamma_1^{(n)}(N))_{\mathbb{Z}_{(p)}}$. Then $\omega_N(F) \equiv k \bmod p-1$. In particular, if $k < p$ and $F \not\equiv c \bmod p$ for any $c \in \mathbb{Z}_{(p)}$, then $k = \omega_N(F)$.*

Let M, N be positive integers with $N \mid M$ and $F \in M_k(\Gamma_1^{(n)}(N))_{\mathbb{Z}_{(p)}}$. Then F can be regarded as an element of $M_k(\Gamma_1^{(n)}(M))_{\mathbb{Z}_{(p)}}$ and hence $\omega_M(F)$ is defined. It is plausible that $\omega_N(F) = \omega_M(F)$ if $p \nmid M$. We remark that we can prove the equality in some cases.

Proposition 2.2 *Let p be a prime, M, N positive integers such that $M \geq 3$, $N \mid M$ and $p \nmid M$, and $F \in M_k(\Gamma_1^{(n)}(N))_{\mathbb{Z}_{(p)}}$.*

(i) *Define a positive integer M' by $M' := N\prod_{q \mid M, q\nmid N} q$, where q runs over the prime factors of M such that $q \nmid N$. Then we have $\omega_{M'}(F) = \omega_M(F)$. Moreover, if $M'/N = \prod_{i=1}^t q_i$ is a prime factorization of M'/N and $p \nmid \prod_{\nu=1}^n (q_i^{2\nu} - 1)$ for all i with $1 \leq i \leq t$, then we have $\omega_N(F) = \omega_M(F)$.*

(ii) *If $n = 1$, then we have $\omega_N(F) = \omega_M(F)$.*

Proof For $F \in M_k(\Gamma_1^{(n)}(M))$ ($N \mid M$), we define the trace and the norm as follows:

$$\mathrm{Tr}_{\Gamma_1^{(n)}(M)\backslash\Gamma_1^{(n)}(N)} F := \sum_{\gamma \in \Gamma_1^{(n)}(M)\backslash\Gamma_1^{(n)}(N)} F|_k\gamma \in M_k(\Gamma_1^{(n)}(N)),$$

$$\mathrm{N}_{\Gamma_1^{(n)}(M)\backslash\Gamma_1^{(n)}(N)} F := \prod_{\gamma \in \Gamma_1^{(n)}(M)\backslash\Gamma_1^{(n)}(N)} F|_k\gamma \in M_{ki}(\Gamma_1^{(n)}(N)).$$

Here i is the index $[\Gamma_1^{(n)}(N) : \Gamma_1^{(n)}(M)]$. It is known that the q-expansion principal holds for Siegel modular forms [14]. Therefore, if F has p-integral rational Fourier coefficients, then $\mathrm{Tr}_{\Gamma_1^{(n)}(M)\backslash\Gamma_1^{(n)}(N)} F$ and $\mathrm{N}_{\Gamma_1^{(n)}(M)\backslash\Gamma_1^{(n)}(N)} F$ also do.

(i) We first prove that if $i := [\Gamma_1^{(n)}(N) : \Gamma_1^{(n)}(M)]$ is coprime to p, then $\omega_N(F) = \omega_M(F)$. Let $l = \omega_M(F)$, $G \in M_l(\Gamma_1^{(n)}(M))_{\mathbb{Z}_{(p)}}$, and $F \equiv G \mod p$. By taking the trace of both sides of $F \equiv G \bmod p$, we have $iF \equiv H \bmod p$ for some $H \in M_l(\Gamma_1^{(n)}(N))_{\mathbb{Z}_{(p)}}$. This implies $l = \omega_M(F) \geq \omega_N(F)$. The opposite inequality follows from the definition. Therefore we obtain $\omega_N(F) = \omega_M(F)$.

The strong approximation theorem for the symplectic group indicates that

$$[\Gamma_1^{(n)}(M') : \Gamma_1^{(n)}(M)] = \prod_{l\text{: prime}} [C_{1,l}^{(n)}(M') : C_{1,l}^{(n)}(M)].$$

Here, for a positive integer M, $C_{1,l}^{(n)}(M)$ is the closure of $\Gamma_1^{(n)}(M)$ in $\mathrm{GL}_{2n}(\mathbb{Z}_l)$ by the l-adic topology. Thus $[\Gamma_1^{(n)}(M') : \Gamma_1^{(n)}(M)]$ is coprime to p. Therefore $\omega_{M'}(F) =$

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$\omega_M(F)$ follows. Note that for a prime q, we have $|\mathrm{Sp}(n,\mathbb{F}_q)| = q^{n^2}\prod_{i=1}^n(q^{2i}-1)$, where $\mathbb{F}_q$ is the prime field of order q. If $p \nmid \prod_{\nu=1}^n(q^{2\nu}-1)$, then for a prime q with $q \nmid pN$, $[\Gamma_1^{(n)}(Nq) : \Gamma_1^{(n)}(N)]$ is coprime to p. Hence we have $\omega_N(F) = \omega_{M'}(F)$ if $p \nmid \prod_{\nu=1}^n(q_i^{2\nu}-1)$ for all $1 \le i \le t$.

(ii) Let $n = 1$ and suppose that $F \in M_k(\Gamma_1^{(1)}(N))_{\mathbb{Z}_{(p)}}$, $G \in M_l(\Gamma_1^{(1)}(M))_{\mathbb{Z}_{(p)}}$, $l = \omega_M(F)$, and $F \equiv G \mod p$. Taking the norm of both sides of $F \equiv G \bmod p$, we have $F^i \equiv H \mod p$, where $i = [\Gamma_1^{(1)}(N) : \Gamma_1^{(1)}(M)]$ and H is a modular form such that $H \in M_{li}(\Gamma_1^{(1)}(N))_{\mathbb{Z}_{(p)}}$. Therefore we have $\omega_N(F^i) \le li = i\omega_M(F)$. Applying Katz's result [15], we have $\omega_N(F^i) = i\omega_N(F)$ for any $i \in \mathbb{Z}_{\ge 1}$. This implies $i\omega_N(F) \le i\omega_M(F)$ and hence $\omega_N(F) \le \omega_M(F)$. The opposite inequality follows from the definition. Therefore we obtain $\omega_N(F) = \omega_M(F)$. ■

Definition 2.3 A formal power series of the form $F = \sum_{T\in\Lambda_n} a_F(T)q^T$ with $a_F(T) \in \mathbb{Q}_p$ is called a *p-adic modular form (of degree n)* if there exists a sequence $\{G_l \in M_{k_l}(\Gamma_n)_{\mathbb{Q}}\}$ of Siegel modular forms such that $\lim_{l\to\infty} G_l = F$ (p-adically). In other words, $\inf\{v_p(a_F(T) - a_{G_l}(T)) \mid 0 \le T \in \Lambda_n\} \to \infty$ as $l \to \infty$, where v_p is the usual additive p-valuation of $\mathbb{Q}$, normalized by $v_p(p) = 1$.

Theorem 2.4 ([8]) *Let p be a prime with $p \ge n+3$. Then any $F \in M_k(\Gamma_0^{(n)}(p^m))_{\mathbb{Z}_{(p)}}$ ($m \ge 0$) is a p-adic modular form. In particular, we have $\widetilde{M}(\Gamma_0^{(n)}(p^m))_{p^l} \subset \widetilde{M}(\Gamma_n)_{p^l}$ for any $l \ge 0$ and $m \ge 1$.*

2.4 Theta Operators and Their Properties

To define the operators $\Theta^{[j]}$ we need some notation. For a square matrix T of size n, we denote by $T^{[j]}$ the matrix of size $\binom{n}{j} \times \binom{n}{j}$ whose entries are given by the determinants of all submatrices of size j. For the reader's convenience, we give an example for $n = 3$.

Example 2.5 Let $T = (t_{ij})$ be a square matrix of size 3. Then we have $T^{[1]} = T$, $T^{[3]} = \det T$ and

$$T^{[2]} = \begin{pmatrix} \begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix} & \begin{vmatrix} t_{11} & t_{13} \\ t_{21} & t_{23} \end{vmatrix} & \begin{vmatrix} t_{12} & t_{13} \\ t_{22} & t_{23} \end{vmatrix} \\ \begin{vmatrix} t_{11} & t_{12} \\ t_{31} & t_{32} \end{vmatrix} & \begin{vmatrix} t_{11} & t_{13} \\ t_{31} & t_{33} \end{vmatrix} & \begin{vmatrix} t_{12} & t_{13} \\ t_{32} & t_{33} \end{vmatrix} \\ \begin{vmatrix} t_{21} & t_{22} \\ t_{31} & t_{32} \end{vmatrix} & \begin{vmatrix} t_{21} & t_{23} \\ t_{31} & t_{33} \end{vmatrix} & \begin{vmatrix} t_{22} & t_{23} \\ t_{32} & t_{33} \end{vmatrix} \end{pmatrix}.$$

Here we fixed an order to determine the entries of these matrices. The definition of $T^{[j]}$ depends on the choice of the order.

Using this notation, we can explain $\Theta^{[j]}$ by

$$F = \sum_T a_F(T)q^T \longmapsto \Theta^{[j]}(F) := \sum_T T^{[j]} \cdot a_F(T)q^T$$

for any formal power series F of the type above (*cf.* [5]).

These operators were introduced in [8], and it was shown that they define (vector-valued) modular forms modulo p, when applied to modular forms. Note that $\Theta^{[n]}$ is the Θ-operator defined in [6], *i.e.*, for a formal Fourier series $F = \sum_{T\in\Lambda_n} a_F(T)q^T$, we have $\Theta^{[n]}(F) = \sum_{T\in\Lambda_n}(\det T)a_F(T)q^T$. Hence, we simply write Θ for $\Theta^{[n]}$.

For $0 \le j \le n$, we observe the following (obvious) properties of these operators with respect to congruences.

Proposition 2.6 (i) *$\Theta^{[j]}(F) \not\equiv 0$ mod p is equivalent to the existence of $T \in \Lambda_n$ and a $j \times j$ submatrix R of T such that $a_F(T) \not\equiv 0$ mod p and $p \nmid \det R$.*

(ii) *$\Theta^{[j]}(F) \equiv 0$ mod p implies $\Theta^{[j+1]}(F) \equiv 0$ mod p.*

(iii) *$\Theta^{[j]}(F)$ is mod p singular if and only if $T^{[j]}a_F(T) \equiv 0$ mod p for all $T \in \Lambda_n^+$.*

(iv) *Let $j \ge 0$ be an integer. Assume that $\Theta^{[j]}(F)$ is mod p singular with p-rank r_p, where the p-rank is the maximum of r for $0 \le r \le n$ such that there exists $T \in \Lambda_n$ with $\operatorname{rank}(T) = r$ satisfying $T^{[j]}a_F(T) \not\equiv 0$ mod p. Then we have $\Theta^{[r_p+1]}(F) \equiv 0$ mod p.*

The following theorem is due to Katz [15], Serre [23], and Swinnerton-Dyer [24].

Theorem 2.7 *Let p be an arbitrary prime and N a positive integer with $p \nmid N$. For $f \in M_k(\Gamma_1^{(1)}(N))_{\mathbb{Z}_{(p)}}$, ($k \in \mathbb{Z}_{\ge 1}$), suppose that $\Theta^{[1]}(f) \equiv 0$ mod p. Then we have $p \mid \omega_N(f)$.*

Remark 2.8 In this case (the degree is 1), the operator $\Theta^{[1]}$ is the usual Ramanujan operator θ.

Swinnerton-Dyer and Serre proved the case $N = 1$ and Katz proved the case $N \ge 3$. The case $N = 2$ is obtained by the case $N \ge 3$ and Proposition 2.2.

Theorem 2.9 ([6]) *Let p be a prime with $p \ge n + 3$ and N a positive integer. If $F \in M_k(\Gamma^{(n)}(N))_{\mathbb{Z}_{(p)}}$, then $\widetilde{\Theta(F)} \in \widetilde{M}_{k+p+1}(\Gamma^{(n)}(N))_p$. Therefore we can regard Θ as a map $\Theta\colon \widetilde{M}(\Gamma^{(n)}(N))_p \to \widetilde{M}(\Gamma^{(n)}(N))_p$.*

We introduce a relation between mod p singular modular forms (see §3.1) and the mod p kernel of $\Theta^{[j]}$.

Proposition 2.10 *Let p be a prime with $p \ge 3$ and N a positive integer with $p \nmid N$. For a positive integer k, assume that $F \in M_k(\Gamma^{(n)}(N))_{\mathbb{Z}_{(p)}}$ is mod p singular of p-rank r_p with $k \not\equiv r_p/2$ mod 2. Then we have $\Theta^{[r_p]}(\Phi^{(n-r_p)}(F)) \equiv 0 \bmod p$. Here Φ is the Siegel Φ-operator and hence $\Phi^{(n-r_p)}(F) \in M_k(\Gamma^{(r_p)}(N))$.*

Remark 2.11 The p-rank is defined as the maximum of r ($0 \le r \le n$) such that there exists $T \in \Lambda_n$ with $\operatorname{rank}(T) = r$ satisfying $a_F(T) \not\equiv 0$ mod p.

If $F \in M_k(\Gamma^{(n)}(N))$ is mod p singular of p-rank r_p, then we have $2k - r_p \equiv 0$ mod $p - 1$ by the result of [4]. Therefore in this case r_p should automatically be even.

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Proof By applying the Φ-operator several times, we may suppose that the p-rank r_p is $n-1$. Then it suffices to prove that $\Theta^{[n-1]}(\Phi(F)) \equiv 0 \bmod p$. Moreover, by taking

$$F(NZ) \in M_k(\Gamma_1^{(n)}(N^2))$$

when F is of $\Gamma^{(n)}(N)$, it suffices to prove it for $F \in M_k(\Gamma_1^{(n)}(N))$.

Let $T_0 \in \Lambda_n$ be of $\operatorname{rank}(T_0) = n-1$ satisfying $a_F(T_0) \not\equiv 0 \bmod p$. If necessary, replacing F by $F|_k\left(\begin{smallmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{smallmatrix}\right) \in M_k(\Gamma_1^{(n)}(N))$ for a suitable $\left(\begin{smallmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{smallmatrix}\right) \in \Gamma_n$, we can assume that $T_0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & M_0 \end{smallmatrix}\right)$ for some $M_0 \in \Lambda_{n-1}^+$ (since

$$F' = F|_k\begin{pmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{pmatrix}$$

is mod p singular such that $a_{F'}(T_0) = \det(u)^k a_F(uT_0{}^t u)$). We shall prove $p \mid \det M_0$ for any such M_0.

We recall the well-known fact [12, 13, 29] that F has a Fourier–Jacobi expansion of the form

$$F(Z) = \sum_{M \in \Lambda_{n-1}} \varphi_T(\tau, \mathfrak{z}) e^{2\pi i \operatorname{tr}(M \cdot \tau')}.$$

Here we decomposed $\mathbb{H}_n \ni Z = \left(\begin{smallmatrix} \tau & {}^t\mathfrak{z} \\ \mathfrak{z} & \tau' \end{smallmatrix}\right)$ for $\tau \in \mathbb{H}_1$, $\tau' \in \mathbb{H}_{n-1}$. Pick up the M_0-th Fourier–Jacobi coefficient and consider its theta expansion

$$\varphi_{M_0}(\tau, \mathfrak{z}) = \sum_{\mu} h_\mu(\tau) \Theta_{M_0}[\mu](\tau, \mathfrak{z}),$$

where μ runs over all elements of $\mathbb{Z}^{(1,n-1)} \cdot (2M_0) \backslash \mathbb{Z}^{(1,n-1)}$ and

$$h_\mu(\tau) = \sum_{l=0}^{\infty} a_F\begin{pmatrix} l & \mu/2 \\ {}^t\mu/2 & M_0 \end{pmatrix} e^{2\pi i (l - \frac{1}{4} M_0^{-1}[{}^t\mu])\tau}.$$

From the mod p singularity of F, the above h_0 satisfies that $h_0 \equiv c \not\equiv 0 \bmod p$. Moreover, h_0 is a modular form of weight $k - \frac{n-1}{2} \in \mathbb{Z}_{\geq 1}$ for $\Gamma_0^{(1)}(NL)$ by [4, Lemma 5.1]. Here L is the level of M_0. If $p \nmid L$, then we have $k - \frac{n-1}{2} \equiv 0 \bmod p-1$ by Katz [15]. However this is impossible because of $k \not\equiv \frac{r_p}{2} \bmod 2$. Hence $p \mid L$ follows. In particular we have $p \mid \det M_0$. This completes the proof of Proposition 2.10. ∎

3 Main Results and Their Proofs

3.1 Main Results

For any $T \in \Lambda_n$, we denote by $\varepsilon(T)$ the content of T defined as

$$\varepsilon(T) := \max\{d \in \mathbb{Z}_{\geq 1} \mid d^{-1}T \in \Lambda_n\}.$$

Let F be a scalar-valued modular form. If $F \not\equiv 0 \bmod p$ and $\Theta(F) \equiv 0 \bmod p$, then there are three possibilities.

(a) For any $T \in \Lambda_n^+$ we have $a_F(T) \equiv 0 \bmod p$.
(b) For any $T \in \Lambda_n^+$ with $a_F(T) \not\equiv 0 \bmod p$, we have $p \mid \varepsilon(T)$.
(c) There exists $T \in \Lambda_n^+$ such that $a_F(T) \not\equiv 0 \bmod p$ and $p \nmid \varepsilon(T)$.

A modular form F of the type (a) is called *mod p singular*. In this case, the authors discussed the possible weight in [4]. Therefore, the main purpose of this paper is to consider types (b) and (c).

The first main result concerns F of type (b), but under a condition on k. Note that condition (b) is equivalent to that $\Theta^{[1]}(F)$ is mod p singular, namely the vector-valued modular form $\Theta^{[1]}(F)$ (mod p) satisfies the same condition for vector-valued modular forms as (a) above, because of Proposition 2.6 (iii).

Theorem 3.1 *Let p be a prime with $p \geq 3$ and N a positive integer with $p \nmid N$. For a positive integer k, let $F \in M_k(\Gamma_1^{(n)}(N))_{\mathbb{Z}_{(p)}}$. Assume that $F \not\equiv c \bmod p$ for any $c \in \mathbb{Z}_{(p)}$.*

(i) *If $\Theta^{[1]}(F) \equiv 0 \bmod p$ and*

$$\begin{cases} 0 < k < 2p-1 & (k \text{ odd}), \\ 0 < k < 3p-1 & (k \text{ even}) \end{cases} \text{then } k = \begin{cases} p & (k \text{ odd}), \\ 2p & (k \text{ even}). \end{cases}$$

(ii) *If $\Theta^{[1]}(F) \equiv 0 \bmod p$, $0 < k < p^2 - p + 1$ and $p \mid k$, then $k = \omega_N(F)$.*

(iii) *If $\Theta^{[1]}(F)$ is non-trivial mod p singular of p-rank r_p (Proposition 2.6 (iv)), then $2k - r_p \equiv 0 \bmod p-1$.*

Remark 3.2 A modular form F satisfying $\Theta^{[1]}(F) \equiv 0 \bmod p$ is called *totally p-singular* by Weissauer [26] who also obtained similar statements (at least for the case of level 1) under a certain condition on the largeness of p in geometrical terminology. Our statement is phrased in classical (elementary) language.

There exists a mod p singular modular form F ($\not\equiv 0 \bmod p$) such that $\Theta^{[1]}(F) \equiv 0 \bmod p$. In fact, we can construct such an example in the following way. For any mod p singular modular form $F \in M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$, we consider

$$G := \sum_{T \in \Lambda_n} a_F(pT) q^{pT} \in M_k\big(\Gamma_0^{(n)}(p^2)\big).$$

Applying Theorem 2.4, we can take $H \in M_{k'}(\Gamma_n)_{\mathbb{Z}_{(p)}}$ such that $H \equiv G \bmod p$. Then H is a mod p singular modular form such that $\Theta^{[1]}(H) \equiv 0 \bmod p$. For the existence of mod p singular modular forms and for their possible weights, see [4].

Statement (iii) in this theorem follows immediately from a property on mod p singular vector-valued Siegel modular forms, which is a generalization of the result in [4].

Theorem 3.3 *Let p be a prime with $p \geq 3$ and k a positive integer. Let N be a positive integer with $p \nmid N$ and $F \in M_k(\Gamma_1^{(n)}(N))_{\mathbb{Z}_{(p)}}$. Suppose that $\Theta^{[j]}(F)$ is mod p singular of p-rank r_p, where the p-rank is defined in Proposition 2.6 (iv). Then $2k \equiv r_p \bmod p-1$ holds.*

Remark 3.4 We may allow the modular group to be of type $\Gamma_1^{(n)}(N) \cap \Gamma_0^{(n)}(p^l)$ with N coprime to p. We may also allow quadratic nebentypus modulo p.

The second main result concerns F of type (c). We remark that $p \nmid \varepsilon(T)$ for $T \in \Lambda_n$ is equivalent to the existence of j with $1 \leq j \leq n-1$ such that $p \nmid T^{[j]}$,

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where we write $p \mid T^{[j]}$ if p divides all entries of $T^{[j]}$, otherwise we write $p \nmid T^{[j]}$. Moreover the existence of j and $T \in \Lambda_n^+$ with $p \nmid T^{[j]}$ such that $a_F(T) \not\equiv 0 \bmod p$, implies $\Theta^{[j]}(F) \not\equiv 0 \bmod p$. Namely we have

$$\begin{aligned} & \exists T \in \Lambda_n^+ \text{ such that } a_F(T) \not\equiv 0 \bmod p, p \nmid \varepsilon(T) \\ \iff & \exists j (1 \leq j \leq n-1), \exists T \in \Lambda_n^+ \text{ such that } a_F(T) \not\equiv 0 \bmod p,\ p \nmid T^{[j]} \\ \implies & \exists j\ (1 \leq j \leq n-1) \text{ such that } \Theta^{[j]}(F) \not\equiv 0 \bmod p. \end{aligned}$$

Note also that the converse of the last right arrow is not assured in general.

For any F of the type (c), we can find j such that

$$\Theta^{[j]}(F) \not\equiv 0 \mod p \quad \text{and} \quad \Theta^{[j+1]}(F) \equiv 0 \mod p.$$

Then we have the following statement.

Theorem 3.5 *Let p be a prime with $p \geq 3$ and N a positive integer with $p \nmid N$. Let n, j, and k be positive integers such that $j < n$. Assume that $F \in M_k(\Gamma_1^{(n)}(N))_{\mathbb{Z}_{(p)}}$ satisfies $\Theta^{[j]}(F) \not\equiv 0 \bmod p$ and $\Theta^{[j+1]}(F) \equiv 0 \bmod p$.*

(i) *If*

$$\begin{cases} k < p + (j-1)/2 & (j \text{ odd}), \\ k < 2p + (j-2)/2 & (j \text{ even},\ k - j/2 \text{ odd}), \\ k < 3p + (j-2)/2 & (j \text{ even},\ k - j/2 \text{ even}) \end{cases}$$

then

$$\begin{cases} 2k - j = p & (j \text{ odd}), \\ k - j/2 = p & (j \text{ even},\ k - j/2 \text{ odd}), \\ k - j/2 \equiv 0 \bmod p-1 \quad \text{or} \quad k - j/2 = 2p & (j \text{ even},\ k - j/2 \text{ even}). \end{cases}$$

(ii) *If*

$$\begin{cases} 2k - j < p^2 - p + 1 & (j \text{ odd}), \\ k - j/2 < p^2 - p & (j \text{ even}) \end{cases} \quad \text{and} \quad p \mid (2k - j),$$

then $\quad k = \omega_N(F)$.

In a more general situation, we predict the following property.

Conjecture 3.6 Let p be a prime and n, j, and k be positive integers with $j < n$. Let k be "sufficiently small" compared with p. Assume that $F \in M_k(\Gamma_1^{(n)}(N))_{\mathbb{Z}_{(p)}}$ ($k \in \mathbb{Z}_{\geq 1}$) satisfies $\Theta^{[j]}(F) \not\equiv 0 \bmod p$ and $\Theta^{[j+1]}(F) \equiv 0 \bmod p$. Then we have

$$p \mid (2\omega_N(F) - j).$$

Therefore, we can regard Theorem 3.5 as an example that supports this conjecture.

Remark 3.7 If the weight is large compared with p, the statement of this conjecture is not true. We will show this, by numerical examples (§5.3, §5.4), for the case of degree 2 and level 1, Yamauchi [28] concluded similar statements as in the two theorems above for the case of degree 2, without condition on the smallness of k compared with p, but under a certain geometrical non-vanishing condition. The proof is also algebraic geometrical.

3.2 Proof of Theorem 3.3

Since Theorem 3.1 (iii) follows from Theorem 3.3, we start by proving Theorem 3.3.

We observe that the Fourier expansion of $\Theta^{[j]}(F)$ runs only over elements of Λ_n with $\operatorname{rank}(T) \geq j$. Therefore only the case $j \leq r_p$ is of interest to us. Also it may be convenient to reduce the claim to the case $r_p = n-1$ by applying the Siegel Φ-operator several times; for details on the Siegel Φ-operator in the vector-valued case we refer to [13, 27]. We just mention that for $j < n$ we may identify

$$\Phi\Big(\sum_{T\in\Lambda_n} T^{[j]} \cdot a_F(T) q^T\Big)$$

with $\sum_{S\in\Lambda_{n-1}} S^{[j]} \cdot a_F\left(\begin{smallmatrix} 0 & 0 \\ 0 & S \end{smallmatrix}\right) q^S$.

We introduce some useful notation following [13]. For an n-rowed matrix M and two subsets P, Q of $\{1,\dots,n\}$ with t elements we denote by $M^{[t]}_{(P,Q)}$ the determinant of the t-rowed matrix M^P_Q which we obtain from M by deleting all rows that do not belong to P and all columns that do not belong to Q.

Now starting from the Fourier expansion $F = \sum a_F(T) q^T$, there exists $T_0 \in \Lambda_n$ with $\operatorname{rank}(T_0) = n-1$ such that $T_0^{[j]} \cdot a_F(T_0)$ is not congruent to zero modulo p. If necessary, taking

$$F|_k \left(\begin{smallmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{smallmatrix}\right) \in M_k(\Gamma_1^{(n)}(N))$$

for a suitable

$$\left(\begin{smallmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{smallmatrix}\right) \in \Gamma_n,$$

we can assume that T_0 is of the form $T_0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & M_0 \end{smallmatrix}\right)$ with $M_0 \in \Lambda^+_{n-1}$ (since

$$F' = F|_k \left(\begin{smallmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{smallmatrix}\right)$$

satisfies the assumption of Theorem 3.3 and $a_{F'}(T_0) = \det(u)^k a_F(u T_0 {}^t u)$).

The property of T_0 from above implies that there is at least one entry of the matrix $T_0^{[j]}$ that is not congruent zero modulo p, *i.e.*, there exist subsets a^o, b^o of $\{1,\dots,n\}$ with $\det(T_0{}^{a^o}_{b^o}) \not\equiv 0 \bmod p$. From the special shape of T_0 it follows that both a^o and b^o are subsets of $\{2,\dots,n\}$, *i.e.*, the (a^o, b^o)-entry of $T_0^{[j]}$ is a determinant of a submatrix of M_0, which we call d_0.

We decompose $Z \in \mathbb{H}_n$ as $Z = \left(\begin{smallmatrix} \tau & \mathfrak{z} \\ \mathfrak{z}^t & \tau' \end{smallmatrix}\right)$ with $\tau' \in \mathbb{H}_{n-1}$ and study the Fourier–Jacobi coefficient $\varphi_{M_0}(\tau,\mathfrak{z}) e^{2\pi i \operatorname{tr}(M_0\tau')}$ viewed as a subseries of the Fourier expansion of F. We apply the operator $\Theta^{[j]}$ to this subseries of F; then its (a^o, b^o) entry is just $d_0 \cdot \varphi_{M_0}(\tau,\mathfrak{z}) e^{2\pi i \operatorname{tr}(M_0\tau')}$. Now proceeding as in [4], for all $R \in \Lambda_n$, the R-th Fourier coefficients of this series should be congruent to zero modulo p unless R is of rank $n-1$. This implies that in the theta expansion of φ_{M_0} a modular form h_0 of weight $k - \frac{n-1}{2}$ appears, which should be congruent to a (nonzero) constant modulo p. The requested congruence follows from this as in [4]. ∎

3.3 Proof of Theorem 3.1

(i) From Proposition 2.6, the assumption $\Theta^{[1]}(F) \equiv 0 \bmod p$ implies that, for any $T \in \Lambda_n$ satisfying $a_F(T) \not\equiv 0 \bmod p$, we have $p \mid \varepsilon(T)$. In particular, all diagonal components of such T are divisible by p. Moreover, since F is not congruent to a constant modulo p, there exists $T \not\equiv 0_n$ $(T \in \Lambda_n)$ such that $a_F(T) \not\equiv 0 \bmod p$ and $p \mid \varepsilon(T)$. We fix one of them and denote it by T_0. Let $\operatorname{diag}(T_0) = (d_1, d_2, \dots, d_n)$ (with $p \mid d_i$ for any i), where d_i is the (i,i) entry of T_0. Since $T_0 \not\equiv 0_n$, we can assume that $d_1 > 0$ by changing F to

$$F|_k \begin{pmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{pmatrix} \in M_k(\Gamma_1^{(n)}(N))$$

for a suitable $\begin{pmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{pmatrix} \in \Gamma_n$ (Note that $F' = F|_k \begin{pmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{pmatrix}$ satisfies the assumption of Theorem 3.1 and $a_{F'}(T_0) = \det(u)^k a_F(uT_0{}^t u)$).

Consider the integral extract for T_0 defined by Börcherer and Nagaoka [7]:

$$f(\tau) := \sum_l a(l) q^l, \quad a(l) := \sum_T a_F(T),$$

where T runs over all positive semi-definite elements of Λ_n satisfying

$$T \equiv T_0 \bmod M, \quad \operatorname{diag}(T) = (l, d_2, \dots, d_n)$$

and $M \in \mathbb{N}$ is large enough such that $(p \det T_0, M) = 1$ and $a(d_1) = a_F(T_0)$. The reason why we can find M such that $a(d_1) = a_F(T_0)$ is due to the same argument as in [7, p. 229, lines 4–5]. Then we have $f \in M_k(\Gamma_1^{(1)}(NM^2))$ and $a(d_1) = a_F(T_0) \not\equiv 0 \bmod p$. Moreover, $p \mid l$ when $a(l) \not\equiv 0 \bmod p$ and $f \not\equiv c \bmod p$ for any $c \in \mathbb{Z}_{(p)}$. Therefore we have $\Theta^{[1]}(f) \equiv 0 \bmod p$ and $\omega_{NM^2}(f) > 0$. Applying Theorem 2.7, we obtain $p \mid \omega_{NM^2}(f)$. By Proposition 2.1, $\omega_{NM^2}(f) \equiv k \bmod p-1$ and therefore $\omega_{NM^2}(f)$ and k have the same parity.

If k is odd, then $0 < \omega_{NM^2}(f) \le k < 2p-1$ and $p \mid \omega_{NM^2}(f)$. In this case we have $\omega_{NM^2}(f) = p$. Therefore, $k = p$ or $k = 2p-1$. By the assumption $k < 2p-1$, we obtain $k = p$.

If k is even, then $\omega_{NM^2}(f)$ is even such that $0 < \omega_{NM^2}(f) \le k < 3p-1$ and $p \mid \omega_{NM^2}(f)$. In this case we have $\omega_{NM^2}(f) = 2p$. Therefore $k = 2p$ or $k = 3p-1$. Since the assumption $k < 3p-1$, we obtain $k = 2p$. This completes the proof of Theorem 3.1 (i).

(ii) As in the proof of (i), we can take $f \in M_k(\Gamma_1^{(1)}(NM^2))$ $(p \nmid M)$ such that $\Theta^{[1]}(f) \equiv 0 \bmod p$, $f \not\equiv c \bmod p$ for any $c \in \mathbb{Z}_{(p)}$ and $0 < \omega_{NM^2}(f) \le \omega_N(F) \le k$. To prove $k = \omega_N(F)$, we may prove $k = \omega_{NM^2}(f)$. Now we obtain $p \mid \omega_{NM^2}(f)$ by Theorem 2.7. By the assumption, we have $\omega_{NM^2}(f) \equiv k \bmod p(p-1)$, because $\omega_{NM^2}(f) \equiv k \equiv 0 \bmod p$ and $\omega_{NM^2}(f) \equiv k \bmod p-1$. Then there exists $t \ge 0$ such that $k = \omega_{NM^2}(f) + tp(p-1)$. However, from $0 < k < p^2 - p + 1$, we have $t = 0$ and hence $k = \omega_{NM^2}(f)$. This completes the proof of Theorem 3.1 (ii).

(iii) The statement follows immediately from Theorem 3.3. ∎

3.4 Proof of Theorem 3.5

(i) By Proposition 2.6 and the assumption $\Theta^{[j]}(F) \not\equiv 0 \bmod p$, there exist $T \in \Lambda_n$ and a $j \times j$ submatrix M of T such that $a_F(T) \not\equiv 0 \bmod p$ and $p \nmid \det M$.

Replacing F by $F|_k\left(\begin{smallmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{smallmatrix}\right) \in M_k(\Gamma_1^{(n)}(N))$ for a suitable $\left(\begin{smallmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{smallmatrix}\right) \in \Gamma_n$, we can assume that there is a principal submatrix M of size j in T such that $p \nmid \det M$, where a principal matrix is defined as a matrix obtained by omitting the same columns and rows (since $F' = F|_k\left(\begin{smallmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{smallmatrix}\right)$ satisfies the assumption of Theorem 3.5 and $a_{F'}(T_0) = \det(u)^k a_F(uT_0{}^t u)$). In particular, we may suppose that T is of the form $T_0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & M_0 \end{smallmatrix}\right)$ with $M_0 \in \Lambda_j$, $p \nmid \det M_0$, and $a_F(T_0) \not\equiv 0 \bmod p$.

The Fourier–Jacobi expansion of F can be written in the form

$$F(Z) = \sum_{M \in \Lambda_j} \varphi_M(\tau, \mathfrak{z}) e^{2\pi i \operatorname{tr}(M \cdot \tau')}.$$

Here we decomposed $\mathbb{H}_n \ni Z = \left(\begin{smallmatrix} \tau & {}^t\mathfrak{z} \\ \mathfrak{z} & \tau' \end{smallmatrix}\right)$ for $\tau \in \mathbb{H}_{n-j}$ and $\tau' \in \mathbb{H}_j$.

We consider the M_0-th Fourier–Jacobi coefficient

$$\varphi_{M_0}(\tau, \mathfrak{z}) = \sum_{\mu} h_\mu(\tau) \Theta_{M_0}[\mu](\tau, \mathfrak{z}),$$

where μ runs over all elements of $\mathbb{Z}^{(n-j,j)} \cdot (2M_0) \backslash \mathbb{Z}^{(n-j,j)}$ and

$$h_\mu(\tau) = \sum_{L \in \Lambda_{n-j}} a_F \begin{pmatrix} L & \frac{\mu}{2} \\ \frac{{}^t\mu}{2} & M_0 \end{pmatrix} e^{2\pi i \operatorname{tr}((L - \frac{1}{4} M_0^{-1}[{}^t\mu])\tau)}.$$

Then h_μ is a modular form of weight $k - \frac{j}{2}$ for $\Gamma^{(n-j)}(4NQ)$, where Q is the level of M_0 and we have $p \nmid Q$. Hence we have

$$H_\mu := h_\mu(4NQ\tau) \in M_{k-\frac{j}{2}}(\Gamma_1^{(n-j)}(4^2N^2Q^2))_{\mathbb{Z}_{(p)}}.$$

Now we prove the following lemma.

Lemma 3.8 *If the Fourier coefficient of H_μ at $L - \frac{1}{4}M_0^{-1}[{}^t\mu]$ is nonzero modulo p, then all entries of $L - \frac{1}{4}M_0^{-1}[{}^t\mu]$ are divisible by p. In particular, by Proposition 2.6, we have $\Theta^{[1]}(H_\mu) \equiv 0 \bmod p$.*

Proof We have by a direct calculation

$$\begin{aligned} S &= \begin{pmatrix} L & \frac{\mu}{2} \\ \frac{{}^t\mu}{2} & M_0 \end{pmatrix} \\ &= \begin{pmatrix} 1_{n-j} & \frac{\mu}{2} M_0^{-1} \\ 0 & 1_j \end{pmatrix} \begin{pmatrix} L - \frac{1}{4} M_0^{-1}[{}^t\mu] & 0 \\ 0 & M_0 \end{pmatrix} \begin{pmatrix} 1_{n-j} & 0 \\ M_0^{-1} \frac{{}^t\mu}{2} & 1_j \end{pmatrix}. \end{aligned}$$

(Multi-) linear algebra shows that for $J := \{n-j+1, \dots, n\}$ and $i, i' \in \{1, \dots, n-j\}$, we have

$$S^{[j]}_{(\{i\} \cup J, \{i'\} \cup J)} = \left(L - \frac{1}{4} M_0^{-1}[{}^t\mu]\right)^{[1]}_{(i,i')} \cdot \det M_0.$$

Here the notation $S^{[t]}_{(P,Q)}$ is the same as in the proof of Theorem 3.3.

We observe that the left-hand side is divisible by p and $\det M_0$ is coprime to p, therefore all entries of $L - \frac{1}{4}M_0^{-1}[{}^t\mu]$ are divisible by p. ∎

We return to the proof of Theorem 3.5. Let j be odd. By $\Theta^{[1]}(H_\mu) \equiv 0 \bmod p$, we can easily prove $\Theta^{[1]}(H_\mu^2) \equiv 0 \bmod p$. Then H_μ^2 is of weight $2k-j < 2p-1$ and $2k-j$ is odd. Therefore $2k-j \not\equiv 0 \bmod p-1$. This implies $H_\mu^2 \not\equiv c \bmod p$ for any $c \in \mathbb{Z}_{(p)}$ (see [7]). Hence we can apply Theorem 3.1 (i) to H_μ^2. We conclude that $2k-j = p$ in the case where j is odd.

Let j be even and $k-j/2$ odd. The weight of H_μ is $k-j/2 < 2p-1$ and therefore $H_\mu \not\equiv c \bmod p$ for any $c \in \mathbb{Z}_{(p)}$, because of $2k-j \not\equiv 0 \bmod p-1$. In this case, we can directly apply Theorem 3.1 (i) to H_μ. Hence we obtain $k-j/2 = p$.

Let j be even and $k-j/2$ even. Note that $k-j/2 < 3p-1$ by the assumption. If $H_\mu \equiv c \bmod p$ for some $c \in \mathbb{Z}_{(p)}$, then we have $k-j/2 \equiv 0 \bmod p-1$, otherwise we can apply Theorem 3.1 (i) to H_μ. Therefore, we obtain $k-j/2 \equiv 0 \bmod p-1$ or $k-j/2 = 2p$. This completes the proof of (i) in Theorem 3.5.

(ii) Let $H_\mu \in M_{k-\frac{j}{2}}(\Gamma_1^{(n-j)}(4^2N^2Q^2))_{\mathbb{Z}_{(p)}}$ be the function appearing in the proof of (i). Note that $\omega_{4^2N^2Q^2}(H_\mu) + j/2 \le \omega_N(F) \le k$. By the assumption, we have

$$\begin{cases} \omega_{4^2N^2Q^2}(H_\mu^2) \le 2k-j < p^2-p+1 & (j \text{ odd}), \\ \omega_{4^2N^2Q^2}(H_\mu) \le k-j/2 < p^2-p & (j \text{ even}) \end{cases} \quad \text{and} \quad p \mid (2k-j).$$

To apply Theorem 3.1 (ii) to H_μ^2 (j odd) and H_μ (j even), we need to confirm that they are not congruent to constants modulo p.

If j is odd, then $2k-j \not\equiv 0 \bmod p-1$. This implies $H_\mu^2 \not\equiv c$ for any $c \in \mathbb{Z}_{(p)}$. Hence we can apply Theorem 3.1 (ii) to H_μ^2. It follows that

$$2k-j = \omega_{4^2N^2Q^2}(H_\mu^2) \le 2\omega_{4^2N^2Q^2}(H_\mu) \le 2\omega_N(F) - j \le 2k-j.$$

This indicates $k = \omega_N(F)$.

Let j be even. Assume that $H_\mu \equiv c \bmod p$ for some $c \in \mathbb{Z}_{(p)}$. Then we have both conditions $k-j/2 \equiv 0 \bmod p-1$ and $p \mid (k-j/2)$ because of the assumption of the theorem. Then there exists $t \ge 1$ such that $2k-j = tp(p-1)$. However this is impossible because of $2k-j < p^2-p$. This means that $H_\mu \not\equiv c \bmod p$ for any $c \in \mathbb{Z}_{(p)}$.

Hence we can apply Theorem 3.1 (ii) to H_μ and then

$$k = \omega_{4^2N^2Q^2}(H_\mu) + j/2 \le \omega_{4^2N^2Q^2}(F) \le k.$$

Therefore we obtain $k = \omega_N(F)$.

This completes the proof of (ii) in Theorem 3.5. ∎

4 On Operators $A^{(j)}(p)$

Following Choi, Choie, and Richter [10], for a Siegel modular form $F \in M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$ with a Fourier expansion $F = \sum_T a_F(T)q^T$, we define an operator $A(p)$ (their notation is $U(p)$) as

$$F|A(p) := \sum_{\substack{T \in \Lambda_n \\ p \mid \det T}} a_F(T)q^T.$$

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We remark that this operator $A(p)$ is different from the usual $U(p)$-type Hecke operator investigated in [3] and elsewhere. If $p \geq n+3$, it is easy to see that $\widetilde{F|A(p)} = \widetilde{FE_{p-1}^{p+1}} - \Theta^{p-1}(\widetilde{F}) \in \widetilde{M}_{k+p^2-1}(\Gamma_n)_p$, because $\Theta(\widetilde{F}) \in \widetilde{M}_{k+p+1}(\Gamma_n)_p$ (see Theorem 2.9). Here $E_{p-1} \in M_{p-1}(\Gamma_n)_{\mathbb{Z}_{(p)}}$ is such that $E_{p-1} \equiv 1 \bmod p$ obtained in [6]. Essentially, this formula appeared in Dewar–Richter [11]. It was not explicitly proved that $F|A(p)$ is a true modular form; we prove it here for more general operators $A^{(j)}(M)$.

Let M be a positive integer. For a formal Fourier series of the form

$$F = \sum_{T \in \Lambda_n} a_F(T) q^T,$$

we set $F|A^{(j)}(M) := \sum_{T^{[j]} \equiv 0 \bmod M} a_F(T) q^T$. Then we can prove its modularity as follows.

Theorem 4.1 *Let k, j, n ($j \leq n$), M, and N be positive integers. If $F \in M_k(\Gamma_1^{(n)}(N))$, then $F|A^{(j)}(N) \in M_k(\Gamma_1^{(n)}(NM^2))$. In particular, if $F \in M_k(\Gamma_0^{(n)}(N), \chi)$ for a Dirichlet character χ modulo N, then $F|A^{(j)}(M) \in M_k(\Gamma_0^{(n)}(NM^2), \chi)$.*

Remark 4.2 As a special case in the above, we have

$$F|A^{(n)}(M) = F|A(M) = \sum_{\substack{T \in \Lambda_n \\ M | \det T}} a_F(T) q^T,$$

$$F|A^{(1)}(M) = F|U(M)V(M) = \sum_{T \in \Lambda_n} a_F(MT) q^{MT}.$$

Here $U(M)$ and $V(M)$ are the usual operator described as

$$F|U(M) = \sum_{T \in \Lambda_n} a_F(MT) q^T, \quad F|V(M) = \sum_{T \in \Lambda_n} a_F(T) q^{MT}.$$

For more details, see [3].

Proof We put $J := \{T \bmod M \mid T \in \Lambda_n\}$. Note that J is a finite set. Then, as in [7], we can find that $\sum_{T \equiv T_0 \bmod M} a_F(T) q^T \in M_k(\Gamma_1^{(n)}(NM^2))$ for any $T_0 \in J$. Now we consider $J_0^{(j)} := \{T \bmod M \mid T \in \Lambda_n,\ T^{[j]} \equiv 0 \bmod M\} \subset J$. Then we have

$$F|A^{(j)}(M) = \sum_{T_0 \in J_0^{(j)}} \sum_{T \equiv T_0 \bmod M} a_F(T) e^{2\pi i \operatorname{tr}(TZ)}.$$

Hence $F|A^{(j)}(M) \in M_k(\Gamma_1^{(n)}(NM^2))$.

Assume that $F \in M_k(\Gamma_0^{(n)}(N), \chi)$. Using the standard procedure of twisting, we show that $F|A^{(j)}(M) \in M_k(\Gamma_0^{(n)}(NM^2), \chi)$. If $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(n)}(NM^2)$, using [7], we get,

$$F|A^{(j)}(M)|_k\, g = \sum_{T_0 \in J_0^{(j)}} \sum_S F|_k \begin{pmatrix} 1_n & \frac{S}{M} \\ 0_n & 1_n \end{pmatrix} e^{-2\pi i tr(T_0 \frac{S}{M})}|_k g, \tag{4.1}$$

where S runs over all symmetric integral matrices of size n modulo M. Then we easily get

$$\begin{pmatrix} 1_n & \frac{S}{M} \\ 0_n & 1_n \end{pmatrix} \cdot g = \widetilde{g} \cdot \begin{pmatrix} 1_n & \frac{\widetilde{S}}{M} \\ 0_n & 1_n \end{pmatrix}$$

with $\widetilde{g} \in \Gamma_0^{(n)}(N)$ and some integral symmetric $\widetilde{S}$ satisfying $\widetilde{S} \equiv {}^tDSD \bmod M$. Note here that $\widetilde{g}$ satisfies $\widetilde{g} \equiv \left(\begin{smallmatrix} * & * \\ * & D \end{smallmatrix}\right) \bmod N$. Keeping in mind that $A^tD \equiv 1_n \bmod M$, we may rewrite (4.1) as

$$\sum_{T_0} \sum_{\widetilde{S}} (F|_k \widetilde{g})|_k \begin{pmatrix} 1_n & \frac{\widetilde{S}}{M} \\ 0_n & 1_n \end{pmatrix} e^{-2\pi i tr(-{}^tAT_0A\frac{\widetilde{S}}{M})}.$$

We observe that $F|_k\widetilde{g} = \chi(\det D)F$ and $T_0 \longmapsto \widetilde{T_0} := -{}^tAT_0A$ just permutes the set $J_0^{(j)}$; this proves the assertion. ∎

Remark 4.3 The proof actually shows that $F|A^{(j)}(M)$ is a modular form of level $\mathrm{lcm}(M^2, N)$.

We have the same statement as in Theorem 4.1 for F of half integral weight in the following way. We consider $G(Z) := F \cdot \theta^{(n)}(MZ)$, where $\theta^{(n)}(Z)$ is the theta function introduced in Subsection 2.1. This is of integral weight and (obviously) we have $F|A^{(j)}(M) = (F|A^{(j)}(M)) \cdot \theta^{(n)}(MZ)$. Therefore, the statement for F follows from that for G.

For any $F \in M_k(\Gamma_n)$, we have $F|A^{(j)}(p^m) \in M_k(\Gamma_0^{(n)}(p^{2m}))$ by Theorem 4.1. By Theorem 2.4, we can regard as $A^{(j)}(p^m)\colon \widetilde{M}(\Gamma_n)_{p^l} \to \widetilde{M}(\Gamma_n)_{p^l}$.

Proposition 4.4 *For any $l \geq 1$, $m \geq 1$ and j $(1 \leq j \leq n)$, we can decompose $\widetilde{M}(\Gamma_n)_{p^l}$ as $\widetilde{M}(\Gamma_n)_{p^l} = \operatorname{Ker} A^{(j)}(p^m) \oplus \operatorname{Im} A^{(j)}(p^m)$.*

Proof Let $\widetilde{F} \in \widetilde{M}(\Gamma_n)_{p^l}$. We set

$$\widetilde{F_1} := \sum_{T[j] \not\equiv 0 \bmod p^m} \widetilde{a_F(T)} q^T, \quad \widetilde{F_2} := \sum_{T[j] \equiv 0 \bmod p^m} \widetilde{a_F(T)} q^T.$$

Namely $\widetilde{F_1} := \widetilde{F} - \widetilde{F}|A^{(j)}(p^m)$ and $\widetilde{F_2} := \widetilde{F}|A^{(j)}(p^m)$. Then $\widetilde{F}$ can be written as $\widetilde{F} = \widetilde{F_1} + \widetilde{F_2}$. Then $\widetilde{F_1} \in \operatorname{Ker} A^{(j)}(p^m)$, $\widetilde{F_2} \in \operatorname{Im} A^{(j)}(p^m)$. This shows

$$\widetilde{M}(\Gamma_n)_{p^l} \subset \operatorname{Ker} A^{(j)}(p^m) + \operatorname{Im} A^{(j)}(p^m).$$

The converse inclusion is trivial. Therefore $\widetilde{M}(\Gamma_n)_{p^l} = \operatorname{Ker} A^{(j)}(p^m) + \operatorname{Im} A^{(j)}(p^m)$.

We shall prove that the summation of the right-hand side is direct. Let

$$\widetilde{F} \in \operatorname{Ker} A^{(j)}(p^m) \cap \operatorname{Im} A^{(j)}(p^m).$$

Then $\widetilde{F} = \widetilde{G}|A^{(j)}(p^m)$ for some $\widetilde{G} \in \widetilde{M}(\Gamma_n)_{p^l}$. This implies

$$\widetilde{F} = \sum_{T[j] \equiv 0 \bmod p^m} \widetilde{a_G(T)} q^T.$$

On the other hand, it follows from $\widetilde{F} \in \operatorname{Ker} A^{(j)}(p^m)$ that $\widetilde{F}|A^{(j)}(p^m) = \widetilde{F} = 0$. Hence we have $\operatorname{Ker} A^{(j)}(p^m) \cap \operatorname{Im} A^{(j)}(p^m) = 0$. ∎

Remark 4.5 Similarly we have also $\widetilde{M}(\Gamma_n)_{p^l} = \operatorname{Ker}\Theta^m \oplus \operatorname{Im}\Theta^m$, for any l and m with $1 \le l \le m$.

We consider the action of $A^{(j)}(p^m)$ on the space of p-adic modular forms.

Proposition 4.6 *If F is a p-adic modular form of degree n, then $F|A^{(j)}(p^m)$ is a p-adic modular form of degree n for any $1 \le j \le n$ and $m \ge 1$.*

Proof Since F is a p-adic modular form, there exists a sequence $\{G_l \in M_{k_l}(\Gamma_n)_{\mathbb{Z}_{(p)}}\}_l$ such that $F \equiv G_l \bmod p^l$. Then we have $F|A^{(j)}(p^m) \equiv G_l|A^{(j)}(p^m) \bmod p^l$. By Theorem 4.1, $G_l|A^{(j)}(p^m) \in M_{k_l}(\Gamma_0^{(n)}(p^{2m}))_{\mathbb{Z}_{(p)}}$ holds. By Theorem 2.4, $G_l|A^{(j)}(p^m)$ is also a p-adic modular form. Therefore $F|A^{(j)}(p^m)$ is a limit of a sequence of p-adic modular forms. This implies the assertion. ∎

5 Examples

In this section, we introduce some examples of elements of the mod p kernel of $\Theta^{[j]}$ and analyze the filtrations of some of them.

5.1 Siegel–Eisenstein Series

Let $E_k^{(n)}$ be the Siegel–Eisenstein series of weight k of degree n, where $k > n+1$ is an even integer. Let p be a prime and n a positive even integer such that $p \equiv (-1)^{\frac{n}{2}} \bmod 4$ and $p > n+3$. We set $k_{(n,p)} := \frac{n+p-1}{2}$. By Nagaoka's result [21], $E_{k_{(n,p)}}^{(n)}$ is an element of the mod p kernel of Θ. Note that this is mod p non-singular by the result of [4]. It follows from $k_{(n,p)} < p$ that $\omega_1(E_{k_{(n,p)}}^{(n)}) = k_{(n,p)}$.

As an easy application of Theorem 3.5 (i), we can prove $\Theta^{[n-1]}(E_{k_{(n,p)}}^{(n)}) \not\equiv 0 \bmod p$ as follows. We can find an integer $1 \le j_0 \le n-1$ such that j_0 is the maximum of positive integers j satisfying $\Theta^{[j]}(E_{k_{(n,p)}}^{(n)}) \not\equiv 0 \bmod p$. Applying Theorem 3.5 (i), we have $n+p-1-j_0 = p$. This implies $j_0 = n-1$.

5.2 Theta Series

In [5], we use certain theta series attached to quadratic forms to construct several types of Siegel modular forms in the mod p kernel of $\Theta^{[j]}$. We compute ω_N for some cases: here N can be an arbitrary number coprime to p. In this way we confirm that the constructions in [5] are the best possible ones in the sense that the level one forms obtained are of smallest possible weight.

First case: Here S is an even positive definite quadratic form of (even) rank n, exact level p and $\det(S) = p^2$.

We showed that the normalized theta series

$$\theta_S^{(n)}(Z) := \frac{1}{\sharp \operatorname{Aut}_{\mathbb{Z}}(S)} \sum_{X \in \mathbb{Z}^{(n,n)}} e^{\pi i \operatorname{tr}({}^tXSXZ)}$$

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is congruent modulo p to a level one form F of weight $k = n/2 + (p-1)$, where $\mathrm{Aut}_{\mathbb{Z}}(S) = \{A \in \mathbb{Z}^{(n,n)} \mid {}^tASA = S\}$. Then $a_F(S) = 1$, in particular, $F \not\equiv 0 \bmod p$ and

$$\Theta^{[n-1]}(F) \equiv 0 \bmod p, \qquad \Theta^{[n-2]}(F) \not\equiv 0 \bmod p.$$

Here the second statement follows from $S^{[n-2]} \cdot a_F(S) \not\equiv 0 \bmod p$. Since $j = n-2$ (even) and $k - \frac{j}{2} = k - \frac{n-2}{2} = p < p^2 - p$, we can apply Theorem 3.1 (ii) to F. This implies $k = \omega_1(F)$.

Second case: Here S is an even positive quadratic form of (even) rank n, exact level p with $\det(S) = p$. In this case, $\theta_S^{(n)}$ is congruent modulo p to a level one form of weight $k = \frac{n}{2} + \frac{p-1}{2}$. Then $\Theta^{[n]}(F) \equiv 0 \bmod p$, but $\Theta^{[n-1]}(F) \not\equiv 0 \bmod p$. Since $j = n-1$ (odd) and $2k - j = 2k - n + 1 = p < p^2 - p + 1$, we can apply Theorem 3.1 (ii). Therefore, in this case also we have $k = \omega_1(F)$.

Second case, with harmonic polynomial: Let S be as before and consider

$$\theta_{S,\det}^{(n)}(Z) := \sum_{X \in \mathbb{Z}^{(n,n)}} \det(X) e^{\pi i \mathrm{tr}({}^tXSXZ)}.$$

Here we must assume in addition that $\mathrm{Aut}_{\mathbb{Z}}(S)$ does not contain improper automorphisms, *i.e.*, all automorphisms have determinant +1. Then it was shown [5] that this theta series is congruent modulo p to a (cuspidal) level one modular form F of weight $k = \frac{n}{2} + 1 + 3\frac{p-1}{2}$. The proof was much more complicated than in the other cases. Again F satisfies $\Theta^{[n]}(F) \equiv 0 \bmod p$, but $\Theta^{[n-1]}(F) \not\equiv 0 \bmod p$.

In this case, from $j = n-1$ (odd) we have $2k - j = n + 2 + 3(p-1) - (n-1) = 3p$. Then $p \mid (2k - j)$ and $2k - j = 3p < p^2 - p + 1$ (when $p \geq 5$). Applying Theorem 3.5 (ii), we have $k = \omega_1(F)$.

Remark 5.1 There is a missing case here, namely $\theta_{S,\det}^{(n)}$ with S of level p and $\det(S) = p^2$. Here we do not yet know a good explicit construction of a level one form F congruent modulo p to $\theta_{S,\det}^{(n)}$ with low weight. A consideration similar to the one above suggests that $\omega_1(F) = \frac{n}{2} + 1 + 2(p-1)$ should hold.

5.3 Operators $A^{(j)}(p)$

For any $F \in M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$, we have $\Theta^{[j]}(F|A^{(j)}(p)) \equiv 0 \bmod p$ by the definition of $A^{(j)}(p)$. Namely, we can always construct elements of the mod p kernel of $\Theta^{[j]}$ for any prime p. In particular, if $p \geq n+3$ and $j = n$, then we get weight $k + p^2 - 1$ for any $k \in \mathbb{Z}_{\geq 1}$ (see Section 4). Moreover these examples are not necessarily of type (b) introduced in Subsection 3.1. Because, if we take a suitable F, then there exists $T \in \Lambda_n$ with $p \nmid \varepsilon(T)$ such that $a_{F|A^{(j)}(p)}(T) \not\equiv 0 \bmod p$. We remark that F is an element of the mod p kernel of $\Theta^{[j]}$ if and only if $F|A^{(j)}(p) \equiv F \bmod p$.

Let $X_{10}^{(2)}, X_{12}^{(2)}$ be cusp forms of degree 2, level 1, and weights 10 and 12, respectively. We normalize them so that $a_{X_{10}}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} = a_{X_{12}}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} = 1$.

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Example 5.2 We have by direct calculation

$$\omega_1(E_8^{(2)} \mid A(7)) = 32, \quad \omega_1(E_{10}^{(2)} \mid A(7)) = 4, \quad \omega_1(X_{10} \mid A(7)) = 46.$$

All of these formulas satisfy $p \mid (2\omega_1(F) - n + 1)$. Therefore Conjecture 3.6 is true for these examples.

We introduce more examples of $\omega_1(F|A(p))$ in tables in Section 6. We shall explain the tables. Let $p \geq 5$ be a prime and $R_p = \mathbb{F}_p[x_4, x_6, x_{10}, x_{12}]$ a polynomial ring over $\mathbb{F}_p$. For a positive integer k, we denote by $R_{p,k} \subset R_p$ the space of isobaric polynomials of weight k. We define a linear map $\psi_k : R_{p,k} \to \widetilde{M}_k(\Gamma_2)_p$ by $\psi_k(f(x_4, x_6, x_{10}, x_{12})) = f(\widetilde{E}_4^{(2)}, \widetilde{E}_6^{(2)}, \widetilde{X}_{10}, \widetilde{X}_{12})$. Then ψ_k is an isomorphism [19]. Let $H \in \widetilde{M}_{p-1}(\Gamma_2)_p$ be a modular form with $H = 1$. Therefore

$$H = \begin{cases} \widetilde{E}_4^{(2)} & \text{if } p = 5, \\ \widetilde{E}_6^{(2)} & \text{if } p = 7. \end{cases}$$

Note that $\psi_k^{-1}(H)$ is a prime element of R_p. For $F \in \widetilde{M}_k(\Gamma_2)_p$, denote by $\operatorname{ord}_H(F)$ the maximum integer e such that $\psi_k^{-1}(F)/(\psi_k^{-1}(H))^e \in R_p$. We understand $\operatorname{ord}_H(0) = \infty$.

We compute images of the linear operator $A(p) : \widetilde{M}_k(\Gamma_2)_p \to \widetilde{M}_{k+p^2-1}(\Gamma_2)_p$ for a basis of $\widetilde{M}_k(\Gamma_2)_p$ for even $k \leq 60$ and $p = 5, 7$. We fix a basis $\mathcal{B}_{k,p} = \{F_1, \dots, F_m\}$ of $\widetilde{M}_k(\Gamma_2)_p$ so that

$$\operatorname{ord}_H\Big(\sum_{G \in S} a_G G\Big) = \min\big\{\operatorname{ord}_H(a_G G) \mid G \in S\big\}, \tag{5.1}$$

for any choice of $a_G \in \mathbb{F}_p$ for each $G \in S$. Here $S = \{F|A(p) : F \in \mathcal{B}_{k,p}\}$. We can take such a basis as follows. In general, let R be a polynomial ring over a field K and h a nonzero element of R. We fix a monomial order of R. Let $\mathcal{M}$ be a subspace of R over K spanned by monomials which are not divisible by the initial term of h. Since $\{h\}$ is a Gröbner basis of the ideal Rh, we can perform the reduction algorithm uniquely. That is, for any $f \in R$, we can uniquely write f as

$$f = \sum_{i=0}^{\infty} g_i(f) h^i, \tag{5.2}$$

where $g_i(f) \in \mathcal{M}$ and $g_i(f) = 0$ for sufficiently large i. Take a basis $\mathcal{B} = \{F_1, \dots, F_m\}$ of $\widetilde{M}_k(\Gamma_2)_p$. We put $f_i = \psi_{k+p^2-1}^{-1}(F_i|A(p))$ and denote $g_j(f_i)$ by the element of $\mathcal{M}$ as in (5.2) for $h = \psi_{p-1}^{-1}(H)$. Take a positive integer a so that $g_j(f_i) = 0$ for all $j > a$. Let $\mathcal{M}'$ be a subspace of $\mathcal{M}$ spanned by $\{g_j(f_i)\}_{1 \leq i \leq m, 0 \leq j}$. We fix a linear isomorphism $\Psi : \mathcal{M}' \cong \mathbb{F}_p^{\nu}$ and put $v(f_i) = \Psi(g_0(f_i)) \oplus \Psi(g_1(f_i)) \oplus \cdots \oplus \Psi(g_a(f_i)) \in \mathbb{F}_p^{\nu(a+1)}$. If we take a basis $\mathcal{B}$ so that the matrix $(v(f_1), \dots, v(f_m))$ is an echelon form, then the basis satisfies (5.1).

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For example, we take $\mathcal{B}_{18,5}$ as $\mathcal{B}_{18,5} = \{\widetilde{E}_4^3\widetilde{E}_6,\ \widetilde{E}_4^3\widetilde{E}_6 + 2\widetilde{E}_6^3,\ \widetilde{E}_6\widetilde{X}_{12},\ \widetilde{E}_4^2\widetilde{X}_{10}\}$. Here we simply write $E_k^{(2)}$ as E_k. Then its images are given as

$$\begin{aligned}\widetilde{E}_4^3\widetilde{E}_6|A(5) &= \widetilde{E}_4^6\widetilde{E}_6(\widetilde{E}_4^3 - 2\widetilde{X}_{12})\\ (\widetilde{E}_4^3\widetilde{E}_6 + 2\widetilde{E}_6^3)|A(5) &= \widetilde{E}_4^6(\widetilde{E}_4^3\widetilde{E}_6 + 2\widetilde{E}_6^3 + \widetilde{E}_4^2\widetilde{X}_{10}),\\ \widetilde{E}_6\widetilde{X}_{12}|A(5) &= \widetilde{E}_4^2\widetilde{X}_{10}|A(5) = 0.\end{aligned}$$

We omit the explicit description of $\mathcal{B}_{k,p}$ for other cases. We note that the multiset $\{\mathrm{ord}_H(F|A(p)) \mid F \in \mathcal{B}_{k,p}\}$ does not depend on the choice of $\mathcal{B}_{k,p}$ satisfying (5.1). Define a map $\alpha_{k,p}\colon \mathcal{B}_{k,p} \to \mathbb{Z}^2 \times \mathbb{F}_p^2$ by

$$\alpha_{k,p}(F) = (\mathrm{ord}_H(F|A(p)), l, l \bmod p, 2l - 1 \bmod p),$$

where $l = \omega_1(F|A(p))$. Tables 1 and 2 show the multiset $\{\alpha_{k,p}(F) \mid F \in \mathcal{B}_{k,p}\}$. Each element $[(a, b, c, d), e]$ in the tables means that there exists exactly e modular forms $F \in \mathcal{B}_{k,p}$ such that $\alpha_{k,p}(F) = (a, b, c, d)$.

Examples show that there exists a modular form F with $\omega_1(F|A(p)) \not\equiv 0 \bmod p$ and $2\omega_1(F|A(p)) - 1 \not\equiv 0 \bmod p$. Filtrations of such modular forms in these tables are $\{24,\ 42,\ 54\}$ if $p = 5$ and $\{24,\ 48,\ 52\}$ if $p = 7$. For example, a modular form of degree 2, weight 24, level 1

$$F = E_4^3E_6^2 + 2E_6^4 + 3E_4^2E_6X_{10} + 3E_4X_{10}^2 + 2E_6^2X_{12} + 3X_{12}^2$$

satisfies $\Theta^{[2]}(F) \equiv 0 \bmod 5$ and $\Theta^{[1]}(F) \not\equiv 0 \bmod 5$, but we have

$$2\omega_1(F) - 1 \not\equiv 0 \mod 5.$$

Bold elements in tables indicate those modular forms.

5.4 More Examples of Filtrations

In this subsection, we use the example of $\omega_1(F)$ for $F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with $\Theta(F) \equiv 0 \bmod p$ to test the validity of Conjecture 3.6. Here we compute the kernel of $\Theta\colon \widetilde{M}_k(\Gamma_2)_p \to \widetilde{M}_{k+p+1}(\Gamma_2)_p$ for $p < 80$ and an even $k \le 100$ with $b_{k+p+1} \le 15$. Here

$$b_k = \begin{cases}[k/10] & \text{if } k \text{ is even},\\ [(k-5)/10] & \text{if } k \text{ is odd}.\end{cases}$$

Note that b_k gives the Sturm bound for $\widetilde{M}_k(\Gamma_2)_p$ (see [9, 17]). We take a basis $\mathcal{B} = \{F_1, \dots, F_m\}$ of $\mathrm{Ker}\,\Theta$ so that (5.1) holds for $S = \mathcal{B}$.

We understand $\mathrm{ord}_H(F)$ for $F \in \widetilde{M}_k(\Gamma_2)_p$ with an odd k as follows: By Nagaoka [20], there uniquely exists $G \in \widetilde{M}_{k-35}(\Gamma_2)_p$ such that $F = X_{35}G$, where X_{35} is the Igusa's generator of weight 35. Then we define $\mathrm{ord}_H(F) = \mathrm{ord}_H(G)$.

Then we have computed the filtration $\omega_1(F_i)$ for $i = 1, \dots, m$. Table 3 lists filtrations. The meaning of the table is as follows: for a prime p, a positive integer k appears in the corresponding cell if and only if $k \le 100$, $b_{k+p+1} \le 15$ and there exists $F \in \widetilde{M}_k(\Gamma_2)_p$ such that $F \not\equiv 0$, $\omega_1(F) = k$ and $\Theta(F) = 0$.

Table 3 also shows that there exists $F \in \widetilde{M}_k(\Gamma_2)_p$ such that $\Theta(F) = 0$, $\omega_1(F) \not\equiv 0 \bmod p$ and $2\omega_1(F) - 1 \not\equiv 0 \bmod p$. The pairs $(p, \omega_1(F))$ for such F in the table are

$$(5,24),(5,42),(5,54),(5,66),(5,72),(5,74),(5,84),(5,92),(5,96),(7,24),$$
$$(7,48),(7,52),(7,72),(7,76),(7,80),(7,94),(7,96),(11,60),(13,84).$$

6 Tables

6.1 Tables for Filtrations of Images of $A(p)$

The following tables are of $[(a,b,c,d),e]$, where

$$(a,b,c,d) := \alpha_{k,p}(F) = (\mathrm{ord}_H(F|A(p)), l, l \bmod p, 2l-1 \bmod p),$$

$l = \omega_1(F|A(p))$, and e is the number of modular forms in $\mathcal{B}_{k,p}$ which have $\alpha_{k,p}(F)$. For more details, see Subsection 5.3

Table 1: Filtrations of Images of $A(5)$

4	$[(7,0,0,4),1]$
6	$[(3,18,3,0),1]$
8	$[(8,0,0,4),1]$
10	$[(4,18,3,0),1]$, $[(\infty,0,0,4),1]$
12	$[(2,28,3,0),1]$, $[(9,0,0,4),1]$, $[(\infty,0,0,4),1]$
14	$[(5,18,3,0),1]$, $[(\infty,0,0,4),1]$
16	$[(3,28,3,0),1]$, $[(10,0,0,4),1]$, $[(\infty,0,0,4),2]$
18	$[(6,18,3,0),2]$, $[(\infty,0,0,4),2]$
20	$[(4,28,3,0),2]$, $[(11,0,0,4),1]$, $[(\infty,0,0,4),2]$
22	$[(2,38,3,0),1]$, $[(7,18,3,0),2]$, $[(\infty,0,0,4),3]$
24	$[(0,48,3,0),2]$, $[(5,28,3,0),2]$, $[(\mathbf{6},\mathbf{24},\mathbf{4},\mathbf{2}),\mathbf{1}]$, $[(12,0,0,4),1]$, $[(\infty,0,0,4),2]$
26	$[(3,38,3,0),1]$, $[(8,18,3,0),2]$, $[(\infty,0,0,4),4]$
28	$[(1,48,3,0),2]$, $[(6,28,3,0),2]$, $[(\mathbf{7},\mathbf{24},\mathbf{4},\mathbf{2}),\mathbf{1}]$, $[(13,0,0,4),1]$, $[(\infty,0,0,4),4]$
30	$[(4,38,3,0),1]$, $[(6,30,0,4),1]$, $[(9,18,3,0),2]$, $[(\infty,0,0,4),7]$
32	$[(2,48,3,0),3]$, $[(7,28,3,0),2]$, $[(\mathbf{8},\mathbf{24},\mathbf{4},\mathbf{2}),\mathbf{1}]$, $[(14,0,0,4),1]$, $[(\infty,0,0,4),5]$
34	$[(0,58,3,0),3]$, $[(5,38,3,0),1]$, $[(7,30,0,4),1]$, $[(10,18,3,0),2]$, $[(\infty,0,0,4),7]$
36	$[(3,48,3,0),4]$, $[(8,28,3,0),2]$, $[(\mathbf{9},\mathbf{24},\mathbf{4},\mathbf{2}),\mathbf{1}]$, $[(15,0,0,4),1]$, $[(\infty,0,0,4),9]$
38	$[(1,58,3,0),3]$, $[(6,38,3,0),1]$, $[(8,30,0,4),1]$, $[(11,18,3,0),2]$, $[(\infty,0,0,4),9]$

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40	$[(4,48,3,0),5], [(9,28,3,0),2], [(\mathbf{10},\mathbf{24},\mathbf{4},\mathbf{2}),\mathbf{1}], [(16,0,0,4),1], [(\infty,0,0,4),12]$
42	$[(2,58,3,0),4], [(\mathbf{6},\mathbf{42},\mathbf{2},\mathbf{3}),\mathbf{1}], [(7,38,3,0),1], [(9,30,0,4),1], [(12,18,3,0),2], [(\infty,0,0,4),13]$
44	$[(0,68,3,0),3], [(5,48,3,0),5], [(10,28,3,0),2], [(\mathbf{11},\mathbf{24},\mathbf{4},\mathbf{2}),\mathbf{1}], [(17,0,0,4),1], [(\infty,0,0,4),12]$
46	$[(3,58,3,0),5], [(\mathbf{7},\mathbf{42},\mathbf{2},\mathbf{3}),\mathbf{1}], [(8,38,3,0),1], [(10,30,0,4),1], [(13,18,3,0),2], [(\infty,0,0,4),17]$
48	$[(1,68,3,0),3], [(6,48,3,0),7], [(11,28,3,0),2], [(\mathbf{12},\mathbf{24},\mathbf{4},\mathbf{2}),\mathbf{1}], [(18,0,0,4),1], [(\infty,0,0,4),17]$
50	$[(4,58,3,0),5], [(6,50,0,4),1], [(\mathbf{8},\mathbf{42},\mathbf{2},\mathbf{3}),\mathbf{1}], [(9,38,3,0),1], [(11,30,0,4),1], [(14,18,3,0),2], [(\infty,0,0,4),20]$
52	$[(2,68,3,0),4], [(7,48,3,0),7], [(12,28,3,0),2], [(\mathbf{13},\mathbf{24},\mathbf{4},\mathbf{2}),\mathbf{1}], [(19,0,0,4),1], [(\infty,0,0,4),22]$
54	$[(0,78,3,0),6], [(5,58,3,0),5], [(\mathbf{6},\mathbf{54},\mathbf{4},\mathbf{2}),\mathbf{1}], [(7,50,0,4),1], [(\mathbf{9},\mathbf{42},\mathbf{2},\mathbf{3}),\mathbf{1}], [(10,38,3,0),1], [(12,30,0,4),1], [(15,18,3,0),2], [(\infty,0,0,4),21]$
56	$[(3,68,3,0),4], [(5,60,0,4),1], [(8,48,3,0),7], [(13,28,3,0),2], [(\mathbf{14},\mathbf{24},\mathbf{4},\mathbf{2}),\mathbf{1}], [(20,0,0,4),1], [(\infty,0,0,4),26]$
58	$[(1,78,3,0),7], [(6,58,3,0),5], [(\mathbf{7},\mathbf{54},\mathbf{4},\mathbf{2}),\mathbf{1}], [(8,50,0,4),1], [(\mathbf{10},\mathbf{42},\mathbf{2},\mathbf{3}),\mathbf{1}], [(11,38,3,0),1], [(13,30,0,4),1], [(16,18,3,0),2], [(\infty,0,0,4),27]$
60	$[(4,68,3,0),4], [(6,60,0,4),2], [(9,48,3,0),7], [(14,28,3,0),2], [(\mathbf{15},\mathbf{24},\mathbf{4},\mathbf{2}),\mathbf{1}], [(21,0,0,4),1], [(\infty,0,0,4),35]$

Table 2: Filtrations of Images of $A(7)$

4	$[(8,4,4,0),1]$
6	$[(9,0,0,6),1]$
8	$[(4,32,4,0),1]$
10	$[(2,46,4,0),1], [(9,4,4,0),1]$
12	$[(0,60,4,0),2], [(10,0,0,6),1]$
14	$[(5,32,4,0),1], [(\infty,0,0,6),1]$
16	$[(3,46,4,0),2], [(10,4,4,0),1], [(\infty,0,0,6),1]$
18	$[(1,60,4,0),2], [(11,0,0,6),1], [(\infty,0,0,6),1]$
20	$[(6,32,4,0),2], [(\infty,0,0,6),3]$
22	$[(4,46,4,0),2], [(11,4,4,0),1], [(\infty,0,0,6),3]$
24	$[(2,60,4,0),3], [(\mathbf{8},\mathbf{24},\mathbf{3},\mathbf{5}),\mathbf{1}], [(12,0,0,6),1], [(\infty,0,0,6),3]$
26	$[(0,74,4,0),2], [(7,32,4,0),2], [(\infty,0,0,6),3]$

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28	$[(5,46,4,0),2]$, $[(8,28,0,6),1]$, $[(12,4,4,0),1]$, $[(\infty,0,0,6),6]$
30	$[(3,60,4,0),4]$, $[(\mathbf{9},\mathbf{24},\mathbf{3},\mathbf{5}),\mathbf{1}]$, $[(13,0,0,6),1]$, $[(\infty,0,0,6),5]$
32	$[(1,74,4,0),3]$, $[(8,32,4,0),3]$, $[(\infty,0,0,6),6]$
34	$[(6,46,4,0),3]$, $[(9,28,0,6),1]$, $[(13,4,4,0),1]$, $[(\infty,0,0,6),9]$
36	$[(4,60,4,0),6]$, $[(\mathbf{10},\mathbf{24},\mathbf{3},\mathbf{5}),\mathbf{1}]$, $[(14,0,0,6),1]$, $[(\infty,0,0,6),9]$
38	$[(2,74,4,0),4]$, $[(9,32,4,0),3]$, $[(\infty,0,0,6),9]$
40	$[(0,88,4,0),7]$, $[(7,46,4,0),3]$, $[(10,28,0,6),1]$, $[(14,4,4,0),1]$, $[(\infty,0,0,6),9]$
42	$[(5,60,4,0),7]$, $[(\mathbf{11},\mathbf{24},\mathbf{3},\mathbf{5}),\mathbf{1}]$, $[(15,0,0,6),1]$, $[(\infty,0,0,6),13]$
44	$[(3,74,4,0),6]$, $[(10,32,4,0),3]$, $[(\infty,0,0,6),15]$
46	$[(1,88,4,0),8]$, $[(8,46,4,0),4]$, $[(11,28,0,6),1]$, $[(15,4,4,0),1]$, $[(\infty,0,0,6),13]$
48	$[(6,60,4,0),7]$, $[(\mathbf{8},\mathbf{48},\mathbf{6},\mathbf{4}),\mathbf{1}]$, $[(\mathbf{12},\mathbf{24},\mathbf{3},\mathbf{5}),\mathbf{1}]$, $[(16,0,0,6),1]$, $[(\infty,0,0,6),21]$
50	$[(4,74,4,0),7]$, $[(11,32,4,0),3]$, $[(\infty,0,0,6),21]$
52	$[(2,88,4,0),9]$, $[(\mathbf{8},\mathbf{52},\mathbf{3},\mathbf{5}),\mathbf{1}]$, $[(9,46,4,0),4]$, $[(12,28,0,6),1]$, $[(16,4,4,0),1]$, $[(\infty,0,0,6),21]$
54	$[(0,102,4,0),8]$, $[(7,60,4,0),7]$, $[(\mathbf{9},\mathbf{48},\mathbf{6},\mathbf{4}),\mathbf{1}]$, $[(\mathbf{13},\mathbf{24},\mathbf{3},\mathbf{5}),\mathbf{1}]$, $[(17,0,0,6),1]$, $[(\infty,0,0,6),21]$
56	$[(5,74,4,0),8]$, $[(8,56,0,6),1]$, $[(12,32,4,0),3]$, $[(\infty,0,0,6),30]$
58	$[(3,88,4,0),11]$, $[(\mathbf{9},\mathbf{52},\mathbf{3},\mathbf{5}),\mathbf{1}]$, $[(10,46,4,0),4]$, $[(13,28,0,6),1]$, $[(17,4,4,0),1]$, $[(\infty,0,0,6),28]$
60	$[(1,102,4,0),10]$, $[(8,60,4,0),9]$, $[(\mathbf{10},\mathbf{48},\mathbf{6},\mathbf{4}),\mathbf{1}]$, $[(\mathbf{14},\mathbf{24},\mathbf{3},\mathbf{5}),\mathbf{1}]$, $[(18,0,0,6),1]$, $[(\infty,0,0,6),30]$

6.2 Table for Filtrations of the Kernel of $\Theta^{[2]}$

Table 3 shows filtration $\omega_1(F)$ for $F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ with $\Theta^{[2]}(F) \equiv 0 \mod p$, $k \le 100$, $p < 80$ and $b_{k+p+1} \le 15$. For more details, see Subsection 5.4.

Table 3: Filtrations of the kernel of $\Theta^{[2]}$

p	k
5	0, 18, 24, 28, 30, 38, 42, 48, 50, 54, 58, 60, 66, 68, 72, 74, 78, 80, 83, 84, 88, 90, 92, 93, 96, 98
7	0, 4, 24, 28, 32, 46, 48, 52, 56, 60, 70, 72, 74, 76, 80, 81, 84, 88, 94, 95, 96, 98
11	0, 6, 28, 44, 50, 60, 66, 72, 83, 88, 94
13	0, 20, 46, 52, 59, 72, 78, 84, 98
17	0, 26, 60, 68, 77, 94

19	0, 10, 48, 76, 86
23	0, 12, 35, 58, 92
29	0, 44
31	0, 16, 47, 78
37	0, 56, 93
41	0, 62
43	0, 22
47	0, 24, 71
53	0, 80
59	0, 30, 89
61	0, 92
67	0, 34
71	0, 36
73	0
79	0, 40

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