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THE SYMMETRIC GENUS OF 2-GROUPS

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Abstract. Let G be a finite group. The *symmetric genus* $\sigma(G)$ is the minimum genus of any Riemann surface on which G acts faithfully. We show that if G is a group of order 2^m that has symmetric genus congruent to 3 (mod 4), then either G has exponent 2^{m-3} and a dihedral subgroup of index 4 or else the exponent of G is 2^{m-2}. We then prove that there are at most 52 isomorphism types of these 2-groups; this bound is independent of the size of the 2-group G. A consequence of this bound is that almost all positive integers that are the symmetric genus of a 2-group are congruent to 1 (mod 4).

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1. Introduction. A finite group G can be represented as a group of automorphisms of a compact Riemann surface. In other words, G acts on a Riemann surface. The *symmetric genus* $\sigma(G)$ is the minimum genus of any compact Riemann surface on which G acts faithfully.

The origins of this parameter can be traced back over a century to the work of Hurwitz, Poincare, Burnside and others. We use the modern terminology introduced in [**16**]. There is now a substantial body of work on the symmetric genus parameter.

A natural problem is to determine the positive integers that occur as the symmetric genus of a group (or a particular type of group). Indeed, whether or not there is a group of symmetric genus n for each value of the integer n remains a challenging open question; see the recent, important article [**4**]. Here, we restrict our attention to 2-groups. The 2-groups are interesting in this context because of the well-known conjecture that, among the finite groups, almost all groups are 2-groups.

The only 2-groups of even genus are the classical 2-groups of genus 0 [**11**, Theorem 9]. In other words, if G is a 2-group with positive symmetric genus, then $\sigma(G)$ is odd. The 2-groups with positive genus are our focus here, and we show that the 2-groups with symmetric genus congruent to 3 modulo 4 are special indeed. In particular, we show that a group G of order 2^m acting on a Riemann surface of genus $g \equiv 3 \pmod 4$ must contain an element of order 2^{m-3} or larger. Further, if $Exp(G) = 2^{m-3}$, then G contains a dihedral subgroup of index 4. This yields the following result.

THEOREM 1. *Let G be a group of order 2^m. If $\sigma(G) \equiv 3 \pmod 4$, then either $Exp(G) = 2^{m-3}$ and G has a dihedral subgroup of index 4 or else $Exp(G) = 2^{m-2}$.*

Thus, if the symmetric genus $\sigma(G) \equiv 3 \pmod 4$, then G is a group of one of two types. First, it may be that G has exponent $Exp(G) = 2^{m-2}$, that is, G has a cyclic subgroup of index 4 but no cyclic subgroup of index 2. The families of 2-groups with this property were classified, long ago, by Burnside [**3**] and Miller [**13**, **14**]. There are two abelian groups and 25 non-abelian groups of this type of order 2^m, as long as $m \geq 6$. It is easy to see that the two abelian groups have symmetric genus 1.

The other possibility for a group G with $\sigma(G) \equiv 3 \pmod 4$ is that $Exp(G) = 2^{m-3}$, and further, G has a dihedral subgroup of index 4. These 2-groups are our main focus here, and we obtain a complete classification of the 2-groups of this type. We show that if $m \geq 7$, there are exactly 27 isomorphism types of these 2-groups (There are fewer for small orders.). The important thing here is that this number of isomorphism types is independent of the size of the 2-group G.

With this classification and the earlier one of Burnside and Miller, our Theorem 1 gives the following.

THEOREM 2. *Let G be a group of order* 2^m. *If* $\sigma(G) \equiv 3 \pmod 4$*, then there are at most* 52 *possible isomorphism types for the group G.*

Of the 52 possible groups of each order, relatively few actually have genus congruent to 3 (mod 4). We do not attempt to classify those families with genus congruent to 3 (mod 4), but such infinite families exist. A consequence of [**4**, Theorem 3.1] is that every group in Miller's family M_5 (see [**12**, Table 2]) has genus congruent to 3 (mod 4). Also, each group in the infinite family H_6 (defined in Table 2) has genus congruent to 3 (mod 4).

The upper bound of Theorem 2 allows us to establish some interesting results using the standard notion of density. We consider the general problem of determining whether there is a 2-group of symmetric genus g, for each value of g. Let T be the set of integers $g \geq 2$ for which there is a 2-group of symmetric genus g; we know that T only contains odd integers. Suppose T_3 is the subset of T consisting of the integers congruent to 3 (mod 4). Then T_3 is infinite, due to the genus formulas for the families $M_5(m)$ and $H_6(m)$. Our main results concerning density are the following.

THEOREM 3. *The set* T_3 *has density 0 in the set of positive integers.*

THEOREM 4. *Almost all positive integers that are the symmetric genus of a* 2*-group are congruent to* 1 *(mod* 4*). Further, the density* $\delta(T)$ *is at most* 1/4.

Theorem 4 has an interesting interpretation in connection with the conjecture that among the finite groups, almost all groups are 2-groups. If this conjecture holds (as it almost certainly does), then our results would imply that almost all groups have symmetric genus congruent to 1 (mod 4).

Not surprisingly, Theorems 3 and 4 agree with the companion results [**12**] about the strong symmetric genus, a closely related parameter. The general approach in [**12**] is along similar lines, but, in fact, the proofs there are easier. This is, however, one instance where work on one parameter suggests the companion results about a related parameter.

2. Preliminaries.

The groups of symmetric genus 0 are the classical, well-known groups that act on the Riemann sphere (possibly reversing orientation) [**8**, Section 6.3.2]. The groups of symmetric genus 1 have also been classified, at least in a sense. These groups act on the torus and fall into 17 classes, corresponding to quotients of the 17 Euclidean space groups [**8**, Section 6.3.3]. Each class is characterized by a presentation, typically a partial one.

For each value of the genus $g \geq 2$, there are only a finite number of groups with symmetric genus g. This is essentially Hurwitz's classical bound for the size of the automorphism group of a Riemann surface. We use the standard well-known approach to group actions on surfaces of genus $g \geq 2$. Let the finite group G act on the (compact) Riemann surface X of genus $g \geq 2$. Then represent $X = U/K$, where K is a Fuchsian surface group and obtain an non-Euclidean crystallographic (NEC) group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that $K = kernel\ \phi$. Associated with the NEC group Γ are its signature and canonical presentation. It is basic that each period and each link period of Γ divide $|G|$. Further, the non-Euclidean area $\mu(\Gamma)$ of a fundamental region for Γ can be calculated directly from its signature [**15**, p. 235]. Then the genus of the surface X on which G acts is given by

$$g = 1 + |G| \cdot \mu(\Gamma)/4\pi. \tag{1}$$

There are four families of non-abelian 2-groups that possess a cyclic subgroup of index 2. A good reference for these groups is [**7**, Section 5.4]. These families can be constructed using the non-trivial automorphisms of a cyclic 2-group. The automorphism group is well-known; for $n \geq 3$, we have

$$Aut(\mathbb{Z}_{2^n}) = \langle -1 \rangle \times \langle 5 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}. \tag{2}$$

These power automorphisms are detailed in [**7**, Lemma 4.1, p. 189]. Three of these families of 2-groups will be needed here, and we describe these three.

For $m \geq 2$, let $D(m)$ be the group with generators x, y and defining relations

$$x^{2^{m-1}} = y^2 = 1, yxy = x^{-1}. \tag{3}$$

The group $D(m)$ is the dihedral group of order 2^m. Each dihedral group has symmetric genus 0.

For $m \geq 4$, let $QD(m)$ be the group with generators x, y and defining relations

$$x^{2^{m-1}} = y^2 = 1, yxy = x^{-1+2^{m-2}}. \tag{4}$$

The group $QD(m)$ of order 2^m is called a *quasi-dihedral* group (or *semi-dihedral* group) [**7**, p. 191]. This group has symmetric genus 1 [**10**, Theorem 2].

For $m \geq 4$, let $QA(m)$ be the group with generators x, y and defining relations

$$x^{2^{m-1}} = y^2 = 1, y^{-1}xy = x^{1+2^{m-2}}. \tag{5}$$

The group $QA(m)$ is a non-abelian group of order 2^m [**7**, p. 190]; we call this group *quasi-abelian* [**10**, p. 237]. This group also has symmetric genus 1 [**10**, Theorem 3].

The fourth family consists of the *dicyclic* groups [**6**, pp. 7, 8]; each dicyclic group has symmetric genus 1 [**10**, p. 236].

The three automorphisms of order 2 of the maximal cyclic group will be called inversion, the quasi-dihedral action and the quasi-abelian action; these actions are given in (3), (4) and (5), respectively. Inversion is also used to construct a dicyclic group, but the element of the group that gives rise to the inner automorphism which is inversion has order 4.

Two additional families of 2-groups will be important here. Each of these groups has a dihedral subgroup of index 2. First, for $m \geq 4$, let $CD(m)$ be the group with generators x, y, z and defining relations

$$x^{2^{m-2}} = y^2 = z^4 = (xy)^2 = 1, xz = zx, yz = zy, z^2 = x^{2^{m-3}}. \tag{6}$$

This group is the central product of the dihedral group $D(m-1)$ and a cyclic group of order 4. We call $CD(m)$ a CD group. Each of these groups is also toroidal, and $\sigma(CD(m)) = 1$ [**11**, Theorem 5].

For $m \geq 5$, let $HD(m)$ be the group with generators x, y, z and defining relations

$$x^{2^{m-2}} = y^2 = z^2 = (xy)^2 = (yz)^2 = 1, zxz = x^{-1+2^{m-3}}. \tag{7}$$

This interesting group of order 2^m has a dihedral subgroup $\langle x, y\rangle$ of index 2 as well as a quasi-dihedral subgroup $\langle x, z\rangle$ of index 2. We call $HD(m)$ a *hyperdihedral* group [**9**, p. 113]. Each group in this family acts on the torus, that is, $\sigma(HD(m)) = 1$ [**11**, Theorem 4].

Each of the groups $HD(m)$ and $CD(m)$ contains a dihedral subgroup of index 2 and has exponent 2^{m-2}. Among the 2-groups with exponent 2^{m-2}, the only other group with a dihedral subgroup of index 2 is the direct product $\mathbb{Z}_2 \times D(m-1)$. For $m \geq 4$, this group has generators x, y, z and defining relations

$$x^{2^{m-2}} = y^2 = z^2 = (xy)^2 = 1, xz = zx, yz = zy. \tag{8}$$

The following classification is in [**9**, Theorem 9]; this result will be important here.

THEOREM A. *Let G be group of order 2^m with a dihedral subgroup M of index 2, with $m \geq 5$. If G has no element of order 2^{m-1}, then G is isomorphic to $\mathbb{Z}_2 \times M$, $HD(m)$ or $CD(m)$.*

We established in [**11**, Theorem 9] that the only 2-groups of even genus are those that act on a Riemann sphere and have genus 0. Important in the proof of the following are the 2-groups with a maximal cyclic subgroup as well as the groups $HD(m)$ and $CD(m)$.

THEOREM B. *Let G be a 2-group with positive symmetric genus. Then $\sigma(G)$ is odd.*

3. 2-groups of odd genus. Here, we consider a 2-group G acting on a Riemann surface of genus $g \equiv 3 \pmod 4$ and obtain a refinement of [**11**, Theorem 7] in this case.

THEOREM 5. *Let G be a group of order 2^m that acts on a Riemann surface X of genus $g \equiv 3 \pmod 4$. Then G contains an element of order 2^{m-3} or larger. If $Exp(G) = 2^{m-3}$, then, further, G contains a dihedral subgroup of index 4.*

Proof. Suppose G acts on the Riemann surface X of genus $g \geq 2$ where $g \equiv 3 \pmod 4$. Represent $X = U/K$, where K is a Fuchsian surface group and obtain an NEC group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that $K = kernel\ \phi$. The NEC group Γ has signature

$$(p; \pm; [\lambda_1, \cdots, \lambda_r]; \{C_1, \cdots, C_k\}),$$

where each period cycle C_i is either empty or contains the link periods $n_{i1}, \cdots, n_{is_i}$. Each link period is the order of a product of involutions in the presentation for Γ. For more information about signatures, see [**15**].

Since K is a surface group, each period λ_i and each link period n_{ij} must be the order of an element of G. The non-Euclidean area is given by

$$\mu(\Gamma)/2\pi = \varepsilon p - 2 + k + \sum\left(1 - \frac{1}{\lambda_i}\right) + \frac{1}{2}\sum\left(1 - \frac{1}{n_{ij}}\right),$$

where $\varepsilon = 1$ or 2 [**15**, p. 235]. Now write $g = 4t + 3$ for some integer t. Then using (1), we have

$$3 + 4t = 1 + 2^{m-1}\left(\varepsilon p - 2 + k + \sum\left(1 - \frac{1}{\lambda_i}\right) + \frac{1}{2}\sum\left(1 - \frac{1}{n_{ij}}\right)\right),$$
$$1 + 2t = 2^{m-2}\left(\varepsilon p - 2 + k + \sum\left(1 - \frac{1}{\lambda_i}\right) + \frac{1}{2}\sum(1 - \frac{1}{n_{ij}})\right).$$

It follows that the sum

$$\sum\left(\frac{2^{m-2}}{\lambda_i}\right)(\lambda_i - 1) + \sum\left(\frac{2^{m-3}}{n_{ij}}\right)(n_{ij} - 1))$$

must be an odd integer. But this clearly would not be the case if $Exp(G) \leq 2^{m-4}$. Hence, $Exp(G) \geq 2^{m-3}$.

Suppose that $Exp(G) = 2^{m-3}$. In this case, an odd number of the link periods must equal to 2^{m-3}. Then suppose that the specific link period $n_{ij} = 2^{m-3}$. Now in the group Γ, there are generating reflections $c_{i,j-1}$ and $c_{i,j}$ with $n_{i,j} = o(c_{i,j-1} \cdot c_{i,j})$. It follows that $\langle \overline{c}_{i,j-1}, \overline{c}_{i,j}\rangle \cong D(m-2)$ in G, and hence, G has a dihedral subgroup of index 4 in this case. □

Proof of Theorem 1. By the previous result, $Exp(G)$ must be at least 2^{m-3}. First, G is not cyclic, since a cyclic group has symmetric genus 0. Suppose then that G contained an element of order 2^{m-1}. If G were abelian, then G would be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-1}}$, a group of genus 0. Thus, G must be non-abelian and either dihedral, dicyclic, quasi-dihedral or quasi-abelian [**7**, Theorem 4.4, p. 193]; but each of these groups has genus 0 or 1. Hence, $Exp(G)$ is either 2^{m-2} or 2^{m-3}. □

Thus, if $\sigma(G) \equiv 3 \pmod 4$, then G is a group of one of two types. First, the families of 2-groups with exponent 2^{m-2} were classified, about a century ago, by Burnside [**3**] and Miller [**13**, **14**]. There are exactly 27 groups of this type of order 2^m, as long as $m \geq 6$; two of these are abelian. First, if G is abelian, then G is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^{m-2}}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$. But each of these groups has symmetric genus 1 [**8**, pp. 291, 292]; these groups are in classes (a) and (h), respectively. The non-abelian groups of this type were studied in [**12**]. In particular, Table 1 of [**12**] gives a presentation for each of the 25 non-abelian groups.

4. Groups with dihedral subgroups of index 4. Here, we study the families of 2-groups that have dihedral subgroups of index 4 but no cyclic subgroups of index 4. There are 27 groups of this type of order 2^m, for each $m \geq 7$.

We use the following notation in all cases. Let G be a group of order 2^m with a dihedral subgroup of index 4 such that $Exp(G) = 2^{m-3}$. Assume that the dihedral subgroup $M \cong D(m-2)$ has generators x and y satisfying the relations (3), with $H = \langle x \rangle$ a cyclic subgroup of index 8. Then, G has a subgroup L of index 2 that

Table 1. Groups with CD subgroup of index 2.

Name	$s^{-1}xs =$	$s^{-1}ys =$	$s^{-1}zs =$	$s^2 =$
J_1	$x^{-1+2^{m-4}}$	y	z	1
J_2	$x^{-1+2^{m-4}}$	y	z	z
J_3	$x^{-1+2^{m-4}}$	y	z^{-1}	1
J_4	x	y	z	1
J_5	x	y	z	z
J_6	x	y	z^{-1}	1
J_7	x	$x^{2^{m-4}}y$	z^{-1}	1

contains the dihedral subgroup M. By Theorem A, L is isomorphic to $CD(m-1)$, $HD(m-1)$ or $\mathbb{Z}_2 \times M$. For each of these three possibilities for the subgroup L, we determine the number of isomorphism types for G.

To construct each group G, we use a standard, well-known technique [**6**, p. 5]. To the group L, we adjoin a new element s, with conjugation by s transforming the elements of L according to an automorphism of order 2. We identify s^2 with a central element u of order j. Then the larger group G has order $2|L|$. The defining relations for G consist of the relations for L, the relations defining the action of s on each generator of L and the relation $s^2 = u$. This general construction suffices in almost all cases.

Proposition 1. *Let G be a group of order 2^m, with $m \geq 7$ and $Exp(G) = 2^{m-3}$. If G contains a subgroup $L \cong CD(m-1)$, then G is isomorphic to one of seven groups; each group is an extension of L with an added generator s and added relations listed in Table* 1.

Proof. The subgroup $L \cong CD(m-1)$ has generators x, y and z satisfying (6). Then the centre $Z(L) = \langle z \rangle$ and M is the unique dihedral subgroup of L with index 2. The group L contains two cyclic subgroups of maximal order. These subgroups are $\langle x \rangle$, which is contained in M, and $\langle xz \rangle$, which is contained in the quasi-dihedral subgroup $\langle xz, y \rangle$. Thus, $\langle z \rangle$, H and M are characteristic in L, and these three subgroups are normal in G. Let C be the centralizer of H in G. Clearly, $\langle x, z \rangle \subseteq C$, but $C \neq G$, since y is not in C. Hence, $[G : C]$ is 2 or 4. In either case, G/C is isomorphic to a subgroup of the automorphism group $Aut(H)$, and $Aut(H)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-5}}$, where the $\mathbb{Z}_2$ factor is generated by the inversion $\alpha(x) = x^{-1}$ [**7**, p. 189].

Case I. Suppose first that $[G : C] = 4$. Then we must have $C = \langle x, z \rangle$. In this case, $G/C \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, with one $\mathbb{Z}_2$ factor generated by inversion and the other $\mathbb{Z}_2$ factor generated by the automorphism $\beta(x) = x^{-1+2^{m-4}}$ (the quasi-dihedral action) [**7**, p. 189]. Hence, there is an element $s \in G - L$ such that $s^{-1}xs = \beta(x) = x^{-1+2^{m-4}}$. Then, easily, $s^{-2}xs^2 = x$, so that s^2 is in the centralizer C. Now we have $G = \langle x, y, z, s \rangle$.

Since $\langle z \rangle$ is normal in G, we must have either $s^{-1}zs = z$ or $s^{-1}zs = z^{-1}$. Also, since the dihedral subgroup $M = \langle x, y \rangle$ is normal, $s^{-1}ys = x^{\ell}y$ for some integer ℓ.

Assume first that $s^{-1}zs = z$. Then s^2 commutes with z so that s^2 is in $Z(\langle s, x, z \rangle) = \langle x^{2^{m-4}}, z \rangle = \langle z \rangle$. By replacing s with sz, if necessary, we may assume that either $s^2 = 1$ or $s^2 = z$. In either case, s^2 commutes with y. Since M is normal, $s^{-1}ys = x^{\ell}y$ for some integer ℓ. Now $y = s^{-2}ys^2 = s^{-1}x^{\ell}ys = x^{2^{m-4}\ell}y$ and so ℓ is even. Write $\ell = 2k$, and then replace y by $x^k y$, and we get the same relations with either $s^{-1}ys = y$ or $s^{-1}ys = x^{2^{m-4}}y$. In the latter case, replace y by $x^{2^{m-5}}y$ and we get the relation $s^{-1}ys = y$. This gives the two groups J_1 and J_2, with $s^2 = 1$ and $s^2 = z$, respectively.

Next, assume that $s^{-1}zs = z^{-1}$. Now $s^2 \in Z(\langle s, x, z \rangle) = \langle z^2 \rangle$ and so $s^2 = 1$ or $s^2 = z^2$. In both cases, by the same argument as before, $s^{-1}ys = y$. Then with $s^2 = 1$,

Table 2. Groups with hyperdihedral subgroup of index 2.

Name	$s^{-1}xs =$	$s^{-1}ys =$	$s^{-1}zs =$	$s^2 =$
H_1	x	y	z	1
H_2	x	y	$zx^{2^{m-4}}$	1
H_3	x	$yx^{2^{m-4}}$	z	1
H_4	x	$yx^{2^{m-4}}$	$zx^{2^{m-4}}$	1
H_5	$x^{1+2^{m-5}}$	y	z	yz
H_6	xyz	z	y	1
H_7	$x^{1+2^{m-5}}yz$	$zx^{2-2^{m-5}}$	$yx^{2+2^{m-5}}$	x^2
H_8	$x^{1+2^{m-5}}yz$	$zx^{2+2^{m-5}}$	$yx^{2-2^{m-5}}$	x^2
H_9	$x^{1+2^{m-5}}yz$	$zx^{2^{m-5}}$	$yx^{-2^{m-5}}$	1
H_{10}	$x^{1+2^{m-5}}yz$	$zx^{-2^{m-5}}$	$yx^{2^{m-5}}$	1

we have the group J_3. The group G with $s^2 = z^2$ is isomorphic to J_3 by the map $\psi : G \to J_3$ defined by $x \mapsto x$, $y \mapsto x^{-1}y$, $z \mapsto z$ and $s \mapsto sx^{1+2^{m-5}}z$.

Case II. Suppose that $[G : C] = 2$. Since inversion is the only non-trivial action on H by any element of G, we may choose $s \in G - L$ so that $s \in C$ and $sx = xs$. Now $s^2 \in \langle x, z\rangle$, since $s^2 \in L \bigcap C$. It is clear that $s^2 = x^{2k}$ or $s^2 = x^{2k}z$, since $o(s) \leq o(x)$. We can replace s by $(x^{-k}s)$ and assume without loss of generality that $s^2 = 1$ or $s^2 = z$. Furthermore, since $\langle z\rangle$ is normal in G, we know that $s^{-1}zs = z$ or $s^{-1}zs = z^{-1}$. Finally, since M is normal in G, we have $s^{-1}ys = x^t y$ for some integer t. Now $y = s^{-2}ys^2 = s^{-1}x^t ys = x^{2t}y$, and $t = 0$ or $t = 2^{m-4}$.

Suppose that $s^{-1}zs = z$. There are two possibilities for s^2 and two possibilities for the action of s on y. Assume first that $s^{-1}ys = y$. Then with $s^2 = 1$ and $s^2 = z$, we obtain the groups J_4 and J_5, respectively. The group G with $s^2 = 1$ and $s^{-1}ys = x^{2^{m-4}}y$ is isomorphic to J_4 by the map $\theta : G \to J_4$ defined by $x \mapsto x$, $y \mapsto y$, $z \mapsto z$ and $s \mapsto szx^{-2^{m-5}}$. The group G with $s^2 = z$ and $s^{-1}ys = x^{2^{m-4}}y$ is isomorphic to J_5 by the map $\varphi : G \to J_5$ defined by $x \mapsto x$, $y \mapsto y$, $z \mapsto z^{-1}$ and $\mapsto sx^{-2^{m-5}}$.

Suppose that $s^{-1}zs = z^{-1}$. It is easy to see that this forces $s^2 = 1$, and we have two groups, J_6 with $s^{-1}ys = y$ and J_7 with $s^{-1}ys = x^{2^{m-4}}y$. These are the final two groups of this type.

It is not hard to check that in each of the seven presentations, the action of s does define an automorphism of L, and it follows that G is a group of order 2^m by the general construction of [**6**, p. 5].

No two of these seven groups are isomorphic. These groups can be distinguished using three group invariants. First, the centre and the abelian quotient invariants suffice to distinguish all but J_3, J_6 and J_7; these two invariants agree for these three groups. These groups have different numbers of involutions; these counts are not difficult, just from the presentations. □

Next, we consider the second type of group, one with a hyperdihedral subgroup of index 2. We omit some details.

Proposition 2. *Let G be a group of order 2^m, with $m \geq 7$ and $Exp(G) = 2^{m-3}$. If G contains a subgroup $L \cong HD(m-1)$, then G is isomorphic to one of 10 groups; each group is an extension of L with an added generator s and added relations listed in Table* 2.

Proof. The subgroup L has generators x, y and z satisfying (7). Then L has two cyclic subgroups of maximal order, namely, $\langle x\rangle$ and $\langle xyz\rangle$. The proof splits into two cases, depending upon whether or not these subgroups are normal in G.

CASE I. Suppose $\langle x\rangle$ is normal in G. Thus, $G/C_G(x)$ has order 4 or 8.

Assume first that $|G/C_G(x)| = 4$. We can find an element s in $G - L$ that centralizes x. Since $s^2 \in C_L(x)$, we see that $s^2 = x^k$ and k must be even. Since $\langle x, s\rangle$ is abelian, choose s so that $s^2 = 1$. A consideration of the possible actions of s on y and of s on z gives four possible presentations and the groups H_1 to H_4 in Table 2.

Suppose that $|G/C_G(x)| = 8$, and let s be any element of $G - L$. Then the automorphism of $\langle x\rangle$ given by conjugation by s has order 4. By multiplying s by the appropriate element of L, we may suppose that $s^{-1}xs = x^{1+2^{m-5}}$. It follows that we must have $s^{-1}ys = yx^k$ and also $s^{-1}zs = zx^k$, with $k = 0$ or $k = 2^{m-4}$. The two choices for k yield two presentations, but both lead to the group H_5.

CASE II. Suppose that $\langle x\rangle$ is not normal in G. Any element s in $G - L$ must interchange the subgroups $\langle x\rangle$ and $\langle xyz\rangle$, and hence $s^{-1}xs = x^k yz$, where k is odd. Therefore, $s^{-1}x^2s = (x^k yz)^2 = (x^2)^{k(1+2^{m-5})}$ and $\langle x^2\rangle$ is normal in G. It follows that $k \equiv 1, -1, 1 + 2^{m-5}$ or $-1 + 2^{m-5}$ (mod 2^{m-4}). By choosing a suitable element s in $G - L$, we have either $s^{-1}xs = xyz$ or $s^{-1}xs = x^{1+2^{m-5}}yz$. Either way, we may assume that $s^2 = x^{2\ell}$, where $\ell = 1$ or ℓ is even, and also that $s^{-1}ys = zx^{2t}$.

Suppose that $s^{-1}xs = xyz$. If $s^2 = x^2$, we can derive a contradiction. Therefore, $s^2 = x^{4k}$ for some integer k. By replacing s with the appropriate element, we have $s^2 = 1$ and $1 = x^{-4t+t2^{m-4}}$. Consequently, G either has relations $s^{-1}ys = zx^{2^{m-4}}$ and $s^{-1}zs = yx^{2^{m-4}}$ or else $s^{-1}ys = z$ and $s^{-1}zs = y$. We now have two complete presentations; each defines the group H_6 in Table 2.

Suppose that $s^{-1}xs = x^{1+2^{m-5}}yz$. We also know that $s^{-1}ys = zx^{2t}$ and $s^2 = x^{2\ell}$, where $\ell = 1$ or ℓ is even. In either case, we have $s^{-1}zs = yx^{2t-2^{m-4}}$.

First, suppose that $s^2 = x^2$. It can be shown that $s^{-1}ys = zx^{2\pm 2^{m-5}}$ and $s^{-1}zs = yx^{2\mp 2^{m-5}}$. This leads to groups H_7 and H_8 in Table 2. Finally, if $s^2 = x^{4k}$, the same type of calculation gives the final two groups, H_9 and H_{10}, in Table 2.

In seven of the presentations, the action of s defines an automorphism of L, and the general construction of [**6**, p. 5] shows that we have a group of order 2^m. The general construction will also handle the groups H_7 and H_8, if we first replace s by $s_1 = sxy$. This gives alternate presentations of H_7 and H_8. Finally, by eliminating the redundant generator z from the presentation for H_5, we see that H_5 is isomorphic to a semi-direct product of $D(m-2)$ by $\mathbb{Z}_4$.

No two of these 10 groups are isomorphic. The first five groups can be distinguished using the centre, the abelian quotient invariants and the fact that H_2 and H_3 are isomorphic to J_6 and J_3, respectively. These invariants also distinguish the first five groups from the second five groups. Among the remaining five groups, H_7 is the only one not generated by involutions and only $H_9(m)$ has a subgroup isomorphic to $\mathbb{Z}_2 \times D(m-2)$. The groups H_6 and H_{10} can be distinguished from H_8 because the quotients of each by $\langle x^4\rangle$ are different groups of order 32. The group $\langle x^4\rangle$ is contained in the intersection of all of the cyclic subgroups of maximal order. Finally, the third centre of H_6 is an abelian subgroup of order 32, and the third centre of H_{10} is a non-abelian subgroup of order 32. □

Table 3. Groups with $\mathbb{Z}_2 \times D(m-2)$ of index 2.

Name	$s^{-1}xs =$	$s^{-1}ys =$	$s^{-1}zs =$	$s^2 =$
A_1	$x^{1+2^{m-4}}$	y	z	1
A_2	$x^{1+2^{m-4}}$	y	z	z
A_3	$x^{1+2^{m-4}}$	zy	z	1
A_4	$x^{1+2^{m-4}}$	zy	z	z
A_5	$x^{1+2^{m-4}}$	y	$zx^{2^{m-4}}$	1
A_6	$x^{1+2^{m-4}}$	$x^{2^{m-5}}zy$	$zx^{2^{m-4}}$	1
A_7	x	y	z	1
A_8	x	y	z	z
A_9	x	yz	z	1
A_{10}	x	yz	z	z
A_{11}	x	$x^{2^{m-4}}y$	z	1
A_{12}	x	$x^{2^{m-4}}yz$	z	z
A_{13}	$x^{-1}z$	y	z	1
A_{14}	$x^{-1}z$	y	z	$x^{2^{m-4}}$
A_{15}	$x^{-1}z$	y	z	z
A_{16}	$x^{-1}z$	y	z	$x^{2^{m-4}}z$
A_{17}	$x^{-1+2^{m-5}}z$	y	$zx^{2^{m-4}}$	1

Now we consider the third type of index 2 subgroup. Again, we provide an outline of the proof but omit quite a few details.

PROPOSITION 3. *Let G be a group of order 2^m, with $m \geq 7$ and $Exp(G) = 2^{m-3}$. If G contains a subgroup $L \cong \mathbb{Z}_2 \times D(m-2)$, then G is isomorphic to one of 17 groups; each group is an extension of L with an added generator s and added relations listed in Table* 3.

Proof. The subgroup L has generators x, y and z satisfying (8). The centre $Z(L) = \langle x^{2^{m-4}}, z\rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. There are two maximal cyclic subgroups of L, namely $\langle x\rangle$ and $\langle zx\rangle$. The proof splits into two cases, depending upon whether or not these subgroups are normal in G.

CASE I. Suppose that $\langle x\rangle$ is normal in G. Now $C_L(x) = \langle x, z\rangle$ and it follows that $G/C_G(x)$ has order 2 or order 4. We consider these two possibilities in two subcases.

Subcase a. Suppose that $G/C_G(x)$ has order 4. Then we can find an element s in $G - L$ such that $s^{-1}xs = x^{1+2^{m-4}}$. It is easy to see that either $s^{-1}zs = z$ or $s^{-1}zs = zx^{2^{m-4}}$. Further, by replacing s by an element of the form $x^{-t}s$ if necessary, we may assume that either $s^2 = 1$ or $s^2 = z$.

Since y acts on x by inversion, so does $s^{-1}ys$. Therefore, $s^{-1}ys = x^ky$ or $s^{-1}ys = x^kyz$. First, suppose that $s^{-1}zs = z$. Then since $y = s^{-2}ys^2 = x^{2k+k2^{m-4}}y$, we see that $k = 0$ or $k = 2^{m-4}$, and there are four possible actions of s on y. With the two possibilities for s^2, there are eight presentations. With $k = 0$, we obtain the four groups A_1 through A_4 in Table 3. Each of the four presentations with $k = 2^{m-4}$ leads to groups isomorphic to one of these four.

Now suppose that $s^{-1}zs = zx^{2^{m-4}}$. First, it is clear that $s^2 = z$ is not possible and so $s^2 = 1$. Again, it follows that $s^{-1}ys = x^ky$ or $s^{-1}ys = x^kyz$. These two relations (with appropriate values for k) lead to groups A_5 and A_6, respectively.

Subcase b. Suppose that $G/C_G(x)$ has order 2. Then we can find an element s in $G - L$ that $s^{-1}xs = x$. Also, $s^2 \in C_L(x) = \langle x, z\rangle$, and therefore $s^2 = x^\ell$ or $s^2 = x^\ell z$ for some integer ℓ. As before, we can get $s^2 = 1$ or $s^2 = z$. It is also easy to see that either $s^{-1}zs = z$ or $s^{-1}zs = zx^{2^{m-4}}$. If $s^{-1}zs = zx^{2^{m-4}}$, then with some work, we are back in subcase (a) and so $s^{-1}zs = z$.

As before, $s^{-1}ys = x^k y$ or $s^{-1}ys = x^k yz$ and $k = 0$ or $k = 2^{m-4}$. So, there are four possible actions of s on y and with the two possibilities for s^2, again there are eight presentations. In this subcase, there are six different groups, the groups A_7–A_{12}.

CASE II. Suppose that $\langle x\rangle$ is not normal in G. Any element s in $G - L$ conjugates $\langle x\rangle$ to $\langle xz\rangle$. As before, $\langle x^2\rangle$ is normal in G. Choosing s carefully, we get $s^{-1}xs = x^{-1+\ell 2^{m-5}}z$ for ℓ equal to 0, 1, 2 or 3. As before, $s^{-1}zs = z$ or $s^{-1}zs = zx^{2^{m-4}}$.

If $s^{-1}zs = z$, without loss of generality, $s^{-1}xs = x^{-1}z$ and we have $s^2 \in C_{\langle x,z\rangle}(s) = \langle x^{2^{m-4}}, z\rangle$. We only need to consider two actions of s on y, $s^{-1}ys = y$ and $s^{-1}ys = yz$. With the relation $s^{-1}ys = y$, we obtain the groups A_{13}–A_{16} in Table 3. The other action $s^{-1}ys = yz$ yields the same four groups.

Finally, suppose that $s^{-1}zs = zx^{2^{m-4}}$. By replacing generators, if needed, we may assume that $s^{-1}xs = x^{-1+2^{m-5}}z$ and $s^{-1}ys = y$, and there are two presentations, depending on the value of s^2. Both presentations yield A_{17}. This completes the listing of the groups in Table 3.

In each of the 17 presentations, the action of s defines an automorphism of L, and consequently, in each case, we obtain a group of order 2^m by the general construction of [**6**, p. 5].

Finally, we need to check that no two of these 17 groups are isomorphic. First, the centre and the abelian quotient invariants distinguish A_1, A_2, A_5, A_7, A_8 and A_{11}. Further, these two invariants separate the other 11 into two sets, the pair A_6 and A_{17} and the remaining nine. Case I and case II groups cannot be isomorphic. This distinguishes A_6 from A_{17} and helps with the others. Of these nine groups, only A_{12} (case I) and A_{16} (case II) are not generated by involutions. This leaves seven groups to consider. Then using the second centre and separating groups by the two cases distinguishes A_{15} and divides the others into three pairs, A_3 and A_9, A_4 and A_{10}, and the pair A_{13} and A_{14}. In the groups A_3 and A_4, the centralizer of a maximal order cyclic subgroup that is normal has index 4, whereas in A_9 and A_{10}, it has index 2. Finally, to separate the pair A_{13} and A_{14}, we use quotient groups by the three central subgroups of order 2. Two of the corresponding subgroups give isomorphic quotients. However, the group A_{13} has a third quotient group isomorphic to $\mathbb{Z}_2 \times D(m-2)$, but the corresponding quotient of A_{14} is isomorphic to $CD(m-1)$). These are the quotients by $\langle z\rangle$ in our presentations. □

THEOREM 6. *Let G be a group of order 2^m, with $m \geq 7$. If G has a dihedral subgroup of index 4 such that $Exp(G) = 2^{m-3}$, then G is isomorphic to one of 27 groups, independent of m.*

Proof. Theorem A and Propositions 1–3 show that there are at most 34 groups. Table 4 gives seven isomorphisms among these three types of groups.

Finally, it is necessary to show that there are no further isomorphisms among the remaining 27 groups. A careful consideration of the centre and the abelian quotient invariants for these groups distinguishes nine of the groups and separates the others into three sets, the trio J_3, J_6 and J_7, the set of nine A_is considered in the proof of Proposition 3, and a final set of six, A_6 and the set of five H_is considered in the proof of Proposition 2. Since the groups of each type have been classified, the only possible

Table 4. Isomorphisms of groups.

Map	Image of x	Image of y	Image of z	Image of s
$\phi : H_1 \to A_1$	x	y	ys	z
$\phi : H_2 \to A_5$	x	y	ys	z
$\phi : H_2 \to J_6$	xs	y	yzs	s
$\phi : H_3 \to J_3$	x	y	z	$zx^{2^{m-5}}$
$\phi : H_4 \to J_1$	x	y	s	$zx^{2^{m-5}}$
$\phi : H_9 \to A_{17}$	xys	xy	xyz	sy
$\phi : J_4 \to A_{11}$	x	y	$sx^{2^{m-5}}$	z

remaining isomorphism is between A_6 and some H_j. But it is not hard to see that the group A_6 does not have a hyperdihedral subgroup of index 2. This completes the classification. □

Now Theorem 1, the classification of Burnside and Miller, and the classification of Theorem 6 combine to establish Theorem 2.

Of the 52 possible 2-groups of each order, relatively few actually have genus congruent to 3 (mod 4). Some have symmetric genus 1, and some have higher genus $\sigma(G) \equiv 1 \pmod 4$. For example, among the groups of order 128, there are 10 groups with genus congruent to 3 (mod 4); the symmetric genus of each group was calculated using MAGMA.

Among the 52 infinite families, there are some containing groups with genus congruent to 3 (mod 4). In [**4**], Conder and Tucker define the following group of order $16n$.

$$V_n = \langle x, y | x^4 = y^4 = [x^2, y] = [y^2, x] = 1, (xy)^{2n} = x^2 \rangle.$$

They prove that $\sigma(V_n) = 4n - 1$ for all $n > 1$ [**4**, Theorem 3.1]. This gives examples of order 2^m with genus congruent to 3 (mod 4), for all $m \geq 7$. A little bit of work suffices to show that

$$V_{2^{m-4}} \cong M_5(m),$$

one of Miller's 2-groups from [**13**]. The family H_6 is another family of groups with genus congruent to 3 (mod 4). We omit the proof.

PROPOSITION 4. *Suppose that G is the group $H_6(m)$ of order 2^m, where $m \geq 7$. Then G has symmetric genus $\sigma(G) = 2^{m-4} - 1$.*

5. Density. Now we consider the general problem of determining whether there is a 2-group of symmetric genus g for each value of g, and describe our results using the standard notion of density.

Let T be the set of integers $g \geq 2$ for which there is a 2-group of symmetric genus g. By Theorem B, all the integers in T are odd. For an integer n, let $f(n)$ denote the number of integers in T that are less than or equal to n. Then the natural density $\delta(T)$ of T in the set of positive integers is

$$\delta(T) = \lim_{n \to \infty} \frac{f(n)}{n}.$$

Also, let T_3 be the subset of T consisting of the integers congruent to 3 (mod 4), with the companion "counting" function denoted by f_3.

Although the set T_3 is infinite, the upper bound of Theorem 2 suffices to prove that the density of T_3 in the set of positive integers is zero.

Proof of Theorem 3. First, among the 2-groups of order 64 or less, there are exactly 11 groups with genus congruent to 3 (mod 4). Assume $n = 2^m$, with $m \geq 7$, and let G be a 2-group with genus n or less such $\sigma(G) \equiv 3 \pmod 4$. From the basic lower bound for the genus of a 2-group, we have

$$|G| \leq 32(\sigma(G) - 1) \leq 32(n - 1) < 2^{m+5},$$

so that $|G| \leq 2^{m+4}$. For each of the possible $m - 2$ orders in the range 128, 256, ..., 2^{m+4}, there are at most 52 groups with genus congruent to 3 (mod 4), by Theorem 2. Thus, $f_3(n) = f_3(2^m) \leq 52(m - 2) + 11$, counting the 11 groups of small order. Hence, $\delta(T_3) = 0$. □

Together, Theorems B and 3 clearly imply Theorem 4.

Another interesting interpretation of our results is possible by considering group counting functions, together with the abundance of 2-groups. Here, see the recent survey article [**5**], together with [**1**] and the book [**2**].

For the positive integer n, the *group number* of n, denoted by $gnu(n)$, is the number of distinct abstract groups of order n [**5**]. The values of $gnu(n)$ are given for all $n < 2,048$ in the Appendix of [**5**].

Let F be a family of finite groups. For a positive integer n, let $f(n)$ denote the number of groups in the family F that have order n or less, and let $t(n)$ be the total number of groups of these orders. Then the natural *group density* $\Delta(F)$ of the family F in the collection of finite groups is

$$\Delta(F) = \lim_{n \to \infty} \frac{f(n)}{t(n)}.$$

For small n, values of the counting function t may be obtained by summing values of the *gnu* function, of course.

Now let F_2 be the family of finite 2-groups, with companion counting function f_2. As is well understood, the number of 2-groups simply overwhelms the number of other groups. In fact, the following conjecture is well-known.

CONJECTURE. The group density of the 2-groups is 1, that is, $\Delta(F_2) = 1$.

We call this conjecture the density of 2-groups (D2G) conjecture. If the D2G conjecture holds, then in this sense, almost all finite groups are 2-groups.

Theorem 2 can now be given another interpretation. Let F_1 be the family of finite groups, each of which has symmetric genus congruent to 1 (mod 4). Now the following clearly holds; for a detailed proof of a very similar result, see [**12**, Theorem 9].

THEOREM 7. *If the D2G conjecture holds, then* $\Delta(F_1) = 1$.

Among the finite groups, then, almost all groups would have symmetric genus congruent to 1 (mod 4) (assuming that the D2G conjecture holds). On the other hand, there is the conjecture that for every integer $g \geq 0$, there is a group G with symmetric genus $\sigma(G) = g$ [**4**, p. 273].

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